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Muntz Rational Approximation for Special Function Classes in Orlicz Spaces

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Abstract. Using the method of construction, with the help of inequalities, we research the Muntz rational approximation of two kinds of special function classes, and give the corresponding estimates of approximation rates of these classes under widely conditions. Because of the Orlicz Spaces is bigger than continuous function space and the Lp space, so the results of this paper has a certain expansion significance.

Key Words: Muntz rational approximation, bounded variation function class, Sobolev function class, Orlicz space.

AMS Subject Classifications: 41A25, 43A90

1 Introduction

For any given real sequence $\{\lambda_n\}_{n=1}^{\infty}$, denote by $\prod_n(\Lambda)$ the set of Muntz polynomials of degree n, that is, all linear combinations of $\{x^{\lambda_1},x^{\lambda_2},\cdots,x^{\lambda_n}\}$, and let $R_n(\Lambda)$ be the Muntz rational functions of degree n, that is,

$$R_n(\Lambda)=\left\{\frac{P_n(x)}{Q_n(x)}:P_n(x)\in\prod_n(\Lambda),\ Q_n(x)\in\prod_n(\Lambda),\ Q_n(x)>0,\ x\in[0,1]\right\}.$$

If $Q(0)=0$, we assume that

$$\lim_{x\to 0^+}\frac{P(x)}{Q(x)}$$

exists and is finite.

Study on the Muntz rational approximation rate is a new research field of rational approximation, see [5] is a pioneer in this work, he got a Jackson type theorem of continuous function space in the first by using the method of construction, that is

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Theorem 1.1. *Assume $f(x)\in C[0,1]$ and given $M>0$, if $\lambda_{n+1}-\lambda_n\geq Mn$ for all $n\geq 1$, then there is a $r(x)\in R_n(\Lambda)$ and a positive constant C_M which only depend on M, such that*

$$\|f-r\|\leq C_M\omega\Big(f,\frac{1}{n}\Big).$$

Here $\omega(f,\frac{1}{n})$ is modulus of continuity of $f(x)$ in normal sense.

Under the expanded condition of real sequence $\{\lambda_n\}_{n=1}^{\infty}$, document [1] obtained following

Theorem 1.2. *Assume $f(x)\in C[0,1]$, $\alpha\geq\frac{1}{2}$ and given $M>0$, if $\lambda_{n+1}-\lambda_n\geq Mn^{\alpha}$ for all $n\geq 1$, then there is a $r(x)\in R_n(\Lambda)$ and a positive constant $C_{M,\alpha}$ which only depend on M and α, such that*

$$\|f-r\|\leq C_{M,\alpha}\omega\Big(f,\frac{1}{n}\Big).$$

Document [2] considered Muntz rational approximation problem of bounded variation function class and Sobolev class, obtained following Jackson type estimate

Theorem 1.3. *Assume $f(x)\in BV[0,1]$, $\alpha\geq\frac{1}{2}$ and given $M>0$, if $\lambda_{n+1}-\lambda_n\geq Mn^{\alpha}$ for all $n\geq 1$, then there is a $r(x)\in R_n(\Lambda)$ and a positive constant $C_{M,\alpha}$ which only depend on M and α, such that*

$$\|f-r\|_1\leq\frac{C_{M,\alpha}V(f)}{n},$$

where $V(f)$ represents the total variation of f on $[0,1]$.

Theorem 1.4. *Assume $f(x)\in W_p^1[0,1]$, $\alpha\geq\frac{1}{2}$ and given $M>0$, if $\lambda_{n+1}-\lambda_n\geq Mn^{\alpha}$ for all $n\geq 1$, then there is a $r(x)\in R_n(\Lambda)$ and a positive constant $C_{M,P,\alpha}$ which only depend on M, P and α, such that*

$$\|f-r\|_P\leq\frac{C_{M,P,\alpha}}{n}\|f'\|_p.$$

The purpose of this paper is to discuss the Muntz rational approximation problem of bounded variation classes and Sobolev class in Orlicz spaces.

In this paper, $M(u)$ and $N(v)$ denote the mutually complementary N function, the definition and properties of the N function can be seen [4]. The Orlicz Space $L_M^*[0,1]$ generated by N function $M(u)$ is all measurable functions $\{u(x)\}$ that have a finite Orlicz norm

$$\|u\|_M=\sup_{\rho(v,N)\leq 1}\left|\int_0^1 u(x)v(x)dx\right|, \tag{1.1}$$

where $\rho(v,N)=\int_0^1 N(v(x))dx$ is the modulus of $v(x)$ with respect to $N(v)$.

For $f(x)\in L_M^*[0,1]$, define the best Muntz rational approximation as

$$R_n(f)_M=\inf_{r\in R_n(\Lambda)}\|f-r\|_M.$$

Our main results are following:

Theorem 1.5. *Assume $f(x)\in BV[0,1]$, $\alpha\ge\frac{1}{2}$ and given $A>0$, if $\lambda_{n+1}-\lambda_n\ge An^{\alpha}$ for all $n\ge 1$, then there is a $r(x)\in R_n(\Lambda)$ and a positive constant $C_{A,\alpha}$ which only depend on A, α and independ on n, such that*

$$\|f-r\|_M\le\frac{C_{A,\alpha}V(f)}{n}.$$

Theorem 1.6. *Assume $f(x)\in W_M^1[0,1]$, $\alpha\ge\frac{1}{2}$ and given $A>0$, if $\lambda_{n+1}-\lambda_n\ge An^{\alpha}$ for all $n\ge 1$, then there is a $r(x)\in R_n(\Lambda)$ and a positive constant $C_{A,\alpha}$ which only depend on A, α and independ on n, such that*

$$\|f-r\|_M\le\frac{C_{A,\alpha}}{n}\|f'\|_M,$$

where $W_M^1[0,1]=\{f: f\in AC[0,1],\ f'\in L_M^*[0,1]\}$.

Note that constants C appeared in the paper in different places represent different values.

2 Auxiliary lemmas

For any $x\in[0,1]$, let

$$\begin{aligned}&x=1+\cos\theta,\quad \frac{\pi}{2}\le\theta\le\pi,\\&x_j=1+\cos\theta_j,\quad \theta_j=\frac{2n-2j+1}{2n}\pi,\quad j=1,2,\cdots,\left[\frac{n}{2}\right].\end{aligned}$$

For convenience, we denote, $x_0=0$, $x_{[\frac{n}{2}]+1}=1$.

Furthermore, set

$$P_j(x)=x^{\lambda_j}\prod_{l=1}^{j}x_l^{-\Delta\lambda_l},\quad r_k(x)=\frac{P_k(x)}{\sum\limits_{l=1}^{[\frac{n}{2}]}P_l(x)},\quad j=1,2,\cdots,\left[\frac{n}{2}\right],\quad k=1,2,\cdots,\left[\frac{n}{2}\right],$$

where $\Delta\lambda_1=\lambda_1$, $\Delta\lambda_k=\lambda_k-\lambda_{k-1}$, $k=2,3,\cdots$.

We construct the rational operator as following

$$L_n(f,x)=\sum_{k=1}^{[\frac{n}{2}]}f(x_k)r_k(x),$$

where

$$\sum_{k=1}^{[\frac{n}{2}]}r_k(x)=1$$

is evidently.

Lemma 2.1 (see [1]). *Let* $x\in[x_{j-1},x_j]$, $j=1,2,\cdots,[\frac{n}{2}]+1$, *if* $1\leq k\leq[\frac{n}{2}]$, *then*

$$\begin{aligned}&r_k(x)\leq C_A e^{-A|\sqrt{j}-\sqrt{k}|},\\&|x-x_k|\leq C\Big(\frac{(|j-k|+1)^2}{n^2}+\frac{|j-k|+1}{n}\sqrt{x}\Big),\end{aligned}$$

where A *is given in the above two theorems.*

Lemma 2.2 (see [3]). *For any* $f\in L_M^*[0,1]$, *we denote the Hardy-Littlewood maximum function of* f *by*

$$\theta_f(x)=\sup_{0\leq t\leq 1,t\neq x}\frac{1}{t-x}\int_x^t|f(u)|du,$$

then

$$\|\theta_f\|_M\leq C\|f\|_M.$$

Lemma 2.3 (Holder inequality, see [4]). *For any* $u(x)\in L_M^*$, $v(x)\in L_N^*$, *we have*

$$\int_D u(x)v(x)dx\leq\|u\|_M\|v\|_N.$$

3 Proof of theorems

Proof of Theorem 1.5. We need only to prove

$$\|f-L_n(f)\|_M\leq\frac{C_{A,\alpha}V(f)}{n}.$$

Applying Jordan decomposition, for any $f(x)\in BV[0,1]$, there exist two monotonically increasing functions $g(x)$, $h(x)$, such that

$$f(x)=g(x)-h(x),\quad V(f)=V(g)+V(h).$$

Furthermore, suppose $g(0)=h(0)=0$, define the monotonic function $g_m(x)$ as follows

$$g_m(x)=\begin{cases}0, & 0<x<x_m,\\ g(x)-g(x_m), & x_m\leq x\leq x_{m+1},\\ d_m, & x_{m+1}<x\leq 1,\end{cases}$$

where

$$d_m=g(x_{m+1})-g(x_m),\quad x_m=1+\cos\theta_m,\quad \theta_m=\frac{2n-2m+1}{2n}\pi,\quad m=0,1,\cdots,\Big[\frac{n}{2}\Big]+1.$$

Set $x_0=0$, $x_{[\frac{n}{2}]+1}=1$, then

$$g(x)=\sum_{m=0}^{[\frac{n}{2}]+1} g_m(x), \quad V(g)=\sum_{m=0}^{[\frac{n}{2}]+1} V(g_m)=\sum_{m=0}^{[\frac{n}{2}]+1} d_m.$$

Fist of all, we will show

$$\|g-L_n(g)\|_M\le \frac{C_A}{n}V(g).$$

We have

$$\|g-L_n(g)\|_M=\left\|\sum_{m=0}^{[\frac{n}{2}]+1}(g_m-L_n(g_m))\right\|_M \le \sum_{m=0}^{[\frac{n}{2}]+1}\|g_m-L_n(g_m)\|_M.$$

Hence we will estimate $\|g_m-L_n(g_m)\|_M$,

$$\begin{aligned}
\|g_m-L_n(g_m)\|_M&=\sup_{\rho(v,N)\le 1}\left|\int_0^1 (g_m(x)-L_n(g_m,x))v(x)dx\right| \\
&=\sup_{\rho(v,N)\le 1}\left|\sum_{j=1}^{[\frac{n}{2}]+1}\int_{x_{j-1}}^{x_j}\sum_{k=1}^{\left[\frac{n}{2}\right]}(g_m(x)-g_m(x_k))r_k(x)v(x)dx\right| \\
&=\sup_{\rho(v,N)\le 1}\left\{\left|\left(\sum_{j=1}^{m}+\sum_{j=m+2}^{[\frac{n}{2}]+1}\right)\int_{x_{j-1}}^{x_j}\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)v(x)dx\right.\right. \\
&\qquad \left.\left.+\int_{x_m}^{x_{m+1}}\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)v(x)dx\right|\right\} \\
&=:\sup_{\rho(v,N)\le 1}\{K_1+K_2+K_3\} \\
&\le \sup_{\rho(v,N)\le 1}K_1+\sup_{\rho(v,N)\le 1}K_2+\sup_{\rho(v,N)\le 1}K_3.
\end{aligned}$$

From the representation of $g_m(x)$ and the process of proof of Theorem 3 in [6], we easy to see

$$\begin{aligned}
\sup_{\rho(v,N)\le 1}K_1&=\sup_{\rho(v,N)\le 1}\left|\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j}\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)v(x)dx\right| \\
&\le \sup_{\rho(v,N)\le 1}\left|\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j}\sum_{k=m+1}^{[\frac{n}{2}]}d_m r_k(x)v(x)dx\right|
\end{aligned}$$

$$
\begin{aligned}
&= d_m \sup_{\rho(v,N)\le 1}\left|\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j}\sum_{k=m+1}^{[\frac{n}{2}]} r_k(x)v(x)dx\right| \\
&\le C_A d_m \sup_{\rho(v,N)\le 1}\left|\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j}\sum_{k=m+1}^{[\frac{n}{2}]} e^{-A|\sqrt{j}-\sqrt{k}|}v(x)dx\right| \\
&\le C_A d_m \sup_{\rho(v,N)\le 1}\left|\sum_{j=1}^{m}\sum_{k=m+1}^{[\frac{n}{2}]} e^{-A|\sqrt{j}-\sqrt{k}|}\int_{x_{j-1}}^{x_j} v(x)dx\right|.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|x_j-x_{j-1}| &= \left|\cos\frac{2n-2j+1}{2n}\pi-\cos\frac{2n-2j+3}{2n}\pi\right| \\
&= \left|2\sin\frac{n-j+1}{n}\pi\sin\frac{1}{2n}\pi\right| \\
&\le C\frac{\pi}{2n}\le Cn^{-1}.
\end{aligned}
$$

Due to almost everywhere boundedness of $v(x)$ and the

$$
\sum_{j=1}^{m}\sum_{k=m+1}^{[\frac{n}{2}]} e^{-A|\sqrt{j}-\sqrt{k}|}
$$

is a part sum of a convergent series, hence we have

$$
\sup_{\rho(v,N)\le 1} K_1 \le \frac{C_A}{n} d_m.
$$

Similarly,

$$
\begin{aligned}
\sup_{\rho(v,N)\le 1} K_2 &= \sup_{\rho(v,N)\le 1}\left|\sum_{j=m+2}^{[\frac{n}{2}]+1}\int_{x_{j-1}}^{x_j}\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)v(x)dx\right| \\
&\le \sup_{\rho(v,N)\le 1}\left|\sum_{j=m+2}^{[\frac{n}{2}]+1}\int_{x_{j-1}}^{x_j}\sum_{k=1}^{m} d_m r_k(x)v(x)dx\right| \\
&\le C_A d_m \sup_{\rho(v,N)\le 1}\left|\sum_{j=m+2}^{[\frac{n}{2}]+1}\int_{x_{j-1}}^{x_j}\sum_{k=1}^{m} e^{-A|\sqrt{j}-\sqrt{k}|}v(x)dx\right| \\
&\le C_A d_m \sup_{\rho(v,N)\le 1}\left|\sum_{j=m+2}^{[\frac{n}{2}]+1}\sum_{k=1}^{m} e^{-A|\sqrt{j}-\sqrt{k}|}\int_{x_{j-1}}^{x_j} v(x)dx\right| \\
&\le \frac{C_A}{n} d_m,
\end{aligned}
$$

$$\begin{aligned}
\sup_{\rho(v,N)\le 1} K_3 &= \sup_{\rho(v,N)\le 1}\left|\int_{x_m}^{x_{m+1}}\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)v(x)dx\right| \\
&= \sup_{\rho(v,N)\le 1}\left|\int_{x_m}^{x_{m+1}}\Big(g_m(x)-\sum_{k=m+1}^{[\frac{n}{2}]} d_m r_k(x)\Big)v(x)dx\right| \\
&\le d_m \sup_{\rho(v,N)\le 1}\left|\int_{x_m}^{x_{m+1}}\Big(1+\sum_{k=m+1}^{[\frac{n}{2}]} r_k(x)\Big)v(x)dx\right| \\
&\le 2d_m \sup_{\rho(v,N)\le 1}\left|\int_{x_m}^{x_{m+1}} v(x)dx\right| \left(\left|\sum_{k=m+1}^{[\frac{n}{2}]} r_k(x)\right|\le 1\right) \\
&\le \frac{C}{n}d_m.
\end{aligned}$$

To sum up above, we get

$$\|g_m-L_n(g_m)\|_M\le \frac{C_A}{n}d_m,$$

therefore

$$\|g-L_n(g)\|_M\le C_A\sum_{m=0}^{[\frac{n}{2}]+1} d_m\frac{1}{n}=\frac{C_A}{n}V(g).$$

Since $h(x)$ is also a monotonically increasing function, so we also have

$$\|h-L_n(h)\|_M\le \frac{C_A}{n}V(h).$$

Hence

$$\|f-L_n(f)\|_M\le \|g-L_n(g)\|_M+\|h-L_n(h)\|_M\le \frac{C_A}{n}(V(g)+V(h))\le \frac{C_A}{n}V(f).$$

Theorem 1.5 is proved. □

Proof of Theorem 1.6. Using Lemma 2.3, choosing $\zeta_j\in\Delta_j=[x_{j-1},x_j]$, such that

$$\theta_{f'}(\zeta_j)=\inf\{\theta_{f'}(x):x\in\Delta_j\}.$$

Modifying the construction of $L_n(f,x)$ suitable

$$L_n^*(f,x)=\sum_{j=1}^{[\frac{n}{2}]} f(\zeta_j)r_j(x),$$

then we need only to show

$$
\begin{aligned}
&\|f-L_n^*(f)\|_M\le \frac{C_A}{n}\|f'\|_M,\\
&\|f-L_n^*(f)\|_M\le \sup_{\rho(v,N)\le 1}\left|\int_0^1\sum_{j=1}^{[\frac{n}{2}]}|f(x)-f(\zeta_j)|r_j(x)v(x)dx\right|\\
&\qquad\qquad\qquad = \sup_{\rho(v,N)\le 1}\left|\int_0^1\sum_{j=1}^{[\frac{n}{2}]}|x-\zeta_j|\frac{1}{|x-\zeta_j|}\left|\int_{\zeta_j}^x f'(t)dt\right|r_j(x)v(x)dx\right|\\
&\qquad\qquad\qquad = \sup_{\rho(v,N)\le 1}\left|\int_0^1\sum_{j=1}^{[\frac{n}{2}]}|x-\zeta_j||\theta_{f'}(\zeta_j)|r_j(x)v(x)dx\right|\\
&\qquad\qquad\qquad \le \sup_{\rho(v,N)\le 1}\left|\int_0^1\sum_{j=1}^{[\frac{n}{2}]}|x-\zeta_j|\frac{1}{|\Delta_j|}\int_{\Delta_j}|\theta_{f'}(t)|dt r_j(x)v(x)dx\right|\\
&\qquad\qquad\qquad \le Cn\sup_{\rho(v,N)\le 1}\left|\int_0^1\sum_{j=1}^{[\frac{n}{2}]}|x-\zeta_j|\int_{\Delta_j}|\theta_{f'}(t)|dt r_j(x)v(x)dx\right|\\
&\qquad\qquad\qquad \le Cn\sup_{\rho(v,N)\le 1}\left|\sum_{j=1}^{[\frac{n}{2}]}\int_{\Delta_j}|\theta_{f'}(t)|dt\int_0^1 r_j(x)|x-\zeta_j|v(x)dx\right|.
\end{aligned}
$$

Using Lemma 2.1, we have

$$
\begin{aligned}
&\int_0^1 r_j(x)|x-\zeta_j|v(x)dx\\
=&\sum_{k=1}^{[\frac{n}{2}]+1}\int_{\Delta_k} r_j(x)|x-\zeta_j|v(x)dx\\
\le&\sum_{k=1}^{[\frac{n}{2}]+1}\int_{\Delta_k} C\Big(\frac{(|j-k|+1)^2}{n^2}+\frac{|j-k|+1}{n}\Big)C_Ae^{-A|\sqrt{j}-\sqrt{k}|}v(x)dx\\
\le&C\frac{1}{n}\sum_{k=1}^{[\frac{n}{2}]+1}C_Ae^{-A|\sqrt{j}-\sqrt{k}|}\int_{\Delta_k}v(x)dx\\
\le&C_A\frac{1}{n^2}\sum_{k=1}^{[\frac{n}{2}]+1}e^{-A|\sqrt{j}-\sqrt{k}|}\\
\le&C_A\frac{1}{n^2},
\end{aligned}
$$

from Lemma 2.3, we have

$$\|f-L_n^*(f)\|_M=\frac{C_A}{n}\left|\sum_{j=1}^{[\frac{n}{2}]}\int_{\Delta_j}|\theta_{f'}(t)|dt\right|=\frac{C_A}{n}\left|\int_0^1|\theta_{f'}(t)|dt\right|$$
$$\leq\frac{C_A}{n}\|\theta_{f'}\|_M\|1\|_N\leq\frac{C_A}{n}\|f'\|_M.$$

Theorem 1.6 is proved. □

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