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# Regularity of Viscous Solutions for a Degenerate Non-linear Cauchy Problem

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**Abstract.** We consider the Cauchy problem for a class of nonlinear degenerate parabolic equation with forcing. By using the vanishing viscosity method it is possible to construct a generalized solution. Moreover, this solution is a Lipschitz function on the spatial variable and Hölder continuous with exponent 1/2 on the temporal variable.

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## 1 Introduction

In this paper we consider the Cauchy problem for nonlinear degenerate parabolic equation

$$u_t = u\Delta u - \gamma|\nabla u|^2 + f(t,u), \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \tag{1.1}$$
$$u(x,0) = u_0(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \tag{1.2}$$

where $\gamma$ is a nonnegative constant. Eq. (1.1) arises in severals applications of biology and physics, see [1,2]. Eq. (1.1) is of degenerate parabolic type: parabolicity it is loss at

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points where $u=0$, see [1,3] for a most detailed description. In [4] a weak solution for the homogeneous equation (1.1) is constructed by using the vanishing viscosity method [5], the regularity of the weak solutions for the homogeneous Cauchy problem (1.1)-(1.2) was studied by the author in [6] and an extension for the inhomogeneous case is given in [7]. In this paper we extend the above results for the inhomogeneous case, this extension is interesting from physical viewpoint, since the Eq. (1.1) is related with non-equilibrium process in porous media due to external forces. We obtain the following main theorem,

**Theorem 1.1.** *If* $\gamma \geq \sqrt{2N}-1$, $|\nabla(u_0^{1+\frac{\alpha}{2}})| \leq M$, *where* $M$ *is a positive constant such as*

$$\alpha^2+(\gamma+1)\alpha+\frac{N}{2}\leq 0,$$

*then the viscosity solutions of the Cauchy problem* (1.1)-(1.2) *satisfies*

$$|\nabla(u^{1+\frac{\alpha}{2}})| \leq M. \tag{1.3}$$

We principally followed the ideas of the authors in [6,7], where the particular reaction therm $Ku^m$ was considered.

## 2 Preliminaries

We begin this section with the definition of solutions in weak sense.

**Definition 2.1.** *A function* $u \in L^{\infty}(\Omega)\bigcap L^2_{loc}([0,+\infty);H^1_{Loc}(\mathbb{R}^N))$, *is called a weak solution of* (1.1)-(1.2) *if it satisfies the following conditions:*

*(i)* $u(x,t)\geq 0$ *, a.e in* $\Omega$.

*(ii)* $u(x,t)$ *satisfies the following relation*

$$\int\limits_{\mathbb{R}^N} u_0\psi(x,0)\,\mathrm{d}x+\iint\limits_{\Omega}(u\psi_t-u\nabla u\cdot\nabla\psi-(1+\gamma)|\nabla u|^2\psi-f(t,u)\psi)\,\mathrm{d}x\mathrm{d}t=0, \tag{2.1}$$

*for any* $\psi \in C^{1,1}(\overline{\Omega})$ *with compact support in* $\overline{\Omega}$.

For the construction of a weak solution to the Cauchy problem (1.1)-(1.2), we use the viscosity method: we add the term $\epsilon\Delta u$ in the Eq. (1.1) and we consider the following Cauchy problem

$$u_t=u\Delta u-\gamma|\nabla u|^2+f(t,u)+\epsilon\Delta u, \qquad u\in\Omega, \tag{2.2}$$

$$u(x,0)=u_0(x), \qquad x\in\mathbb{R}^N, \tag{2.3}$$

where $\gamma\geq 0$. The existence of solutions for (2.2)-(2.3) follows by the Maximum principle and vanishing viscosity method ensures the convergence of the weak solutions when $\epsilon\to 0$ to the Cauchy problem (1.1)-(1.2).

**Definition 2.2.** *The weak solution for the Cauchy problem* (1.1)-(1.2) *constructed by the vanishing viscosity method is called viscosity solution.*

## 3 Estimates of Hölder

In this section we are going to proof the main theorem of this paper. We begin with some a priori estimates for the function $u$.

*Proof.* Let

$$w=\frac{1}{2}\sum_{i=1}^{N}u_{x_i}^2. \tag{3.1}$$

Deriving with respect $t$ in (3.1) and replacing in (1.1) we have

$$w_t=\sum_{i=1}^{N}u_{x_i}\left[u_{x_i}\Delta u+u\left(\sum_{j=1}^{N}u_{x_ix_jx_j}\right)-2\gamma w_{x_i}+f_u u_{x_i}\right].$$

By other hand

$$\Delta w=\frac{1}{2}\sum_{j=1}^{N}\left(\sum_{i=1}^{N}u_{x_i}^2\right)_{x_jx_j}=\frac{1}{2}\left[\sum_{j=1}^{N}(2u_{x_1}u_{x_1x_j})_{x_j}+\sum_{j=1}^{N}(2u_{x_2}u_{x_2x_j})_{x_j}+\cdots+\sum_{j=1}^{N}(2u_{x_N}u_{x_Nx_j})_{x_j}\right],$$

$$\Delta w=\sum_{i,j=1}^{N}u_{x_ix_j}^2+\sum_{i,j=1}^{N}u_{x_i}u_{x_ix_jx_j}, \tag{3.2}$$

thereby,

$$w_t=2w\Delta u+u\Delta w-u\sum_{i,j=1}^{N}u_{x_ix_j}^2-2\gamma\sum_{i=1}^{N}u_{x_i}w_{x_i}+2f_u w. \tag{3.3}$$

Set,

$$z=g(u)w. \tag{3.4}$$

After take twice derivatives with respect $x_i$ in (3.4) we have

$$w_{x_i}=(g^{-1})_{x_i}z+g^{-1}z_{x_i} \tag{3.5}$$

$$w_{x_ix_i}=(g^{-1})_{x_ix_i}z+2(g^{-1})_{x_i}z_{x_i}+g^{-1}z_{x_ix_i}. \tag{3.6}$$

From Eqs. (3.2), (3.5)-(3.6), we have that,

$$\Delta w=\sum_{i=1}^{N}w_{x_ix_i}=\sum_{i=1}^{N}\left[(g^{-1})_{x_ix_i}z+2(g^{-1})_{x_i}z_{x_i}+g^{-1}z_{x_ix_i}\right],$$

Deriving two times with respect $x_i$ in (3.4) we have

$$(g(u)^{-1})_{x_i}=-g^{-2}g'u_{x_i} \tag{3.7}$$

$$(g(u)^{-1})_{x_ix_i}=\left(\frac{2g^{2\prime}-gg''}{g_4}\right)gu_{x_i}^2-\frac{g'}{g^2}u_{x_ix_i}, \tag{3.8}$$

then,

$$\Delta w=\left(\frac{2g^{2\prime}-gg''}{g^4}\right)g\sum_{i=1}^N u_{x_i}^2 z-\frac{g'}{g^2}\sum_{i=1}^N u_{x_ix_i}z-2g^{-2}g'\sum_{i=1}^N u_{x_i}z_{x_i}+g^{-1}\sum_{i=1}^N z_{x_ix_i}$$

$$=g^{-1}\sum_{i=1}^N z_{x_ix_i}-2g^{-2}g'\sum_{i=1}^N u_{x_i}z_{x_i}+2\left(\frac{2g'-gg''}{g^4}\right)gwz-\frac{g'}{g^2}z\sum_{i=1}^N u_{x_ix_i},$$

$$\Delta w=g^{-1}\Delta z-2g^{-2}g'\sum_{i=1}^N u_{x_i}z_{x_i}+2\left(\frac{2g^{2\prime}-gg''}{g^4}\right)z^2-\frac{g'}{g^2}z\Delta u. \tag{3.9}$$

From (3.3)-(3.5), (3.9), we obtain

$$\begin{aligned} z_t=&u\Delta z-(2g^{-1}ug'+2\gamma)\sum_{i=1}^N u_{x_i}z_{x_i}+(2f_u+g'g^{-1}f(t,u))z\\ &+\left(\frac{4ug^{2\prime}}{g^3}-\frac{2ug''}{g^2}+\frac{2\gamma g'}{g^2}\right)z^2+2z\Delta u-ug(u)\sum_{i,j=1}^N u_{x_ix_j}^2. \end{aligned} \tag{3.10}$$

By choosing $g(u)=u^\alpha$, and since

$$\sum_{i,j=1}^N u_{x_ix_j}^2\geq\frac{1}{N}(\Delta u)^2, \tag{3.11}$$

replacing $g$ in (3.10)-(3.11) we have

$$\begin{aligned} z_t\leq& u\Delta z-2(\alpha+\gamma)\sum_{i=1}^N u_{x_i}z_{x_i}+(2f_u+\alpha u^{-1}f(t,u))z\\ &+2\alpha(\alpha+1+\gamma)u^{-\alpha-1}z^2+2z\Delta u-\frac{u^{\alpha+1}}{N}(\Delta u)^2. \end{aligned} \tag{3.12}$$

For $\gamma\geq\sqrt{2N}-1$, if $\alpha$ satisfies

$$\alpha^2+(\gamma+1)\alpha+\frac{N}{2}\leq 0, \tag{3.13}$$

where $\alpha^2+(\gamma+1)\alpha\leq -N/2$, then,

$$2\alpha(\alpha+\gamma+1)u^{-\alpha-1}z^2+2z\Delta u-\frac{u^{\alpha+1}}{N}(\Delta u)^2\leq 0. \tag{3.14}$$

Therefore from (3.12) and (3.14) we have

$$z_t \leq u\Delta z - 2(\alpha+\gamma)\sum_{i=1}^{N} u_{x_i} z_{x_i} + (2f_u + \alpha u^{-1} f(t,u))z. \tag{3.15}$$

By an application of the maximum principle in (3.15) we have

$$|z|_\infty \leq |z_0|_\infty.$$

Now, from (3.1), (3.4), with $g(u)=u^\alpha$, since the initial data (1.2) satisifes

$$|\nabla(u_0^{1+\frac{\alpha}{2}})| \leq M,$$

with $M$ a positive constant and $\alpha$ satisfies (3.13), we have

$$|\nabla(u^{1+\frac{\alpha}{2}})|^2 = \left|\sum_{i=1}^{N}(u^{1+\frac{\alpha}{2}})_{x_i} e_i\right|^2 = \sum_{i=1}^{N}\left[(u^{1+\frac{\alpha}{2}})_{x_i}\right]^2 = \sum_{i=1}^{N}\left[\left(1+\frac{\alpha}{2}\right)u^{\frac{\alpha}{2}} u_{x_i}\right]^2$$

$$= \left(1+\frac{\alpha}{2}\right)^2 u^\alpha \sum_{i=1}^{N} u_{x_i}^2 = 2\left(1+\frac{\alpha}{2}\right)^2 u^\alpha w = 2\left(1+\frac{\alpha}{2}\right)^2 z,$$

therefore

$$|\nabla(u^{1+\frac{\alpha}{2}})| \leq M. \qquad \square$$

## 4 Hölder continuity of $u(x,t)$

Now using the main theorem, we have the following corollary about the regularity of the viscosity solution $u(x,t)$ to the Cauchy problem (1.1)-(1.2).

**Corollary 4.1.** *Let $f$ be a continuous function such that*

$$|f(t,w)| \leq k|w|^m, \tag{4.1}$$

*where $w$ is a real value function and $m$, $k$ non-negative constants. Under conditions of the Theorem* 3.1 *the viscosity solution $u(x,t)$ of the Cauchy problem* (1.1)-(1.2) *is Lipschitz continuous with respect to $x$ and locally Hölder continuous with exponent* $1/2$ *with respect to $t$ in $\overline{\Omega}$.*

*Proof.* From there exists $\alpha \in \mathbb{R}$ with $\alpha^2 + (\gamma+1)\alpha + \frac{N}{2} \leq 0$, with $\alpha < 0$, or,

$$-\frac{\sqrt{(\gamma+1)^2 - 2N}}{2} - \frac{\gamma+1}{2} \leq \alpha \leq -\frac{\gamma+1}{2} + \frac{\sqrt{(\gamma+1)^2-2N}}{2} < 0.$$

Since $\alpha < 0$, taking $\alpha \neq -2$, we have the estimate,

$$|\nabla(u^{1+\frac{\alpha}{2}})| = \left|(1+\frac{\alpha}{2})u^{\frac{\alpha}{2}}\nabla u\right| = \left|1+\frac{\alpha}{2}\right| u^{\frac{\alpha}{2}} |\nabla u| \leq M.$$

Now, as $u \geq 0$, we have that

$$|\nabla u| \leq \left|1+\frac{\alpha}{2}\right|^{-1} u^{-\frac{\alpha}{2}} M \leq M_1, \qquad \text{in } \overline{\Omega}, \tag{4.2}$$

since $u$ is bounded.

Using the value mean theorem, we have

$$u(x_1,t) - u(x_2,t) = \nabla u(x_1 + \theta(x_2 - x_1),t) \cdot (x_1 - x_2), \tag{4.3}$$

for any $\theta \in (0,1)$. From (4.2)-(4.3) we have,

$$|u(x_1,t) - u(x_2,t)| \leq |\nabla u(x_1 + \theta(x_2 - x_1),t)||x_1 - x_2| \leq M_1 |x_1 - x_2|, \quad \forall (x_1,t),(x_2,t) \in \Omega.$$

Therefore $u(x,t)$ is a Lipschitz continuous with respect to the spatial variable. For Hölder continuity of $u(x,t)$ with respect to the temporary variable, we are going to use the ideas developed in [8]. Let $u_\epsilon(x,t) \in C^{2.1}(\Omega) \bigcap C(\overline{\Omega}) \bigcap L^\infty(\Omega)$ the classical solution to the Cauchy problem problem (1.1)-(1.2), namely,

$$\begin{cases} u_t = u\Delta u - \gamma |\nabla u|^2 + f(t,u), & \text{in } \Omega, \\ u(x,0) = u_0(x) + \epsilon, & \text{in } \mathbb{R}^N. \end{cases}$$

We have that

$$\left|\nabla (u_0+\epsilon)^{1+\frac{\alpha}{2}}\right| = \left|\left(1+\frac{\alpha}{2}\right)(u_0+\epsilon)^{\frac{\alpha}{2}} \nabla u_0\right| \leq \left|1+\frac{\alpha}{2}\right| (u_0)^{\frac{\alpha}{2}} |\nabla u_0| = \left|\nabla\left(u_0^{1+\frac{\alpha}{2}}\right)\right| \leq M.$$

Then, the conditions of Theorem 3.1 holds. Thereby

$$\left|\nabla (u_0+\epsilon)^{1+\frac{\alpha}{2}}\right| \leq M.$$

Since $u_\epsilon$ is a classical solution, $u$ is also a weak solution of the Cauchy problem (2.2)-(2.3). Hence, using the same arguments in the proof of Theorem 3.1, we have that $u_\epsilon$ is a Lipschitz continuous with respect to the spatial variable, with constant $M$, namely

$$|u_\epsilon(x_1,t) - u_\epsilon(x_2,t)| \leq M|x_1 - x_2|, \qquad \forall\, (x_1,t),(x_2,t) \in \Omega. \tag{4.4}$$

The following argument is due to [7], let

$$L(z) = u_\epsilon z - \gamma |u_\epsilon|^2 + f(t,z) - z_t$$

be the parabolic differential operator. Then $z = u_\epsilon$ satisfies $L(z) = 0$, therefore

$$u_\epsilon \Delta z - z_t = \gamma |\nabla u_\epsilon|^2 - f(t,u_\epsilon), \qquad \text{in } B_{2R}(0) \times (0,T], \tag{4.5}$$

where $B_{2R}(0)$ is the open ball centered in 0, with radius $2R$ in $\mathbb{R}^N$. Noticing that

$$u_\epsilon \in C^{2,1}(B_{2R}(0)\times(0,T]).$$

Therefore $u_\epsilon$ and $\nabla u_\epsilon$ are bounded in $\overline{B_{2R}(0)}\times(0,T]$, so exists a constant $\mu>0$ such that

$$\sum_{i=1}^{N} u_\epsilon(x,t) = N u_\epsilon(x,t) \le \mu, \qquad \gamma|\nabla u_\epsilon(x,t)| \le \mu, \qquad \forall (x,t)\in B_{2R}(0)\times(0,T],$$

and by the condition (4.1) we have

$$|f(t,u_\epsilon)| \le k|u_\epsilon|^m \le k(\frac{\mu}{N})^m.$$

From (4.4), we have also

$$|z(x_1,t)-z(x_2,t)| \le M|x_1-x_2|, \qquad \forall\ (x,t)\in B_{2R}(0))\times(0,T].$$

From the main theorem in [8] (page 104), there exists a positive constant $\delta$ (which depends only of $\mu$ and $R$) and a positive constant $K$, which depends only of $\mu$, $R$ and $M$, such that

$$|z(x,t)-z(x,t_0)| \le K|t-t_0|^{\frac{1}{2}},$$

for all $(x,t),(x,t_0)\in B_R(0)\times(0,T]$ with $|t-t_0|<\delta$. That is,

$$|u_\epsilon(x,t)-u_\epsilon(x,t_0)| \le K|t-t_0|^{\frac{1}{2}},$$

for all $(x,t),(x,t_0)\in B_R(0)\times(0,T]$ with $|t-t_0|<\delta$. Whenever $K$ is independent of $\epsilon$, taken $\epsilon \searrow 0$, we obtain

$$|u(x,t)-u(x,t_0)| \le K|t-t_0|^{\frac{1}{2}},$$

for all $(x,t),(x,t_0)\in B_R(0)\times(0,T]$ with $|t-t_0|<\delta$. □

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### References

[1] Vazquéz J. L., The Porous Medium Equation: Mathematical Theory, Oxford Science Publications, 2007.
[2] Mikelic A., Espedal M. S. and Fasano A., Filtration in Porous Media and Industrial Applications, Springer-Verlag, New York, Heidelberg, Berlin, 2000.

[3] DiBenedetto E., Degenerate Parabolic Equations, Springer-Verlag, New York, Heidelberg, Berlin, 1993.

[4] Ughi M., Bertsch M. and Dal Passo R., Discontinuous viscosity solutions of a degenerate parabolic equation. *Trans Amer. Math. Soc.*, **320** (2) (1990), 779-798.

[5] Lions P. L., Crandall M. G., Viscosity Solutions of Hamilton-Jacobi Equations, Transactions of the American Mathematical Society, 1983.

[6] Lu Y. G., Qian L., Regularity of viscosity solutions of a degenerate parabolic equation. *Proc. Amer. Math. Soc.*, **130** (4) (2001), 999-1004.

[7] Romero P., Rendón L., Regularidad de soluciones viscosas de una ecuación parabolica degenerada. *Rev. Col. Mat*, **45** (2011), 19-30.

[8] Gilding B. H., Hölder continuity of solutions of parabolic equations. *J. Landon Math. Soc.*, **13** (1976), 103-106.

[9] Lu Y. G., Hölder estimates of solutions to some doubly nonlinear degenerate parabolic equations. *Comm. Partial Differential, Equations*, **24** (5-6) (1999), 895-913.

[10] John F., Partial Differential Equations. Springer-Verlag, New York, Heidelberg, 1982.