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# THE UNCONDITIONAL STABILITY OF PARALLEL DIFFERENCE SCHEMES WITH SECOND ORDER CONVERGENCE FOR NONLINEAR PARABOLIC SYSTEM*

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Dedicated to Prof. Boling Guo for his 70th Birthday

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**Abstract** For solving nonlinear parabolic equation on massive parallel computers, the construction of parallel difference schemes with simple design, high parallelism and unconditional stability and second order global accuracy in space, has long been desired. In the present work, a new kind of general parallel difference schemes for the nonlinear parabolic system is proposed. The general parallel difference schemes include, among others, two new parallel schemes. In one of them, to obtain the interface values on the interface of sub-domains an explicit scheme of Jacobian type is employed, and then the fully implicit scheme is used in the sub-domains. Here, in the explicit scheme of Jacobian type, the values at the points being adjacent to the interface points are taken as the linear combination of values of previous two time layers at the adjoining points of the inner interface. For the construction of another new parallel difference scheme, the main procedure is as follows. Firstly the linear combination of values of previous two time layers at the interface points among the sub-domains is used as the (Dirichlet) boundary condition for solving the sub-domain problems. Then the values in the sub-domains are calculated by the fully implicit scheme. Finally the interface values are computed by the fully implicit scheme, and in fact these calculations of the last step are explicit since the values adjacent to the interface points have been obtained in the previous step. The existence, uniqueness, unconditional stability and the second order accuracy of the discrete vector solutions for the parallel difference schemes are proved. Numerical results are presented to examine the stability, accuracy and parallelism of the parallel schemes.

**Key Words** Parallel difference scheme; nonlinear parabolic system; unconditional stability; second order convergence.

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# 1. Introduction

There is rich literature on the parallel difference schemes for the parabolic equation (see [1-5] [6-10]).Explicit schemes are naturally parallel and also easy to implement, but they usually require small time steps because of stability constraints. Implicit schemes are necessary for finding steady state solution or computing slowly unsteady problems where one needs to march with large time steps. However, implicit schemes are not inherently parallel. The parallel schemes in [4, 5]and [9, 10] use the explicit scheme and the implicit scheme alternately in the time and space direction, which can implement the parallel computation and are unconditionally stable. These schemes involve three time layers in essence, and have been extended to semi-linear parabolic equation in [7]. Their truncation error is $O(\tau + h)$, where $\tau$ and $h$ are the time and space step respectively. Furthermore they suffer from the defect that the truncation error of the alternating schemes can not be eliminated for the general nonlinear parabolic problem. For clarity we recall the definition of the unconditional stability. Let $r = \frac{\tau}{h^2}$. If a difference scheme is stable for all small $\tau$ and $h$ satisfying $r \leq \Lambda$ with $\Lambda$ being any fixed positive constant, then we call it is unconditionally stable. For the heat equation $u_t = u_{xx}$ the fully implicit scheme is unconditional stable, while the fully explicit scheme is not unconditional stable since the constant $\Lambda$ cannot be taken larger than $\frac{1}{2}$ in this case.

A natural way to solve partial differential equations in parallel is to divide the domain over which the problem is defined into sub-domains, and solves the sub-domain problems in parallel. The major difficulties with such procedures involve defining values on the sub-domain boundaries and piecing the solutions together into a reasonable approximation to the true solution. Once the interface values are available, the global problem is fully decoupled and thus computed in parallel. A parallel scheme was proposed in [1], where instead of using the same spacing $h$ as for the interior points where the implicit scheme is applied, a larger spacing $H_D$ is used at each interface point where the explicit scheme is applied. There are also some other schemes with domain decomposition in [2, 3]. These schemes are conditionally stable. Since unconditional stable schemes are desired in many applications, some unconditional stable schemes were proposed in [8], which firstly take the values of previous time step as the boundary condition, and then solve the sub-domain problems in parallel, and finally update the interface values between sub-domains by the implicit scheme. These schemes can be easily implemented in the parallel computer, but their convergence order is only one order. In order to improve the convergence order, the parallel iterative difference schemes based on interface correction for parabolic equation were proposed in [6], which are complicated in requiring the correction of the interface values in the process of iteration of solving sub-domain problems.

In this paper we propose a new kind of general parallel difference schemes for the nonlinear parabolic problem. The resulting schemes are of second order global accuracy in space and unconditionally stable as well. The general parallel difference schemes include, among others, two new parallel schemes. In one of them, to obtain the interface values on the interface of sub-domains an explicit scheme of Jacobian type is employed, in which the values at the points being adjacent to the interface points are taken as the linear combination of values of previous two time layers at the adjoining points of the inner interface, and then the fully implicit scheme is used in the sub-domains. The main ideas of constructing the another new parallel difference scheme are as follows. First we take a linear combination of the values of previous two time layers at the interface points among the sub-domains as the (Dirichlet) inner boundary condition for solving the sub-domain problems. Then the values in the sub-domains are calculated by using the fully implicit scheme in parallel. Finally the interface values are computed by the fully implicit scheme, and in fact these calculations of the last step are explicit since the values at the points being adjacent to the interface points have been obtained in the previous step. It's obvious that the design of the parallel schemes is very simple. The new kind of schemes is equivalent to a kind of two-layer schemes with intrinsic parallelism, though the values of the previous two time layers at the interface points (or their adjacent points) are used in the statement of the construction.

To show the performance of our new parallel difference schemes for two dimensional problem, we will compare them with the well-known parallelization method, the so-called parallel algebraic method, which uses the parallel preconditioned conjugate gradient method to solve the global (linear or nonlinear) algebraic system of equations arising from the fully implicit scheme on the global space domain. That is, the standard implicit discretization for the nonlinear parabolic equations is used, and then the usual Picard linearization is applied to form a linear algebraic system of equations, and finally the parallel algebraic solver (parallel preconditioned conjugate gradient method, see [11]) is exploited to solve the linear system in parallel on parallel computers. We call it the parallel algebraic method since the difference scheme to be solved is the fully (nonlinear) implicit scheme, not a parallel difference scheme. Our new parallel difference scheme can be regarded as a meaningful modification of both the fully implicit scheme and the parallel difference schemes proposed in previous papers. They differ in the treatment of the values at the points being adjacent to the interface points.

The rest of this paper is organized as follows. In next section, we describe the parallel difference schemes for one dimensional problem. In Section 3, we derive the priori estimate, existence, and convergence of the discrete vector solutions for the parallel difference schemes under the assumption that the unique smooth vector solution of the original problems for the nonlinear parabolic system exist. In Section 4, we prove the unconditional stability and uniqueness of the discrete vector solutions for the parallel schemes. In Section 5, we extend the results to two-dimensional problem. In Section 6, we examine numerically the stability, accuracy, and parallelism of the schemes. The numerical results verify the theoretical results. Moreover it is shown that the super-linear speedup is achieved. In the final section, we present a summary.

# 2. Construction of Parallel Difference Schemes

## 2.1 Problem and notation

Consider the nonlinear parabolic system:

$$u_t = A(x,t,u,u_x)u_{xx} + f(x,t,u,u_x), \quad 0 < x < l, 0 < t \le T \tag{2.1}$$

$$u(0,t) = u(l,t) = 0, 0 < t \le T \tag{2.2}$$

$$u(x,0) = \varphi(x), 0 \le x \le l \tag{2.3}$$

where $u(x,t) = (u_1(x,t), \cdots, u_m(x,t))$ is an $m$–dimensional vector function ($m \ge 1$), $u_t = \frac{\partial u}{\partial t}, u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ are the corresponding vector derivatives. $A(x,t,u,p)$ is a $m \times m$ positive definite coefficient matrix, $f(x,t,u,u_x)$ and $\varphi(x)$ are the $m$–dimensional vector functions.

Let $Q_T = \{0 \le x \le l, 0 \le t \le T\}$, where $l > 0$, $T > 0$. Divide the domain $Q_T$ into small grids by $x = x_j$ ($j = 0,1,\cdots,J$) and $t = t^n$ ($n = 0,1,\cdots,N$), where $x_j = jh$, $t^n = n\tau$, $J$ and $N$ are positive integers, $h$ and $\tau$ are step lengths of the grids. Denote $Q_j^n = \{x_j < x \le x_{j+1}, t^n < t \le t^{n+1}\}$, where $j = 0,1,\cdots,J-1; n = 0,1,\cdots,N-1$. For a function $f(x,t)$ defined at mesh points $(x_j,t^n)$, let $f_j^n = f(x_j,t^n)$. Let $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\cdots,J; n = 0,1,\cdots,N\}$ be the $m$–dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_j,t^n) | j = 0,1,\cdots,J; n = 0,1,\cdots,N\}$.

Define the difference operators

$$\Delta_\tau v_j^{n+1} = \frac{v_j^{n+1} - v_j^n}{\tau}, \qquad \delta v_{j+\frac{1}{2}}^n = \frac{v_{j+1}^n - v_j^n}{h},$$

$$\delta v_j^{n+1} = \frac{1}{2}(\delta v_{j+\frac{1}{2}}^n + \delta v_{j-\frac{1}{2}}^n), \qquad \bar{\delta} v_j^{n+1} = \frac{1}{2h}\left(v_{j+1}^{\bar{n}+\lambda_j} - v_{j-1}^{\bar{n}+\mu_j}\right),$$

$$\overset{*}{\delta}{}^2 v_j^{n+1} = \frac{1}{h^2}\left(v_{j+1}^{\bar{n}+\lambda_j} - 2v_j^{n+1} + v_{j-1}^{\bar{n}+\mu_j}\right), \quad n \ge 1,$$

where

$$v_{j+1}^{\bar{n}+\lambda_j} = \lambda_j v_{j+1}^{n+1} + (1-\lambda_j)(2v_{j+1}^n - v_{j+1}^{n-1}),$$

$$v_{j-1}^{\bar{n}+\mu_j} = \mu_j v_{j-1}^{n+1} + (1-\mu_j)(2v_{j-1}^n - v_{j-1}^{n-1}).$$

When $n = 0$, define the difference operator

$$\overset{*}{\delta}{}^2 v_j^{n+1} = \delta^2 v_j^1 = \frac{1}{h^2}\left(v_{j+1}^1 - 2v_j^1 + v_{j-1}^1\right).$$

For the discrete function $u_h = \{u_j | j = 0,1,\cdots,J\}$, where $u_0 = u_J = 0$, define the discrete norm as follows:

$$\|u_h\|_\infty = \max_{0 \le j \le J} |u_j|, \quad \|u_h\|_2^2 = \sum_{j=1}^{J-1} u_j^2 h, \quad \|\delta u_h\|_2^2 = \sum_{j=0}^{J-1} |\delta u_{j+\frac{1}{2}}|^2 h,$$

$$\left\|\overset{*}{\delta}{}^2 v_h^{n+1}\right\|_2^2=\sum_{j=1}^{J-1}\left|\overset{*}{\delta}{}^2 v_j^{n+1}\right|^2 h,\quad \|\Delta_\tau v_h^{n+1}\|_2^2=\sum_{j=1}^{J-1}\left|\Delta_\tau v_j^{n+1}\right|^2 h.$$

## 2.2 Parallel difference scheme

The general parallel difference schemes for the nonlinear parabolic system (2.1)–(2.3) are as follows:

$$\frac{v_j^{n+1}-v_j^n}{\tau}=A_j^{n+1}\,\overset{*}{\delta}{}^2 v_j^{n+1}+f_j^{n+1},\quad (j=1,\cdots,J-1;n=0,\cdots,N-1),\tag{2.4}$$

$$v_0^{n+1}=v_J^{n+1}=0,\quad (n=0,1,\cdots,N-1),\tag{2.5}$$

$$v_j^0=\varphi_j,\quad (j=0,1,\cdots,J),\tag{2.6}$$

where $\varphi_j=\varphi(x_j)$, $(j=0,1,\cdots,J)$, $\varphi_0=\varphi_J=0$; $A_j^{n+1}=A(x_j,t^{n+1},v_j^{n+1},\bar{\delta}v_j^{n+1})$, $f_j^{n+1}=f(x_j,t^{n+1},v_j^{n+1},\bar{\delta}v_j^{n+1})$.

We can obtain some concrete examples of the parallel schemes by choosing the parameter in the general schemes (2.4).

## 2.3 Assumptions

Introduce the following assumptions.

**(I)** The problem (2.1)–(2.3) has a unique smooth solution $u(x,t)\in C^3(Q_T)$, and the maximum norms of $u(x,t)$ and its first order derivatives and second order derivatives are bounded by a constant $G$, i.e.,

$$|u(x,t)|,|u_x(x,t)|,|u_{xx}(x,t)|,|u_t(x,t)|,|u_{xt}(x,t)|\le G.\tag{2.7}$$

**(II)** There has a positive constant $\sigma_0$, such that for any $\xi\in R^m$, $(x,t)\in Q_T$ and $u,p\in R^m$,

$$(\xi,A(x,t,u,p)\xi)\ge\sigma_0|\xi|^2.\tag{2.8}$$

**(III)** The coefficient matrix $A(x,t,u,p)$ and vector function $f(x,t,u,p)$ are continuous with respect to $(x,t)\in Q_T$, and continuously differentiable with respect to $u,p\in R^m$.

**(IV)** The initial vector function $\varphi(x)\in C^1[0,l]$, and $\varphi(0)=\varphi(l)=0$.

**(V)** Suppose that $r=\frac{\tau}{h^2}\le\Lambda$, which holds for all $\tau$ and $h$ small enough, where $\Lambda$ can be any given positive constant; and there hold $0\le\lambda_j,\mu_j\le1,\lambda_j+\mu_{j+1}\ge1$ for $\forall$ $1\le j\le J-1$.

Note that the constant $\lambda_j$ and $\mu_j$ depend on $j=1,2,\cdots,J-1$. They may be different from different $j$. These schemes (2.4)–(2.6) include many different schemes with intrinsic parallelism, for example, the resulting scheme when $\lambda_j=0$ or $\mu_j=0$ at some mesh points.

Since $u(x,t) \in C^3(Q_T)$ is the smooth solution of the problem (2.1)–(2.3), the discrete vector function $u_\Delta = u_h^\tau = \{u_j^n = u(x_j, t^n) | 0 \le j \le J, 0 \le n \le N\}$ satisfies the difference system:

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \bar{A}_j^{n+1} \overset{*}{\delta}{}^2 u_j^{n+1} + \bar{f}_j^{n+1} + R_j^{n+1},$$
$$(j = 1, \cdots, J-1; n = 0, \cdots, N-1), \tag{2.9}$$
$$u_0^{n+1} = u_J^{n+1} = 0, \quad (n = 0, 1, \cdots, N-1), \tag{2.10}$$
$$u_j^0 = \varphi_j, \quad (j = 0, 1, \cdots, J), \tag{2.11}$$

where $\bar{A}_j^{n+1} = A(x_j, t^{n+1}, u_j^{n+1}, \bar{\delta} u_j^{n+1})$, $\bar{f}_j^{n+1} = f(x_j, t^{n+1}, u_j^{n+1}, \bar{\delta} u_j^{n+1})$, $R_j^{n+1} = O(\tau + h^2)$.

Denote $w_\Delta = v_\Delta - u_\Delta = v_h^\tau - u_h^\tau = \{w_j^n = v_j^n - u_j^n | 0 \le j \le J, 0 \le n \le N\}$. Subtract (2.9) from (2.4), (2.10) from (2.5), and (2.11) from (2.6) to obtain

$$\frac{w_j^{n+1} - w_j^n}{\tau} = A(v)_j^{n+1} \overset{*}{\delta}{}^2 w_j^{n+1} + B(u,v)_j^{n+1} w_j^{n+1} + C(u,v)_j^{n+1} \bar{\delta} w_j^{n+1}$$
$$+ R_j^{n+1}, \quad (j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1), \tag{2.12}$$
$$w_0^{n+1} = w_J^{n+1} = 0, \quad (n = 0, 1, \cdots, N-1), \tag{2.13}$$
$$w_j^0 = 0, \quad (j = 0, 1, \cdots, J), \tag{2.14}$$

where

$$A(v)_j^{n+1} = A_j^{n+1} = A(x_j, t^{n+1}, v_j^{n+1}, \bar{\delta} v_j^{n+1}),$$
$$B(u,v)_j^{n+1} = (A_u)_j^{n+1} \overset{*}{\delta}{}^2 u_j^{n+1} + (f_u)_j^{n+1},$$
$$C(u,v)_j^{n+1} = (A_p)_j^{n+1} \overset{*}{\delta}{}^2 u_j^{n+1} + (f_p)_j^{n+1},$$
$$(A_u)_j^{n+1} = \int_0^1 A_u(x_j, t^{n+1}, \lambda v_j^{n+1} + (1-\lambda) u_j^{n+1}, \bar{\delta} v_j^{n+1}) d\lambda,$$
$$(A_p)_j^{n+1} = \int_0^1 A_p(x_j, t^{n+1}, u_j^{n+1}, \lambda \bar{\delta} v_j^{n+1} + (1-\lambda) \bar{\delta} u_j^{n+1}) d\lambda,$$

$(f_u)_j^{n+1}$ and $(f_p)_j^{n+1}$ are similarly defined.

## 2.4 Lemmas

The proof of our main results relies on the following lemmas (see [12]):

**Lemma 2.1** (The discrete Green formula) *Let $u_j$ and $v_j$ be the discrete function defined on $\{x_j | j = 0, 1, \cdots, J\}$, then*

$$\sum_{j=0}^{J-1} u_j (v_{j+1} - v_j) = -\sum_{j=1}^{J-1} (u_j - u_{j-1}) v_j - u_0 v_0 + u_{J-1} v_J.$$

**Lemma 2.2** (The discrete Gronwall inequality) (i) *Let $w^n \geq 0$ be a discrete function defined on $\{t^n | n = 0, 1, \cdots, N\}$, and satisfy*

$$w^{n+1} - w^n \leq B\tau(w^{n+1} + w^n) + C_n\tau, \quad n = 0, 1, \cdots, N-1,$$

*where $B$ and $C_n$ are nonnegative constants, then*

$$w^n \leq \left( w^0 + \sum_{k=0}^{N} C_k\tau \right) e^{4BT}, \quad n = 0, 1, \cdots, N,$$

*where we take $\tau$ such that $4B\tau \leq \frac{N-1}{N}$.*

(ii) *Suppose that the discrete function $w^\tau = \{w^n \geq 0 | n = 0, 1, \cdots, N\}$, $N\tau = T$, satisfies*

$$w^n \leq B + C\sum_{k=0}^{n} w^k\tau,$$

*where $B$ and $C$ are nonnegative constants, then*

$$w^n \leq B(e^{2CT} + 1),$$

*where we take $\tau$ such that $C\tau \leq \frac{1}{2}$.*

**Lemma 2.3**(The interpolation formula) *For any discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ $(Jh = l)$, the following assertions hold.*

(i) *For $\forall\ \varepsilon > 0$, there are*

$$\|u_h\|_\infty^2 \leq \varepsilon\|\delta u_h\|_2^2 + \frac{C}{\varepsilon}\|u_h\|_2^2,$$

*where $C$ is a constant depending on $l$, and independent of $\varepsilon$, $h$ and $u_h$;*

(ii) *If $u_0 = u_J = 0$, then*

$$\|u_h\|_2 \leq l\|\delta u_h\|_2, \quad \|u_h\|_\infty \leq \|\delta u_h\|_2^{\frac{1}{2}}\|u_h\|_2^{\frac{1}{2}};$$

(iii) *There exists a constant $C$ independent of $h$ and $l$, such that*

$$\|\delta u_h\|_2 \leq C\left( \|u_h\|_2^{\frac{1}{2}}\|\delta^2 u_h\|_2^{\frac{1}{2}} + l^{-1}\|u_h\|_2 \right).$$

Throughout the paper $C$ denotes a generic positive constant, which is independent of $h, \tau$ and $v_h^\tau$.

## 3. Priori Estimate, Existence and Convergence

Construct a mapping $\Phi$ from $m(J+1)(N+1)$ dimensional Euclidean space $R^* = R^{m(J+1)(N+1)}$ to itself.

For $\forall\ z_\Delta = z_h^\tau = \{z_j^n | 0 \le j \le J, 0 \le n \le N\} \in R^*$, let $w_\Delta = \Phi(z_\Delta) \in R^*$ be the solution of the linear difference system:

$$\frac{w_j^{n+1} - w_j^n}{\tau} = A(u+z)_j^{n+1} \overset{*}{\delta}{}^2 w_j^{n+1} + H_j^{n+1},$$
$$(j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1), \tag{3.1}$$
$$w_0^{n+1} = w_J^{n+1} = 0, \quad (n = 0, 1, \cdots, N-1), \tag{3.2}$$
$$w_j^0 = 0, \quad (j = 0, 1, \cdots, J), \tag{3.3}$$

where $H_j^{n+1} = B(u, u+z)_j^{n+1} w_j^{n+1} + C(u, u+z)_j^{n+1} \bar{\delta} w_j^{n+1} + R_j^{n+1}$.

Let $\Omega$ be a close convex set bounded in $R^*$:

$$\Omega = \{z_\Delta | \max_{0 \le n \le N} \|z_h^n\|_\infty \le G, \max_{0 \le n \le N} \|\delta z_h^n\|_\infty \le G\}.$$

Making the scalar product of $\overset{*}{\delta}{}^2 w_j^{n+1} h\tau$ with (3.1), and summing up the resulting products for $j = 1, 2, \cdots, J-1$, we get

$$\sum_{j=1}^{J-1} \left( \overset{*}{\delta}{}^2 w_j^{n+1}, w_j^{n+1} - w_j^n \right) h = \tau \sum_{j=1}^{J-1} \left( \overset{*}{\delta}{}^2 w_j^{n+1}, A_j^{n+1} \overset{*}{\delta}{}^2 w_j^{n+1} + H_j^{n+1} \right) h. \tag{3.4}$$

Obviously

$$\overset{*}{\delta}{}^2 w_j^{n+1} = \delta^2 w_j^{n+1} - r \left[ (1-\lambda_j)(\Delta_\tau w_{j+1}^{n+1} - \Delta_\tau w_{j+1}^n) + (1-\mu_j)(\Delta_\tau w_{j-1}^{n+1} - \Delta_\tau w_{j-1}^n) \right],$$

and by Lemma 2.1,

$$\sum_{j=1}^{J-1} \left( \delta^2 w_j^{n+1}, w_j^{n+1} - w_j^n \right) h = -\frac{1}{2} \|\delta w_h^{n+1}\|_2^2 + \frac{1}{2} \|\delta w_h^n\|_2^2 - \frac{1}{2} \|\delta w_h^{n+1} - \delta w_h^n\|_2^2.$$

When $0 \le \lambda_j \le 1, 0 \le \mu_j \le 1, \lambda_j + \mu_{j+1} \ge 1$, we have

$$\begin{aligned}
&-r \sum_{j=1}^{J-1} \left( (1-\lambda_j)(\Delta_\tau w_{j+1}^{n+1} - \Delta_\tau w_{j+1}^n) + (1-\mu_j)(\Delta_\tau w_{j-1}^{n+1} - \Delta_\tau w_{j-1}^n), w_j^{n+1} - w_j^n \right) h \\
&\quad -\frac{1}{2} \|\delta w_h^{n+1} - \delta w_h^n\|_2^2 \le \frac{\tau^2}{2h} \sum_{j=0}^{J-1} \left[ -(1-\lambda_j) \left| \Delta_\tau w_{j+1}^{n+1} \right|^2 - (1-\mu_{j+1}) \left| \Delta_\tau w_j^{n+1} \right|^2 \right. \\
&\qquad\qquad \left. + (1-\lambda_j) \left| \Delta_\tau w_{j+1}^n \right|^2 + (1-\mu_{j+1}) \left| \Delta_\tau w_j^n \right|^2 \right] \\
&\qquad\qquad = -\frac{\tau r}{2} \sum_{j=1}^{J-1} (2 - \lambda_{j-1} - \mu_{j+1}) \left( \left| \Delta_\tau w_j^{n+1} \right|^2 - \left| \Delta_\tau w_j^n \right|^2 \right) h.
\end{aligned}$$

Hence,

$$\|\delta w_h^{n+1}\|_2^2-\|\delta w_h^n\|_2^2+2\tau\sum_{j=1}^{J-1}\left(\overset{*}{\delta}{}^2 w_j^{n+1},A_j^{n+1}\,\overset{*}{\delta}{}^2 w_j^{n+1}\right)+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})$$
$$\times\left(\left|\Delta_\tau w_j^{n+1}\right|^2-\left|\Delta_\tau w_j^n\right|^2\right)h\le -2\tau\sum_{j=0}^{J-1}\left(\overset{*}{\delta}{}^2 w_j^{n+1},H_j^{n+1}\right)h$$
$$\le 2\tau\left|\sum_{j=1}^{J-1}\left(\overset{*}{\delta}{}^2 w_j^{n+1},H_j^{n+1}\right)h\right|$$
$$\le\frac{\tau}{2}\sum_{j=1}^{J-1}\sigma(A_j^{n+1})\left|\overset{*}{\delta}{}^2 w_j^{n+1}\right|^2 h+2\tau\sum_{j=1}^{J-1}\frac{\left|H_j^{n+1}\right|^2}{\sigma(A_j^{n+1})}h.$$

Notice that

$$\bar\delta w_j^{n+1}=\frac{1}{2h}(w_{j+1}^{n+1}-w_{j-1}^{n+1})$$
$$-\frac{\tau}{2h}\left((1-\lambda_j)(\Delta_\tau w_{j+1}^{n+1}-\Delta_\tau w_{j+1}^n)-(1-\mu_j)(\Delta_\tau w_{j-1}^{n+1}-\Delta_\tau w_{j-1}^n)\right),$$

and then there is

$$\sum_{j=1}^{J-1}|\bar\delta w_j^{n+1}|^2 h\le C\sum_{j=1}^{J-1}|\delta w_{j+\frac12}^{n+1}|^2 h$$
$$+C\tau r\sum_{j=1}^{J-1}\left((1-\lambda_{j-1})^2+(1-\mu_{j+1})^2\right)\cdot\left(|\Delta_\tau w_j^{n+1}|^2+|\Delta_\tau w_j^n|^2\right)h.$$

Hence,

$$\|\delta w_h^{n+1}\|_2^2-\|\delta w_h^n\|_2^2+2\tau\sum_{j=1}^{J-1}\left(\overset{*}{\delta}{}^2 w_j^{n+1},A_j^{n+1}\,\overset{*}{\delta}{}^2 w_j^{n+1}\right)$$
$$+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})\left(\left|\Delta_\tau w_j^{n+1}\right|^2-\left|\Delta_\tau w_j^n\right|^2\right)h$$
$$\le\frac{\tau}{2}\sum_{j=1}^{J-1}\left(\overset{*}{\delta}{}^2 w_j^{n+1},A_j^{n+1}\,\overset{*}{\delta}{}^2 w_j^{n+1}\right)h+C\tau[\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})\left(\left|\Delta_\tau w_j^{n+1}\right|^2\right.$$
$$\left.+\left|\Delta_\tau w_j^n\right|^2\right)+\|w_h^{n+1}\|_2^2+\|\delta w_h^{n+1}\|_2^2+(\tau+h^2)^2],\tag{3.5}$$

where the Cauchy inequality is used, and

$$\sigma(A)=\inf_{\xi\in R^m}\frac{(\xi,A\xi)}{|\xi|^2}.$$

It follows

$$\begin{aligned}&\|\delta w_h^{n+1}\|_2^2-\|\delta w_h^n\|_2^2+\sigma_0\tau\|\overset{*}{\delta}{}^2 w_h^{n+1}\|_2^2\\&\quad+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})\left(\left|\Delta_\tau w_j^{n+1}\right|^2-\left|\Delta_\tau w_j^n\right|^2\right)h\\&\le C\tau\left[\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})\left(\left|\Delta_\tau w_j^{n+1}\right|^2+\left|\Delta_\tau w_j^n\right|^2\right)+\|\delta w_h^{n+1}\|_2^2+(\tau+h^2)^2\right].\end{aligned}$$

Obviously,

$$\|\delta w_h^1\|_2^2+\tau\|\Delta_\tau w_h^1\|_2^2\le C(\|\delta w_h^0\|_2^2+(\tau+h^2)^2)=C(\tau+h^2)^2.$$

Combine the above two inequalities to find

$$\max_{0\le n\le N-1}\|\delta w_h^{n+1}\|_2\le C(\tau+h^2),\quad \max_{0\le n\le N-1}\|w_h^{n+1}\|_\infty\le C(\tau+h^2), \tag{3.6}$$

and

$$\begin{aligned}&\left(\sum_{n=0}^{N-1}\left\|\overset{*}{\delta}{}^2 w_h^{n+1}\right\|_2^2\tau\right)^{\frac12}+\left(\sum_{n=0}^{N-1}\|\Delta_\tau w_h^{n+1}\|_2^2\tau\right)^{\frac12}\le C(\tau+h^2),\\&\left(\sum_{n=0}^{N-1}\|\delta^2 w_h^{n+1}\|_2^2\tau\right)^{\frac12}\le C(\tau+h^2),\quad \max_{0\le n\le N-1}\|\delta w_h^{n+1}\|_\infty\le C(\frac{\tau}{h^{\frac12}}+h^{\frac32}).\end{aligned} \tag{3.7}$$

For $\frac{\tau}{h^{\frac12}}$ and $h^{\frac32}$ small enough,

$$\max_{0\le n\le N-1}\|w_h^{n+1}\|_\infty\le G,\quad \max_{0\le n\le N-1}\|\delta w_h^{n+1}\|_\infty\le G.$$

i.e. $w_\Delta\in\Omega$. So the mapping $\Phi$ is a continuous mapping from $\Omega$ to itself.

By the Brouwer fixed point theorem, we get the following result.

**Theorem 3.1** *Suppose that the conditions* **(I)–(V)** *hold, and step length $\tau$, $h$ and $\frac{\tau}{h^{\frac12}}$ are small enough. Then there exists at least one solution $v_\Delta$ for the parallel difference schemes* (2.4)–(2.6)*, and the estimates* (3.6)–(3.7) *hold.*

## 4. Stability and Uniqueness

Suppose that $m\times m$ coefficient matrix $\tilde{A}(x,t,u,p)$, $m$–dimensional vector functions $\tilde{f}(x,t,u,p)$ and $\tilde{\varphi}(x)$ satisfy the conditions **(I)–(IV)**, and they approximate

$A(x,t,u,p)$, $f(x,t,u,p)$ and $\varphi(x)$ respectively. Suppose that $\tilde{v}_\Delta = \{\tilde{v}_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ satisfies

$$\frac{\tilde{v}_j^{n+1} - \tilde{v}_j^n}{\tau} = \tilde{A}_j^{n+1} \overset{*}{\delta}{}^2 \tilde{v}_j^{n+1} + \tilde{f}_j^{n+1}, \ (j = 1, \cdots, J-1; n = 0, \cdots, N-1), \tag{4.1}$$

$$\tilde{v}_0^{n+1} = \tilde{v}_J^{n+1} = 0 \quad (n = 0, 1, \cdots, N-1), \tag{4.2}$$

$$\tilde{v}_j^0 = \tilde{\varphi}_j = \tilde{\varphi}(x_j), \quad (j = 0, 1, \cdots, J), \tag{4.3}$$

where

$$\tilde{A}_j^{n+1} = \tilde{A}(x_j, t^{n+1}, \tilde{v}_j^{n+1}, \bar{\delta}\tilde{v}_j^{n+1}), \quad \tilde{f}_j^{n+1} = \tilde{f}(x_j, t^{n+1}, \tilde{v}_j^{n+1}, \bar{\delta}\tilde{v}_j^{n+1}).$$

Assume the inequalities (3.6)–(3.7) hold for $\tilde{v}_\Delta$ and $\tilde{v}_\Delta = \{\tilde{v}_j^n\} \in \Omega$.

Denote $w_\Delta = v_\Delta - \tilde{v}_\Delta = \left\{ w_j^n = v_j^n - \tilde{v}_j^n \mid j = 0, 1, \cdots, J; n = 0, 1, \cdots, N \right\}$. Subtract (4.1) from (2.4), (4.2) from (2.5), and (4.3) from (2.6) to obtain

$$\frac{w_j^{n+1} - w_j^n}{\tau} = A_j^{n+1} \overset{*}{\delta}{}^2 w_j^{n+1} + B_j^{n+1} w_j^{n+1} + C_j^{n+1} \bar{\delta} w_j^{n+1} + r_j^{n+1},$$
$$(j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1), \tag{4.4}$$

$$w_0^{n+1} = w_J^{n+1} = 0, \ (n = 0, 1, \cdots, N-1), \tag{4.5}$$

$$w_j^0 = \varphi_j - \tilde{\varphi}_j, \quad (j = 0, 1, \cdots, J), \tag{4.6}$$

where

$$B_j^{n+1} = (\tilde{A}_u)_j^{n+1} \overset{*}{\delta}{}^2 \tilde{v}^{n+1} + (\tilde{f}_u)_j^{n+1}, \ C_j^{n+1} = (\tilde{A}_p)_j^{n+1} \overset{*}{\delta}{}^2 \tilde{v}^{n+1} + (\tilde{f}_p)_j^{n+1},$$
$$r_j^{n+1} = A\,[\tilde{v}]_j^{n+1} \overset{*}{\delta}{}^2 \tilde{v}^{n+1} + f\,[\tilde{v}]_j^{n+1},$$

and

$$(\tilde{A}_u)_j^{n+1} = \int_0^1 A_u(x_j, t^{n+1}, \lambda v_j^{n+1} + (1-\lambda)\tilde{v}_j^{n+1}, \bar{\delta} v_j^{n+1}) d\lambda;$$
$$(\tilde{A}_p)_j^{n+1} = \int_0^1 A_p(x_j, t^{n+1}, \tilde{v}_j^{n+1}, \lambda \bar{\delta} v_j^{n+1} + (1-\lambda)\bar{\delta}\tilde{v}_j^{n+1}) d\lambda;$$
$$A\,[\tilde{v}]_j^{n+1} = A\left(x_j, t^{n+1}, \tilde{v}_j^{n+1}, \bar{\delta}\tilde{v}_j^{n+1}\right) - \tilde{A}\left(x_j, t^{n+1}, \tilde{v}_j^{n+1}, \bar{\delta}\tilde{v}_j^{n+1}\right).$$

There are similar expressions for $(\tilde{f}_u)_j^{n+1}$, $(\tilde{f}_p)_j^{n+1}$ and $f\,[\tilde{v}]_j^{n+1}$.

Making the scalar product of $\overset{*}{\delta}{}^2 w_j^{n+1} h\tau$ with (4.4) and summing up the resulting products for $j = 1, 2, \cdots, J-1$, we get

$$\begin{aligned}
&\|\delta w_h^{n+1}\|_2^2 - \|\delta w_h^n\|_2^2 + 2\tau \sum_{j=1}^{J-1} \left(\overset{*}{\delta}{}^2 w_j^{n+1}, A_j^{n+1} \overset{*}{\delta}{}^2 w_j^{n+1}\right) h \\
&\quad + \tau r \sum_{j=1}^{J-1} (2 - \lambda_{j-1} - \mu_{j+1}) \left(\left|\Delta_\tau w_j^{n+1}\right|^2 - \left|\Delta_\tau w_j^n\right|^2\right) h \\
&\le 2\tau \left|\sum_{j=1}^{J-1} \left(\overset{*}{\delta}{}^2 w_j^{n+1}, B_j^{n+1} w_j^{n+1} + C_j^{n+1} \bar{\delta} w_j^{n+1} + r_j^{n+1}\right)\right| h. \qquad (4.7)
\end{aligned}$$

Furthermore, there holds

$$\begin{aligned}
&\sum_{j=1}^{J-1} \left|B_j^{n+1} w_j^{n+1} + C_j^{n+1} \bar{\delta} w_j^{n+1} + r_j^{n+1}\right|^2 h \\
&\le C \left(\|w_h^{n+1}\|_\infty^2 + \|\delta w_h^{n+1}\|_\infty^2 + \|A[\tilde{v}]_h^{n+1}\|_\infty^2 + \|f[\tilde{v}]_h^{n+1}\|_2^2\right) (1 + \| \overset{*}{\delta}{}^2 \tilde{v}_h^{n+1}\|_2^2) \\
&\quad + C(1 + \| \overset{*}{\delta}{}^2 \tilde{v}_h^{n+1}\|_\infty^2) \tau r \sum_{j=1}^{J-1} (2 - \lambda_{j-1} - \mu_{j+1}) \left(\left|\Delta_\tau w_j^{n+1}\right|^2 + \left|\Delta_\tau w_j^n\right|^2\right) h.
\end{aligned}$$

Suppose that for sufficiently small $\tau$ and $h$

$$\frac{h}{r} \text{ is uniformly bounded.} \qquad (4.8)$$

By the inequality (3.7), $\| \overset{*}{\delta}{}^2 \tilde{v}_h^{n+1}\|_\infty^2 \le C$. For $n \ge 1$, combine these results to find

$$\begin{aligned}
&\|\delta w_h^{n+1}\|_2^2 + \sum_{k=0}^{n} \left\|\overset{*}{\delta}{}^2 w_h^{k+1}\right\|_2^2 \tau + \tau r \sum_{j=1}^{J-1} (2 - \lambda_{j-1} - \mu_{j+1}) \left|\Delta_\tau w_j^{n+1}\right|^2 h \\
&\le C \left(\sum_{k=0}^{n} \|\delta w_h^{k+1}\|_\infty^2 \tau + \|\delta w_h^0\|_2^2 + R_0\right), \qquad (4.9)
\end{aligned}$$

where

$$R_0 \equiv \max_{0 \le n \le N-1} \left\|A[\tilde{v}]_h^{n+1}\right\|_\infty^2 + \sum_{n=0}^{N-1} \left\|f[\tilde{v}]_h^{n+1}\right\|_2^2 \tau.$$

When $k \ge 1$, for $\forall \varepsilon > 0$, we have

$$\begin{aligned}
\|\delta w_h^{k+1}\|_\infty^2 &\le \varepsilon \|\delta^2 w_h^{k+1}\|_2^2 + \frac{C}{\varepsilon} \|\delta w_h^{k+1}\|_2^2 \\
&\le C\varepsilon \left(\left\|\overset{*}{\delta}{}^2 w_h^{k+1}\right\|_2^2 + r^2 \sum_{j=1}^{J-1} (2 - \lambda_{j-1} - \mu_{j+1}) (\left|\Delta_\tau w_j^{k+1}\right|^2 + \left|\Delta_\tau w_j^k\right|^2) h\right) \\
&\quad + \frac{C}{\varepsilon} \|\delta w_h^{k+1}\|_2^2.
\end{aligned}$$

When $k=0$, there holds

$$\|\delta w_h^1\|_\infty^2\tau \le \|\delta^2 w_h^1\|_2^2 + C\|\delta w_h^1\|_2^2\tau \le C(\|\delta w_h^0\|_2^2 + \left\|A[\tilde{v}]_h^1\right\|_\infty^2 + \left\|f[\tilde{v}]_h^1\right\|_2^2\tau).$$

Substitute the above two inequalities into (4.9), and take $\varepsilon$ small enough, to obtain

$$\|\delta w_h^{n+1}\|_2^2 + \sum_{k=0}^{n}\left\|\overset{*}{\delta}{}^2 w_h^{k+1}\right\|_2^2\tau \le C\left(\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\tau + \|\delta w_h^0\|_2^2 + R_0\right).$$

Therefore,

$$\|\delta w_h^{n+1}\|_2^2 + \sum_{k=0}^{n}\left\|\overset{*}{\delta}{}^2 w_h^{k+1}\right\|_2^2\tau \le C\left(\|\delta w_h^0\|_2^2 + R_0\right).$$

Then we have proved the following theorem of unconditional stability.

**Theorem 4.1** *Suppose that the conditions of Theorem 3.1 hold, and the condition (4.8) holds. Then, for $w_\Delta = v_\Delta - \tilde{v}_\Delta$,*

$$\|v_\Delta - \tilde{v}_\Delta\|_{W_2^{2,1}(Q_\Delta)}^2 \le C\left(\|\varphi_h - \tilde{\varphi}_h\|_{H_h^1}^2 + R_0\right),$$

*where $C$ is a constant independent of $\tau$ and $h$, and*

$$\|\varphi_h\|_{H_h^1}^2 \equiv \|\varphi_h\|_2^2 + \|\delta\varphi_h\|_2^2,$$

$$\|w_\Delta\|_{W_2^{2,1}(Q_\Delta)}^2 \equiv \max_{0\le n\le N}\|w_h^n\|_{H_h^1}^2 + \sum_{k=0}^{N-1}\left(\left\|\overset{*}{\delta}{}^2 w_h^{n+1}\right\|_2^2 + \left\|\Delta_\tau w_h^{n+1}\right\|_2^2\right)\tau.$$

## 5. Extension to Two-dimensional Problem

In this section, we extend the results of Sections 2–4 for one-dimensional problem to two-dimensional case. The notations for the two-dimensional problem are similar to those for the one-dimensional case. For the sake of simplicity, we consider the following two-dimensional problem.

$$u_t = u_{xx} + u_{yy} + f(u), \quad (x,y,t)\in\Omega\times[0,T], \tag{5.1}$$

$$u(x,y,t) = 0, \quad (x,y,t)\in\partial\Omega\times[0,T], \tag{5.2}$$

$$u(x,y,0) = \varphi(x,y), \quad (x,y)\in\Omega, \tag{5.3}$$

where $\Omega = (0,l_1)\times(0,l_2)$.

Suppose that the following conditions are fulfilled.

**(i)** The problem (5.1)–(5.3) has a unique smooth solution $u(x,y,t)$.

**(ii)** The $m$-dimensional vector function $f(u)$ is continuously differentiable.

**(iii)** The initial value $m$-dimensional vector function $\varphi(x,y)\in C^1(\bar{\Omega})$, and $\varphi = 0$ on $\partial\Omega$.

### 5.1 Construction of the parallel scheme

Divide the domain $\Omega\times[0,T]$ by $x_i=ih_1, y_j=jh_2, t^n=n\tau$. For a function $\phi(x,y,t)$ defined on the grid point $(x_i,y_j,t^n)$, let $\phi^n_{i,j}=\phi(x_i,y_j,t^n)$. Denote

$$\begin{aligned}
v^{\bar n+\lambda_{i,j}}_{i+1,j}&=\lambda_{i,j}v^{n+1}_{i+1,j}+(1-\lambda_{i,j})(2v^n_{i+1,j}-v^{n-1}_{i+1,j}),\\
v^{\bar n+\mu_{i,j}}_{i-1,j}&=\mu_{i,j}v^{n+1}_{i-1,j}+(1-\mu_{i,j})(2v^n_{i-1,j}-v^{n-1}_{i-1,j}),\\
v^{\bar n+\bar\lambda_{i,j}}_{i,j+1}&=\bar\lambda_{i,j}v^{n+1}_{i,j+1}+(1-\bar\lambda_{i,j})(2v^n_{i,j+1}-v^{n-1}_{i,j+1}),\\
v^{\bar n+\bar\mu_{i,j}}_{i,j-1}&=\bar\mu_{i,j}v^{n+1}_{i,j-1}+(1-\bar\mu_{i,j})(2v^n_{i,j-1}-v^{n-1}_{i,j-1}).
\end{aligned}$$

The general difference scheme with intrinsic parallelism for the problem (5.1)–(5.3) is as follows:

$$\frac{v^{n+1}_{i,j}-v^n_{i,j}}{\tau}=\overset{*}{\Delta}v^{n+1}_{i,j}+f^{n+1}_{i,j},\tag{5.4}$$

where

$$\overset{*}{\Delta}v^{n+1}_{i,j}=\overset{*}{\delta}{}^2_x v^{n+1}_{i,j}+\overset{*}{\delta}{}^2_y v^{n+1}_{i,j}=\frac{v^{\bar n+\lambda_{ij}}_{i+1,j}-2v^{n+1}_{i,j}+v^{\bar n+\mu_{ij}}_{i-1,j}}{h_1^2}+\frac{v^{\bar n+\bar\lambda_{ij}}_{i,j+1}-2v^{n+1}_{i,j}+v^{\bar n+\bar\mu_{ij}}_{i,j-1}}{h_2^2},$$

$$f^{n+1}_{i,j}=f(v^{n+1}_{i,j}).$$

Denote $w^{n+1}_{i,j}=v^{n+1}_{i,j}-u^{n+1}_{i,j}$, where $u^{n+1}_{i,j}=u(x_i,y_j,t^{n+1})$. Then $w^{n+1}_{i,j}$ satisfies

$$\frac{w^{n+1}_{i,j}-w^n_{i,j}}{\tau}=\overset{*}{\Delta}w^{n+1}_{i,j}+(f_u)^{n+1}_{i,j}w^{n+1}_{i,j}+R^{n+1}_{i,j},\tag{5.5}$$

where $(f_u)^{n+1}_{i,j}=\int_0^1 f_u(\lambda v^{n+1}_{i,j}+(1-\lambda)u^{n+1}_{i,j})d\lambda$, $R^{n+1}_{i,j}=O(\tau+h_1^2+h_2^2)$.

The following assumption will be needed.

**(iv)** Suppose that $\frac{h_1}{r_1}+\frac{h_2}{r_2}$ is uniformly bounded as $\tau$, $h_1$ and $h_2$ are small, where $r_1=\frac{\tau}{h_1^2}$, $r_2=\frac{\tau}{h_2^2}$. And there hold $0\le\lambda_{i,j},\bar\lambda_{i,j},\mu_{i,j},\bar\mu_{i,j}\le1$, $\lambda_{i,j}+\mu_{i+1,j}\ge1$ and $\bar\lambda_{i,j}+\bar\mu_{i,j+1}\ge1$.

### 5.2 Existence, convergence and stability

Similar to the proof of the one-dimensional problem, we have the following theorems.

**Theorem 5.1** (Existence and Convergence) *Suppose that the conditions* **(i)–(iv)** *hold, and step length* $\tau$, $h_1$, $h_2$ *and* $\min\left(\frac{\tau}{h_1^{\frac12}},\frac{\tau}{h_2^{\frac12}}\right)$ *are small enough. Then there exists one solution* $\{v^n_{i,j}\}$ *for the parallel difference schemes* (5.1)–(5.3)*, and the solution satisfies the following inequality,*

$$\max_{0\le n\le N-1}(||w_h^{n+1}||_2+||\delta_x w_h^{n+1}||_2+||\delta_y w_h^{n+1}||_2)\le C(\tau+h^2),\tag{5.6}$$

*where* $C$ *is a constant dependent only on the given data.*

**Theorem 5.2** (Stability) *Suppose that the conditions* **(i)–(iv)** *hold. Then*

$$||v_\Delta - \tilde{v}_\Delta||^2_{W_2^{2,1}(Q_\Delta)} \le C(||\varphi_h - \tilde{\varphi}_h||^2_{H_h^1} + \sum_{n=0}^{N-1} ||f(\tilde{v})_h^{n+1}||_2^2 \tau),$$

*where* $\tilde{v}_\Delta = \{\tilde{v}_{i,j}^n\}$*, and* $C$ *is a constant dependent only on the given data.*

# 6. Numerical Results

In this section, we present numerical results examining the accuracy, stability and parallelism.

## 6.1 One-dimensional test

Consider the following one-dimensional problem:

$$\begin{aligned} &u_t = (a(x)u_x)_x + uu_x + f(x,t), \quad x \in (0,1), \; t \in (0,T], \\ &u(0,t) = u(1,t) = 0, \quad t \in (0,T], \\ &u(x,0) = \sin(\pi x), \quad x \in [0,1], \end{aligned}$$

where $a(x) = 0.04$, $f(x,t) = -\frac{\pi}{2} e^{-0.08\pi^2 t} \sin(2\pi x)$, and the exact solution is $u = e^{-0.04\pi^2 t} \sin(\pi x)$.

First, we examine the errors in the solution for the parallel scheme. The errors in the solution are presented in Table 1. Here the time step $\tau = 1.0e-6$, and the time $T = 0.1$. Moreover, the rate is the experimental rate of convergence, four mesh refinements are used and four processors are used. As can be seen in this table, the errors appear to be $O(h^2)$.

Table 1: The accuracy for one-dimensional problem ($\tau = 1.0e-6$, $T = 0.1$)

| $J-1$ | 20 | 40 | 80 | 160 |
|---|---|---|---|---|
| $\max_{j,n} \lvert v_j^n - u_j^n \rvert$ | 5.73E-4 | 1.43E-4 | 3.58E-5 | 8.95E-6 |
| $\max_{j,n} \frac{\lvert v_j^n - u_j^n \rvert}{\lvert u_j^n \rvert}$ | 1.44E-3 | 3.68E-4 | 9.24E-5 | 2.31E-5 |
| *rate* | – | 2.00 | 2.00 | 2.00 |

Next, we examine the stability and parallelism of the parallel scheme. In order to demonstrate the unconditional stability of the scheme, we present the numerical results for $r = 10, 100, 1000, 10000$ in Tables 2 to 5, respectively, where $r = \tau/h^2$, CPUs is the number of processor, $T_{all}$ is the amount of clock time computing 100000 time steps, $S_p$ is the relative speedup and $E_{ff}$ is the parallel efficient. The fully implicit solution is given when CPUs=1.

In these runs a uniform mesh is used with 100000 grid blocks, i.e., $J-1=100000$. The direct solver is used to solve the linear systems on each sub-domain. Table 2 shows that, for this test problem, the parallel difference scheme produces results which are slightly more accurate than the fully implicit scheme, and the amount of clock time of solving the problem is decreased essentially linearly with the number of processors increased. In fact, the speed-up is super-linear. The similar phenomena can be seen for $r=100,1000,10000$ in Tables 3 and 5 respectively. Hence, the parallel scheme is stability for $r=10,100,1000$ and 10000, and these results indicate that the parallel scheme is unconditionally stable.

Table 2: The stability and parallelism for one-dimensional problem ($r=10$)

| CPUs | 1 | 10 | 20 | 40 | 80 |
|---|---|---|---|---|---|
| $max_{j,n}\|v_j^n-u_j^n\|$ | 2.60E-13 | 2.60E-13 | 2.60E-13 | 2.60E-13 | 2.51E-13 |
| $T_{all}(s)$ | 17027 | 1522 | 755 | 382 | 191 |
| $S_p$ | 1 | 11.18 | 22.54 | 44.52 | 88.98 |
| $E_{ff}(100\%)$ | 1 | 1.12 | 1.13 | 1.11 | 1.11 |

Table 3: The stability and parallelism for one-dimensional problem ($r=100$)

| CPUs | 1 | 10 | 20 | 40 | 80 |
|---|---|---|---|---|---|
| $max_{j,n}\|v_j^n-u_j^n\|$ | 8.78E-11 | 8.79E-11 | 8.79E-11 | 8.79E-11 | 8.79E-11 |
| $T_{all}(s)$ | 17009 | 1535 | 762 | 381 | 192 |
| $S_p$ | 1 | 11.08 | 22.32 | 44.67 | 88.64 |
| $E_{ff}(100\%)$ | 1 | 1.11 | 1.12 | 1.12 | 1.11 |

Table 4: The stability and parallelism for one-dimensional problem ($r=1000$)

| CPUs | 1 | 10 | 20 | 40 | 80 |
|---|---|---|---|---|---|
| $max_{j,n}\|v_j^n-u_j^n\|$ | 7.80E-11 | 7.73E-11 | 7.60E-11 | 7.34E-11 | 6.81E-11 |
| $T_{all}(s)$ | 17901 | 1523 | 767 | 379 | 199 |
| $S_p$ | 1 | 11.75 | 23.33 | 47.17 | 90.03 |
| $E_{ff}(100\%)$ | 1 | 1.18 | 1.17 | 1.18 | 1.13 |

## 6.2 Two-dimensional test

Consider the following two-dimensional problem:

$$u_t=u(u_{xx}+u_{yy})+f(x,y,t),\ \ x\in(0,1),\ y\in(0,1),\ t\in[0,T].$$

Table 5: The stability and parallelism for one-dimensional problem ($r = 10000$)

| CPUs | 1 | 10 | 20 | 40 | 80 |
|---|---|---|---|---|---|
| $max_{j,n}\|v_j^n - u_j^n\|$ | 7.60E-9 | 6.47E-9 | 5.22E-9 | 2.77E-9 | 2.15E-9 |
| $T_{all}(s)$ | 16907 | 1525 | 757 | 380 | 192 |
| $S_p$ | 1 | 11.09 | 22.32 | 44.48 | 87.66 |
| $E_{ff}(100\%)$ | 1 | 1.11 | 1.12 | 1.11 | 1.10 |

The solution is chosen to be

$$u(x, y, t) = e^{-2\pi^2 t}(2 + \sin(\pi x)\sin(\pi y)).$$

The initial function is $u(x, y, 0) = 2 + \sin(\pi x)\sin(\pi y)$, $f(x, y, t) = -2\pi^2 e^{-2\pi^2 t}(2 + \sin(\pi x)\sin(\pi y)) + 2\pi^2 e^{-4\pi^2 t}(2 + \sin(\pi x)\sin(\pi y))\sin(\pi x)\sin(\pi y)$, and the Dirichlet boundary conditions are used.

First, we examine the errors in the solution for the parallel scheme. The errors in the solution are presented in Table 6. Here the rate is the experimental rate of convergence, five mesh refinements are used and four processors ($2 \times 2$) are used. We use the Picard iteration method to linearize the nonlinear systems resulting from the parallel scheme, and use the diagonally preconditioned conjugate gradient method in [11] to solve the linear systems on each sub-domains. As can be seen in this table, the errors appear to be $O(h^2)$.

Table 6: The accuracy for two-dimensional problem ($r = 1, T = 0.1$)

| $(J-1) \times (J-1)$ | $10 \times 10$ | $20 \times 20$ | $40 \times 40$ | $80 \times 80$ | $160 \times 160$ |
|---|---|---|---|---|---|
| $\max_{i,j,n} \|v_{i,j}^n - u_{i,j}^n\|$ | 8.09E-2 | 1.67E-2 | 3.96E-3 | 1.30E-3 | 3.68E-4 |
| $\max_{i,j,n} \frac{\|v_{i,j}^n - u_{i,j}^n\|}{\|u_{i,j}^n\|}$ | 9.08E-2 | 2.54E-2 | 1.00E-2 | 3.16E-3 | 8.87E-4 |
| $rate$ | – | 2.28 | 2.08 | 1.61 | 1.82 |

Next, we examine the stability of the parallel schemes for the two-dimensional problem. In order to demonstrate the unconditional stability of the scheme, we present the numerical results for different $r$ in Table 7. Here $\tau = 1.0e-4$, $T = 0.01$, and four processors ($2 \times 2$) are used. This table shows that, for this test problem, the parallel difference scheme produces very good results for different $r$. Notice that the stability constraint for fully explicit scheme is $r \max_{i,j,n} u_{i,j}^n \leq 1/4$, and our parallel difference scheme is still stable for $r = 400$. It indicates that the scheme is unconditionally stable.

Lastly, we examine the parallelism of the scheme for the two-dimensional problem. In Figure 1, we present the speed-up for different number of processor. In these runs a uniform mesh is used with 1000 grid blocks in each direction, i.e., the scale is $1000 \times 1000$.

Table 7: The stability for two-dimensional problem ($\tau = 1.0E-4$, $T = 0.01$)

| $r$ | 1 | 16 | 64 | 100 | 400 |
|---|---|---|---|---|---|
| $\max_{i,j,n} \lvert v_{i,j}^n - u_{i,j}^n \rvert$ | 3.31E-4 | 8.96E-4 | 2.27E-3 | 2.97E-3 | 6.42E-3 |
| $\max_{i,j,n} \frac{\lvert v_{i,j}^n - u_{i,j}^n \rvert}{\lvert u_{i,j}^n \rvert}$ | 2.11E-4 | 1.39E-4 | 3.33E-4 | 4.51E-4 | 1.09E-3 |

The time step $\tau = 1.0e-5$, the time $T = 0.01$ and the mesh ratio $r = 10$. The Picard iteration method is used to linearize the nonlinear systems and a diagonally preconditioned conjugate gradient algorithm is used to solve the linear systems on each sub-domains. The $x$-direction and $y$-direction have the same number of processor $m$, i.e., the number of processor is $m \times m$. Notice that the global problem is fully decoupled using our parallel difference scheme, hence, some small scale systems on each sub-domain are formed. There is no communication between different sub-domains within the process of solving the nonlinear and linear systems on each sub-domain. The communications only exist between the neighboring processors in order to update the interface values, and then these communications are local. So our parallel schemes have high parallelism.

Our methods have also been compared with the parallel algebraic method, which solves a large scale system resulting from fully implicit scheme in parallel. The Picard iteration method is applied to linearize the global nonlinear systems, and a diagonally preconditioned conjugate gradient algorithm in [11] is used to solve the resulting linear system of large scale in parallel. The parallel algebraic method needs both the local communications between the neighboring processors and the global communications among all processors as well. Hence, its parallelism is low compared with our methods, especially when the CPUs number increases. It is worth to point out that, in the numerical experiments, the same stop criterion is exploited to decide whether the solution is convergence for the parallel difference method and parallel algebraic method.

In Figure 1, the solid line expresses the speed-up of our parallel schemes and the dot line expresses that of the parallel algebraic method. As can be seen in this figure, the speed-up of our method is higher than that of parallel algebraic method with the number of processor increased. Moreover, the speed-up is over 213 when 100 processors are used. This is because that the iterative number of small scale system is less than that of the large scale system. In Table 8, we present the iterative number of parallel difference scheme, where $nonlinear\quad it^{\#}$ is the average number of iteration for the nonlinear system, and $linear\quad it^{\#}$ is the average number of iteration for the linear system. For the parallel algebraic method, the iterative number is the same as that for the parallel difference scheme when one processor is used, i.e., the iterative numbers are 3 and 61 for the nonlinear system and the linear system respectively, and there is no difference for different number of processor. When 100 processors are used, the iterative number of parallel algebraic method is 61, however the iterative number of parallel difference scheme is 49, and hence our parallel methods converge still fast.

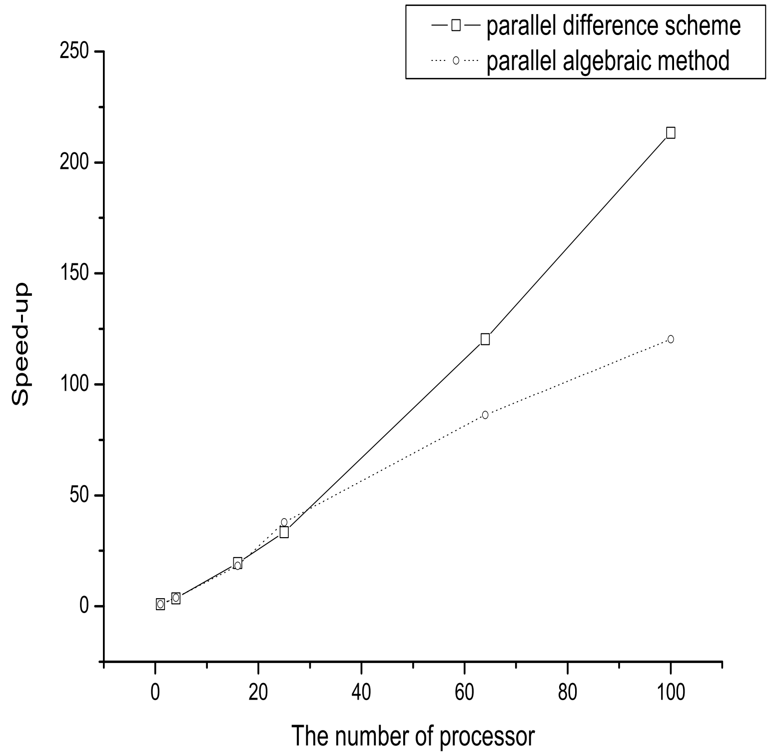


Figure 1: The parallelism for two-dimensional problem

Table 8: The average number of iteration for parallel difference scheme

| $CPUs$ | 1 | 4 | 16 | 25 | 64 | 100 |
|---|---|---|---|---|---|---|
| $nonlinear\ \ it^{\#}$ | 3 | 3 | 3 | 3 | 3 | 3 |
| $linear\ \ it^{\#}$ | 61 | 64 | 57 | 55 | 51 | 49 |

## 7. Summary

We have presented the construction of a new kind of general parallel difference schemes with unconditional stability and second order accuracy for nonlinear parabolic system. A priori error estimate, existence, convergence, stability and uniqueness of the discrete vector solutions for the parallel difference schemes have been derived. The design of the schemes is new and simple as well, and then it can be implemented with little extra effort by using the sequential codes, which are written originally by the fully implicit scheme. The numerical results demonstrate the good performance of the parallel schemes, i.e., they are unconditionally stable, and have second order accuracy and high degree of parallelism. In particular, the super-linear speedup is achieved.

## References

[1] Dawson C N, Du Q, Dupont T F. A finite difference domain decomposition algorithm for numerical solution of the heat equation. *Math. Comp*, 1991, **57**: 63-71.

[2] Dawson C N, Dupont T F. Explicit/implicit conservative domain decomposition procedures for parabolic problems based on block–centered finite differences. *SIAM J. Numer. Anal.*, 1994, **31**: 1045–1061.

[3] Du Q, Mu M, Wu Z N. Efficient parallel algorithms for parabolic problems. *SIAM J. Numer. Anal.*, 2001, **39**: 1469-1487.

[4] Evans D J. Alternating group explicit method for the diffusion equations. *Appl. Math. Modeling*, 1985, **19**: 201–206.

[5] Han Z, Fu H Y, Shen L J. Pure alternating segment explicit–implicit method for the diffusion equations. *Intern. J. Computer Math.*, 1993, **51**: 8–15.

[6] Yuan G W, Hang X D. The Parallel Iterative Difference Schemes Based on Interface Correction for Parabolic Equations. Beijing: ICCP6-CCP, 2003.

[7] Yuan G W, Shen L J, Zhou Y L. Unconditional stability of parallel alternating difference schemes for semilinear parabolic systems. *Applied Mathematics and Computation*, 2001, **117**: 267- 283.

[8] Yuan G W, Shen L J, Zhou Y L. Implicit Differences Domain Decomposition Algotithms for Parabolic Problem. Beijing: Annual Report of LCP, 2000: 472–489.

[9] Zhang B L. Difference graphs of block ADI method. *SIAM J. Numer. Anal*, 2000, **38**: 742-752.

[10] Zhang B L, Li W Z. On alternating segment Crank–Nicolson scheme. *Parallel Computing*, 1994, **20**: 897–902.

[11] Saad Y. Iterative Method for Sparse Linear Systems. New York: PWS publishing, 1996.

[12] Zhou Y L. Applications of Discrete Functional Analysis to Finite Difference Method. Beijing: International Academic Publishers, 1990.