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## INITIAL BOUNDARY VALUE PROBLEM FOR GENERALIZED 2D COMPLEX GINZBURG–LANDAU EQUATION\*

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**Abstract** In this paper we study an initial boundary value problem for a generalized complex Ginzburg–Landau equation with two spatial variables (2D). Applying the notion of the  $\varepsilon$ -regular map we show the unique existence of global solutions for initial data with low regularity and the existence of the global attractor.

**Key Words** Generalized 2D Ginzburg–Landau equation; initial boundary value problem;  $\varepsilon$ -regular map; global solution; global attractor.

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### 1. Introduction

The Ginzburg–Landau equation (GLE) describes various pattern formation and the onset of instabilities in nonequilibrium fluid dynamical systems, as well as in the theory of phase transitions and superconductivity and has drawn great attention to many scientists. The existence of weak and strong solutions, the global attractors and their relative dynamical issues, have been studied by many authors, see, e.g. [1–3] and references therein. A 1D generalized (derivative) GLE has been derived by Doelman [4, 5]) and the global existence of solutions and long time behavior have been studied in [6–8].

In this paper we study the initial boundary value problem for the generalized 2D GLE on a bounded regular domain  $\Omega \subset \mathbf{R}^2$

$$u_t = \gamma u + (1 + i\nu)\Delta u - (1 + i\mu)|u|^{2\sigma}u + \lambda_1 \cdot \nabla(|u|^2u) + (\lambda_2 \cdot \nabla u)|u|^2, \quad (1.1)$$

$$u(t, x) = 0, \quad t \geq 0, \quad x \in \partial\Omega, \quad (1.2)$$

$$u(0, x) = u_0(x), \quad x \in \Omega. \quad (1.3)$$

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Here  $\lambda_1, \lambda_2$  are constant vectors with complex components. The most interesting case for the derivative GLE is  $\sigma = 2$ . Guo and Wang [9] proved the existence of a finite dimensional global attractor. One of their assumptions on  $\sigma$  is  $\sigma \geq 3$ ; the initial data is in  $H^2$ . The Cauchy problem was studied in [7] and the lower bound on  $\sigma$  becomes  $\sigma \geq \frac{1+\sqrt{10}}{2}$ . Later the results were improved in [10, 11]. The initial data are required to have one order (weak) derivatives and the conditions on  $\sigma, \nu$  and  $\mu$  are reduced to

(A1) either (i)  $\sigma > 2$  or (ii)  $\sigma = 2$ ,  $|\lambda_1|$  and  $|\lambda_2|$  are suitably small;

(A2)  $-1 - \nu\mu < \frac{\sqrt{2\sigma+1}}{\sigma}|\nu - \mu|$ .

The main purpose of this paper is to study the existence and uniqueness of the global solution with initial data belonging to some fractional power Sobolev space  $H^s(\Omega)$ ,  $s < 1$ , the existence of the global attractor, and the existence of a time-periodic solution as well. We shall prove

**Main Theorem** *Let  $\sigma, \nu, \mu$  satisfy (A1) and (A2),  $s \in (1 - \frac{1}{2\sigma}, 1)$ . Then for any  $u_0 \in X^1 = D((-\Delta)^{s/2}) \subset H^s(\Omega)$ , (1.1)–(1.3) possess a unique solution  $u$  satisfying  $u \in C([0, \infty); X^1) \cap C((0, \infty); H^2 \cap H_0^1(\Omega))$ . When  $\sigma$  is an integer,  $u \in C^\infty((0, \infty) \times \bar{\Omega})$ . Moreover, (1.1)(1.2) possesses a global attractor  $\mathcal{A}$  which is compact in  $H_0^1$  and attracts bounded subsets of  $H_0^1$  and points of  $X^1$ .*

This paper is arranged as follows. First we prove in Section 2 the local existence of solutions. The idea comes from the so-called  $\varepsilon$ -regular map and  $\varepsilon$ -regular solution developed in [12]. Then in Section 3 we refine the estimates in [10] to show the uniform boundedness of solutions for large time and the existence of the global attractor.

## 2. Local Existence

We put (1.1)–(1.3) into a functional setting

$$u_t + Au = F(u), \quad u(0) = u_0,$$

$A = -(1 + i\nu)\Delta : D(A) = H^2 \cap H_0^1(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ ,  $F(u) = \gamma u + (1 + i\mu)|u|^{2\sigma}u + (\lambda_1 \cdot \nabla)(|u|^2u) + (\lambda_2 \cdot \nabla u)|u|^2$ . It is known that  $A$  is a sectorial operator and generates an analytic semigroup on  $L^2$ . The fractional power of  $A$ ,  $A^\beta$ , with the domain of definition  $E^\beta = D(A^\beta)$ , for any  $\beta \in \mathbf{R}$ , has the following properties [13].

$$\begin{aligned} E^\beta &\hookrightarrow H^{2\beta}(\Omega), \quad \beta \geq 0; E^\beta = H^{2\beta} \cap H_0^1(\Omega), \quad \frac{1}{2} \leq \beta \leq 1, \\ E^\beta &= H_0^1(\Omega), \quad 0 \leq \beta \leq 1/4, E^\beta \hookrightarrow L^s(\Omega), \quad -\frac{1}{2} < \beta \leq 0, \quad s \geq \frac{4}{2 - 4\beta}. \end{aligned}$$

The realization of  $A$  in  $E^\beta$  (still denoted by  $A$ ) is an isometry from  $E^{1+\beta}$  to  $E^\beta$  and is also sectorial on  $E^\beta$ .

Let  $s \in (1 - \frac{1}{2\sigma}, 1)$ ,  $X^\alpha = E^{\alpha + \frac{1}{2}s - 1}$ ,  $\varepsilon = \frac{1}{2}(1 - s)$ ,  $\gamma = (2\sigma + 1)\varepsilon$ . Then  $A$  is a sectorial operator on  $X^0$  with the domain of definition  $X^1$ ,  $0 < 1 - \frac{1}{2}s - \gamma < \frac{1}{2}$ , and

$$\begin{aligned}
X^1 &= E^{s/2} \hookrightarrow H^s(\Omega), X^{1+\varepsilon} = E^{1/2} = H_0^1(\Omega) \hookrightarrow L^p(\Omega), \quad \forall p \geq 1, \\
X^\gamma &= E^{\gamma+s/2-1} \hookrightarrow L^{s_1}(\Omega), s_1 = \frac{2}{3-s-2\gamma}, \quad 1 < s_1 < 2, \\
\|F(u)\|_{X^\gamma} &\leq \|u\|_{X^{1+\varepsilon}} + C\|u\|_{2\sigma+1}^{2\sigma+1} + C\|u\|_{4s_1/(2-s_1)}^2 \|\nabla u\|_2 \\
&\leq C \left( \|u\|_{X^{1+\varepsilon}} + \|u\|_{X^{1+\varepsilon}}^{2\sigma+1} + \|u\|_{X^{1+\varepsilon}}^3 \right), \\
\|F(u) - F(v)\|_{X^\gamma} &\leq C \left( 1 + \|u\|_{X^{1+\varepsilon}}^{2\sigma} + \|u\|_{X^{1+\varepsilon}}^2 \right) \|u - v\|_{X^{1+\varepsilon}},
\end{aligned}$$

$\forall u, v \in X^{1+\varepsilon}$ , here and in the sequel  $C$  is the generic constant,  $\|\cdot\|_p$  denotes the norm of  $L^p(\Omega)$ . Therefore  $F(u)$  is an  $\varepsilon$ -regular map relative to the pair  $(X^1, X^0)$ .

Similarly, for  $s = 1$ , we let  $X^\alpha = E^{\alpha-\frac{1}{2}}$ ,  $\varepsilon \in (0, \frac{1}{2(2\sigma+1)})$ ,  $\gamma = (2\sigma + 1)\varepsilon$ . Then  $A$  is a sectorial operator on  $X^0 = H^{-1}$  with the domain of definition  $X^1 = H_0^1(\Omega)$ ,

$$\begin{aligned}
X^{1+\varepsilon} &= E^{2\varepsilon+1/2} = H^{1+2\varepsilon} \cap H_0^1(\Omega) \hookrightarrow L^\infty(\Omega), \\
X^\gamma &= E^{\gamma-1/2} \hookrightarrow L^{s_2}(\Omega), \quad s_2 = \frac{1}{1-\gamma}, \quad 1 < s_2 < 2,
\end{aligned}$$

and  $F(u)$  is also an  $\varepsilon$ -regular map relative to the pair  $(X^1, X^0)$ .

Thanks to the theory of linear operator semigroups in [12] we have

**Theorem 2.1** *Let  $s \in (1 - \frac{1}{2\sigma}, 1)$ ,  $\varepsilon = \frac{1}{2}(1 - s) > 0$ . For any  $u_0 \in X^1 = D((-\Delta)^{s/2})$ , there exists a  $T > 0$  such that () admits a unique  $\varepsilon$ -regular solution  $u(t) \in C([0, T], X^1) \cap C((0, T], X^{1+\varepsilon})$ .*

**Remark** For  $s = 1$ ,  $\varepsilon \in (0, \frac{1}{2(2\sigma+1)})$ , if  $u_0 \in H_0^1$ , the solution is in  $C([0, T]; H_0^1(\Omega)) \cap C((0, T]; H^{1+2\varepsilon} \cap H_0^1(\Omega))$ . Now if we consider () in  $X^0 = L^2(\Omega)$ ,  $X^1 = D(A) = H^2 \cap H_0^1(\Omega)$ , then  $F(u)$  is locally Lipschitz continuous from  $D(A^{\frac{1}{2}+\varepsilon}) = H^{1+2\varepsilon} \cap H_0^1(\Omega)$  ( $\forall \varepsilon > 0$ ) to  $X^0$ . Thus, from the classical local existence theorem [13], for  $u_0 \in H^{1+2\varepsilon} \cap H_0^1(\Omega)$ , the solution is in  $C((0, T]; D(A))$ . Therefore, the solution stated in the above theorem is in fact in  $C((0, T]; H^2 \cap H_0^1(\Omega))$ .

### 3. Proof of The Main Theorem

**Lemma 3.1** *Assume that  $p, \nu$  satisfy  $2 \leq p < \frac{2\sqrt{1+\nu^2}}{\sqrt{1+\nu^2}-1}$ , or  $\frac{2\sqrt{1+\nu^2}}{\sqrt{1+\nu^2}+1} < p \leq 2$ . Then there is a constant  $K_1$  independent of  $u_0$  and  $t$  such that*

$$\|u\|_p^p \leq \|u_0\|_p^p e^{-t} + K_1(1 - e^{-t}), \quad t \geq 0. \quad (3.1)$$

**Proof** We multiply (1.1) by  $|u|^{p-2}\bar{u}$ , integrate over  $\Omega$ , take real parts and get

$$\begin{aligned}
\frac{1}{p} \frac{d}{dt} \|u\|_p^p &= \gamma \|u\|_p^p - \|u\|_{p+2\sigma}^{p+2\sigma} + \operatorname{Re} \int (1 + i\nu) |u|^{p-2} \bar{u} \Delta u \, dx \\
&\quad + \operatorname{Re} \int |u|^{p-2} \bar{u} \left( (\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla u) |u|^2 \right) dx.
\end{aligned}$$

Note that

$$\operatorname{Re} \int (1 + i\nu)|u|^{p-2}\bar{u}\Delta u \, dx = -\frac{1}{4} \int |u|^{p-4} \sum_{j=1}^2 (\bar{u}\partial_j u, u\partial_j \bar{u}) M(\nu, p) \begin{pmatrix} u\partial_j \bar{u} \\ \bar{u}\partial_j u \end{pmatrix} dx, \quad (3.2)$$

where  $M(\nu, p) = \overline{M(\nu, p)}^{tr} = \begin{pmatrix} p & (1+i\nu)(p-2) \\ * & p \end{pmatrix}$  is a Hermitian matrix. Under the assumption on  $p$ , the smaller eigenvalue of  $M(\nu, p)$   $\lambda_M(\nu, p) = p - |p-2|\sqrt{1+\nu^2} > 0$ , so  $M(\nu, p)$  is positively definite. When  $\sigma > 2$ , by controlling the indefinite terms we obtain

$$\frac{1}{p} \frac{d}{dt} \|u\|_p^p \leq (\gamma + C) \|u\|_p^p - \frac{1}{2} \|u\|_{p+2\sigma}^{p+2\sigma} - \frac{\lambda_M(\nu, p)}{8} \int |u|^{p-2} |\nabla u|^2 \, dx.$$

Since  $\sigma$  is positive,  $\exists C_1 > 0$  such that  $(\gamma + C + \frac{1}{p})z^p \leq \frac{1}{4}z^{p+2\sigma} + C_1, \forall z \in \mathbf{R}$ , so

$$\frac{1}{p} \frac{d}{dt} \|u\|_p^p + \frac{1}{p} \|u\|_p^p + \frac{1}{4} \|u\|_{p+2\sigma}^{p+2\sigma} + \frac{\lambda_M(\nu, p)}{8} \int |u|^{p-2} |\nabla u|^2 \, dx \leq C_1 |\Omega|.$$

Applying Gronwall inequality we get (3.1).

When  $\sigma = 2$ , and  $6|\lambda_1| + 2|\lambda_2| < \lambda_M(\nu, p)$ , (3.1) is also valid.

**Lemma 3.2** *Under the assumptions of (A1) and (A2), there exists a  $K_2$  such that*

$$\|\nabla u(t)\|_2^2 \leq \|\nabla u_0\|_2^2 e^{-t} + K_2(1 - e^{-t}), \quad t \geq 0.$$

**Proof** Similar to (3.2), for any  $\alpha$  with  $|\alpha| < \frac{\sqrt{2\sigma+1}}{\sigma}$ , the smaller eigenvalue  $\lambda_M(\alpha, 2\sigma + 2)$  of  $M(\alpha, 2\sigma + 2)$  is positive and thus  $M(\alpha, 2\sigma + 2)$  is definitely positive. Thus we have

$$\operatorname{Re} \int (1 + i\alpha)|u|^{2\sigma}\bar{u}\Delta u \, dx + \lambda_M(\alpha, 2\sigma) \int |u|^{2\sigma} |\nabla u|^2 \, dx \leq 0.$$

Define  $V_\delta(u(t)) = \int (\frac{1}{2}|\nabla u|^2 + \frac{\delta}{2\sigma+2}|u|^{2\sigma+2}) \, dx$ , multiply (3.3) by  $-\eta$  ( $\delta > 0, \eta > 0$  and  $0 \leq \kappa < 1$  will be suitably chosen), then add it to  $\frac{d}{dt}V_\delta(u(t))$  and get

$$\begin{aligned} \frac{d}{dt}V_\delta(u(t)) &\leq \gamma(\|\nabla u\|_2^2 + \delta\|u\|_{2\sigma+2}^{2\sigma+2}) - (1 - \kappa)(\|\Delta u\|_2^2 + \delta\|u\|_{4\sigma+2}^{4\sigma+2}) \\ &\quad - \eta\lambda_M(\alpha, 2\sigma + 2) \int |u|^{2\sigma} |\nabla u|^2 \, dx \\ &\quad + \frac{1}{2} \operatorname{Re} \int (|u|^{2\sigma} u, \Delta u) \cdot N \cdot \begin{pmatrix} |u|^{2\sigma}\bar{u} \\ \Delta \bar{u} \end{pmatrix} dx \\ &\quad + C(3|\lambda_1| + |\lambda_2|) \int (|u|^2 |\nabla u| |\Delta u| + |u|^{2\sigma+3} |\nabla u|) \, dx, \end{aligned} \quad (3.3)$$

where  $N = \overline{N}^{tr} = \begin{pmatrix} -2\delta\kappa & 1 + \delta - \eta - i(\delta\nu - \mu - \alpha\eta) \\ * & -2\kappa \end{pmatrix}$  is a Hermitian matrix.

When  $\sigma, \nu$  and  $\mu$  satisfy the assumptions (A1) and (A2), we can choose suitable  $\delta, \eta$

positive,  $\kappa \in [0, 1)$  and  $|\alpha| < \frac{\sqrt{2\sigma+1}}{\sigma}$  such that  $N$  is nonpositive (see [3, 10]). Hence the integral involving  $N$  is negative. Carefully estimating the last integral of (3.3), using Lemma 3.1 and interpolation inequalities, we have

$$\begin{aligned} \frac{d}{dt} V_\delta(u(t)) &\leq (\gamma + C)(\|\nabla u\|_2^2 + \delta\|u\|_{2\sigma+2}^{2\sigma+2}) - \frac{1}{2}(1 - \kappa)(\|\Delta u\|_2^2 + \delta\|u\|_{4\sigma+2}^{4\sigma+2}) \\ &\leq -V_\delta(u(t)) - \frac{1}{4}(1 - \kappa)(\|\Delta u\|_2^2 + \|u\|_{4\sigma+2}^{4\sigma+2}) + K_2, \end{aligned}$$

where  $K_2$  may depend on  $\|u\|_{L^2}$ . By Gronwall inequality we have the lemma.

**Lemma 3.3** *Let  $t_0 > 0$ ,  $u(t_0) \in H^2 \cap H_0^1(\Omega)$ . Under the assumptions of (A1) and (A2), there exists a  $K_3$  such that*

$$\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leq 5\|\Delta u(t_0)\|_2^2 e^{-(t-t_0)} + K_3, \quad t \geq t_0. \quad (3.4)$$

**Proof** Differentiate (1.1) with respect to  $t$ , multiply both sides by  $\bar{u}_t$ , integrate over  $\Omega$ , and then take real parts to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_t\|_2^2 + \|\nabla u_t\|_2^2 &\leq -\operatorname{Re} \int (1 + i\mu)(|u|^{2\sigma} u)_t \bar{u}_t dx \\ &\quad + C(3|\lambda_1| + |\lambda_2|) \int (|u| |\nabla u| |u_t| + |u|^2 |\nabla u_t|) dx. \end{aligned} \quad (3.5)$$

Note that, similar to (3.2) and (3.3), under the assumption (A2),  $\mu$  satisfies  $|\mu| < \frac{\sqrt{2\sigma+1}}{\sigma}$ , the smaller eigenvalue  $\lambda_M(\mu, 2\sigma + 2)$  of the matrix  $M(\mu, 2\sigma + 2)$  is positive and thus  $M(\mu, 2\sigma + 2)$  is definitely positive. Thus we have

$$\operatorname{Re} \int (1 + i\mu)(|u|^{2\sigma} u)_t \bar{u}_t dx = \frac{1}{2} \int |u|^{2\sigma-2} (u \bar{u}_t, \bar{u} u_t) \cdot M(\mu, 2\sigma + 2) \cdot \begin{pmatrix} \bar{u} u_t \\ u \bar{u}_t \end{pmatrix} dx \geq 0.$$

Therefore

$$\frac{1}{2} \frac{d}{dt} \|u_t\|_2^2 + \|\nabla u_t\|_2^2 \leq C(\|u\|_4 \|\nabla u\|_2 \|u_t\|_4 + \|u\|_4^2 \|\nabla u_t\|_2) \leq \frac{1}{2} \|\nabla u_t\|_2^2 + C,$$

where  $C = C(\|u\|_4^2 \|\nabla u\|_2^2 + \|u\|_4^4)$ . By Poincaré inequality  $\lambda \|u_t\|_2^2 \leq \|\nabla u_t\|_2^2$ , constant  $\lambda > 0$ , and Gronwall inequality we have  $\|u_t(t)\|_2^2 \leq \|u_t(t_0)\|_2^2 e^{-(t-t_0)} + \tilde{K}_3$  for all  $t \geq t_0$ . By (1.1) and Lemmas 3.1 and 3.2, we have  $\|u_t(t_0)\|_2 \leq 2\|\Delta u(t_0)\|_2 + C$ ,  $\|\Delta u(t)\|_2 \leq 2\|u_t(t)\|_2 + C$ , and thus (3.4).

Now we prove the main theorem. From Theorem 2.1 and Lemmas 3.1–3.3 we see that, for every initial data  $u_0 \in X^1$ , the solution exists globally and remains bounded in  $H^2$  for  $t \geq t_0 \forall t_0 > 0$ . When  $\sigma$  is an integer, the nonlinear term is analytic. Using bootstrapping method we see that  $u \in C^j((0, +\infty); H^k(\Omega)), \forall j, k \geq 0$ . The solution operator  $S(t) : u_0 \mapsto u(t) = S(t)u_0$  forms a continuous dynamic system generated by the derivative GLE.

From Lemma 3.1 we can choose  $\rho_1 = 2K_1$  and  $t_1(R)$  such that  $\|u(t)\|_2 \leq \rho_1, \forall t \geq t_1(R)$  whenever  $\|u_0\|_2 \leq R$ . Thus, by Lemma 3.2, when  $t \geq t_1$ ,  $K_2$  is independent of

$u_0$ . Let  $\rho_2 = 2(K_1 + K_2)$ . From Lemma 3.2 we can choose  $t_2(R) > t_1(R)$  such that  $\|u(t)\|_{H^1} \leq \rho_2$ ,  $\forall t \geq t_2(R)$  whenever  $\|u_0\|_{H^1} \leq R$ . That is,  $B(0, \rho_2)$ , the ball in  $H_0^1(\Omega)$  of radius  $\rho_2$  centered at 0, is a bounded absorbing set of  $S(t)$ . Moreover  $S(t)$  is compact in  $X^1$  for  $t > 0$  and compact in  $H_0^1(\Omega)$  uniformly for  $t$  large. The  $\omega$ -limit set of  $B(0, \rho_2)$  under the action of  $S(t)$ ,  $\mathcal{A} = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B(0, \rho_2)}^{H_0^1}$ , is a global attractor of the derivative GLE.  $\mathcal{A}$  attracts points in  $X^1$  and bounded subsets of  $H_0^1(\Omega)$ . It can be shown that  $\mathcal{A}$  has finite Hausdorff and fractal dimension [14]. The proof of the main theorem is completed.

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