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WELL-POSEDNESS, DECAY ESTIMATES AND BLOW-UP THEOREM FOR THE FORCED NLS*

Charles Bu[1] Randy Shull[2] Hefei Wang[1] and Millie Chu[1]
([1]Department of Mathematics, Wellesley College, Wellesley, MA 02481, U.S.A.)
([2] Department of Computer Science, Wellesley College, Wellesley, MA 02481, U.S.A.)
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Abstract In this article we prove that the following NLS $iu_t = u_{xx} - g|u|^{p-1}u, g > 0, x, t > 0$ with either Dirichlet or Robin boundary condition at $x = 0$ is well-posed. L^{p+1} decay estimates, blow-up theorem and numerical results are also given.

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1. Introduction

Boundary value problems for important evolution equations often are called forced problems. Often these problems have significant physical implications. For example, in ionospheric modification experiments, one directs a radio frequency wave at the ionosphere. At the reflection point of the wave, a sufficient level of electron plasma waves is excited to make the nonlinear behavior important [1,2]. This may be described by the NLS equation with the cubic nonlinear term and a nonlinear boundary condition

$$\begin{cases} iq_t = q_{xx} \pm 2|q|^2 q, & x, t \in \mathbf{R}^+ \\ q(x,0) = h(x), q(0,t) = g(t) \end{cases} \tag{1.1}$$

where $h(x)$ decays for large x and the given functions $h(x), g(t)$ have appropriate smoothness, and satisfy the necessary compatibility conditions. For (1.1), global existence, well-posedness and blow-up result were established when $h \in H^2[0,\infty), Q \in C^2[0,\infty)$[3,4].

In this paper, we study the following NLS with a general nonlinear term $-g|u|^{p-1}u$ for $p > 1, g > 0$:

$$\begin{cases} iu_t = u_{xx} - g|u|^{p-1}u, & x, t \in \mathbf{R}^+ \\ u(x,0) = h(x) \end{cases} \tag{1.2}$$

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with either Dirichlet boundary condition $u(0,t) = Q(t)$ or Robin boundary condition $u_x(0,t) + \alpha u(0,t) = R(t)$, where α is real. Under the assumption that $h \in H^2[0,\infty), Q$ or $R \in C^2[0,\infty)$, there exists a unique global classical solution $u \in C^1([0,\infty), L^2[0,\infty)) \cap C^0([0,\infty), H^2[0,\infty))$[3]. Let $P(t) = u_x(0,t)$, the following three identities can be easily verified (in the case of Robin boundary condition, P is replaced by $R - \alpha Q$):

$$\partial_t \int_0^\infty |u|^2 dx = -2\mathrm{Im}(P\bar{Q}) \tag{1.3}$$

$$\partial_t \int_0^\infty \left(|u_x|^2 + \frac{2g}{p+1}|u|^{p+1} \right) dx = -2\mathrm{Re}P\bar{Q}' \tag{1.4}$$

and

$$\partial_t \int_0^\infty u\bar{u}_x dx = -Q\bar{Q}' + i|P|^2 - i\frac{2g}{p+1}|Q|^{p+1} \tag{1.5}$$

In the following, we prove well-posedness for the above problem with either boundary condition, give L^{p+1} decay estimates *via* a pseudoconformal identity and present blow-up result.

2. Well-Posedness Results

Consider (1.2) for $0 \le t \le T$ and assume that for some $M > 0, \|Q\|_{C^2[0,T]} < M$ or $\|R\|_{C^2[0,T]} < M$, depending on the type of boundary condition. Also we assume that $\|h\|_{H^2(\mathrm{R}^+)} \le M$. For the Dirichlet boundary value problem, assume that u, v solve (1.2) with boundary-initial data (Q, u_0) and (Q_1, v_0) both lying in $C^2[0,T] \times H^2(\mathrm{R}^+) = X$. By global existence theorem, there exists a constant $\tilde{\lambda} > 0$ that only depends on M and T such that $\|u\|_{H^1(\mathrm{R}^+)} \le \tilde{\lambda}$ for $t \in [0,T]$ thus $\|u\|_\infty \le c_0\|u'\|_2^{\frac{1}{2}}\|u\|_2^{\frac{1}{2}} \le \lambda$. Clearly, the map $f : X \to Y = C^1(L^2,[0,T]) \cap C^0(H^2,[0,T]) via (Q,u_0) \mapsto u$ is well-defined. Let $z = (Q,u_0), z_1 = (Q_1, v_0) \in X, \|z\|_X = \max\{\|Q\|_{C^2[0,T]}, \|h\|_{2,2}\} < M, \|z_1\|_X < M$ and

$$w = \Delta u = v - u, \Delta z = z_1 - z = (\Delta Q, w_0) = (Q_1 - Q, v_0 - u_0) \tag{2.1}$$

Since $v = w + u$ satisfies (1.2) as well, one has $i(w_t + u_t) = w_{xx} + u_{xx} - g|w + u|^{p+1}(w+u)$ where w satisfies the following variable-coefficient, initial-value, boundary-value problem:

$$\begin{cases} iw_t = w_{xx} - g|w+u|^{p+1}(w+u) + u_{xx} - iu_t = w_{xx} + G(w,t) \\ w(0,t) = \Delta Q, w_0 = v_0 - u_0 \end{cases} \tag{2.2}$$

Let $\Delta P = P_1 - P = v_x(0,t) - u_x(0,t)$. From (2.2) one has

$$i\partial_t|w|^2 = iw_t\bar{w} + iw\bar{w}_t = 2i\mathrm{Im}(w_{xx}\bar{w} + \bar{w}G(w,t)) \tag{2.3}$$

Thus

$$\begin{aligned}\int_0^\infty |w|^2dx =& \|w_0\|_2^2 + 2\mathrm{Im}\int_0^t \left(\int_0^\infty (w_{xx}\bar{w} + \bar{w}G(w,t))dxd\tau\right.\\ =& \|w_0\|_2^2 - 2\mathrm{Im}\int_0^t \Delta P\Delta\bar{Q}d\tau + 2\mathrm{Im}\int_0^t\int_0^\infty \bar{w}G(w,t)dxd\tau\\ \le& \|w_0\|_2^2 + 2\left(\int_0^t |\Delta Q(\tau)|^2d\tau\right)^{\frac{1}{2}}\left(\int_0^t |\Delta P(\tau)|^2d\tau\right)^{\frac{1}{2}}\\ &+\int_0^t\int_0^\infty 2g(|u|^{p+2}|w| + |w+u|^{p+1}|(w+u)w|)dxd\tau \qquad (2.4)\end{aligned}$$

We note that $\sup|w(x,t)| \le \sup(|u(x,t)| + |v(x,t)|) \le 2\lambda$ for $0\le t\le T, 0\le x<\infty$ by boundedness of u, v in H^1 norm. The proof of global existence [3] demonstrated that $\left(\int_0^t |P(\tau)|^2d\tau\right)^{\frac{1}{2}} \le c' + \dfrac{\|u_x\|}{\sqrt{2}}$ for some $c'>0$, thus

$$\begin{aligned}\left(\int_0^t |\Delta P(\tau)|^2d\tau\right)^{\frac{1}{2}} \le& \left[\int_0^t (2|P_1(\tau)|^2 + 2|P(\tau)|^2)d\tau\right]^{\frac{1}{2}}\\ \le& \sqrt{2}\left(\int_0^t |P_1(\tau)|^2d\tau\right)^{\frac{1}{2}} + \sqrt{2}\left(\int_0^t |P(\tau)|^2d\tau\right)^{\frac{1}{2}}\\ \le& \sqrt{2}\left(\frac{\|v_x\|_2}{\sqrt{2}} + 2c' + \frac{\|u_x\|_2}{\sqrt{2}}\right)\\ =& 2c'\sqrt{2} + \|v_x\|_2 + \|u_x\|_2 \le \hat{c} \qquad (2.5)\end{aligned}$$

Also,

$$\begin{aligned}\int_0^\infty |w|^2dx \le& \|w_0\|_2^2 + 2\sqrt{T}c_0\|\Delta Q\|_{C[0,T]}\\ &+\int_0^t\int_0^\infty 2g(\lambda^{p+1}|uw| + (2\lambda)^{p+1}|(w+u)w|)dxd\tau\\ \le& \|w_0\|_2^2 + c_0\|\Delta Q\|_{C[0,T]} + \tilde{c}\int_0^t\int_0^\infty |w|^2dxd\tau \qquad (2.6)\end{aligned}$$

By Gronwall's lemma,

$$\int_0^\infty |w|^2dx \le (\|w_0\|_2^2 + c_0\|\Delta Q\|_{C[0,T]})e^{\tilde{c}t} \le (\|w_0\|_2^2 + c_0\|\Delta Q\|_{C[0,T]})e^{\tilde{c}T} \qquad (2.7)$$

Therefore

$$\begin{aligned}\sup_{0\le t\le T}\left(\int_0^\infty |w|^2dx\right)^{\frac{1}{2}} \le& \sqrt{(\|w_0\|_2^2 + c_0\|\Delta Q\|_{C[0,T]})e^{\tilde{c}T}}\\ \le& m_0\left(\|w_0\|_2^{\frac{1}{2}}\|w_0\|_2^{\frac{1}{2}} + c\|\Delta Q\|_{C[0,T]}^{\frac{1}{2}}\right)\\ \le& m_0\left(\sqrt{2M}\|w_0\|_2^{\frac{1}{2}} + c\|\Delta Q\|_{C[0,T]}^{\frac{1}{2}}\right) \le m\|\Delta z\|_{X_0}^{\frac{1}{2}} \qquad (2.8)\end{aligned}$$

which implies that $\sup_{0\le t\le T}\|v-u\|_2 \le m\|z_1-z\|^{\frac{1}{2}}_{X_0}$ where $X_0 = C[0,T]\times L^2(\mathbf{R}^+)$ for some $m>0$. Let $Y_2 = C^1(L^2,[0,T])$. The norm of u on Y_2 is $\sup_{0\le t\le t}(\|u_t\|_2+\|u\|_2)$. A change of variables $viaw = W+\Delta Q(t)e^{-x}$ leads to the following equation $iW_t = W_{xx}+G(W,t)$ with $W(x,0)=w_0(x)-\Delta Q(0)e^{-x} = W_0(x), W_0(0)=w_0(0)-\Delta Q(0)=0$. Since $Q\in C^2(\mathbf{R}^+), \partial_t G(W,t)$ is continuous in $L^2(0,\infty)$. By [4], one has $\int_0^t N(t-s)G(W,s)ds \in D(A)$, where $A=-D_x^2$ with $D(A)=\{W,W_{xx}\in L^2(\mathbf{R}^+); W(0)=0\}$ and $N(t)=\exp\{At\}$ being a strongly continuous contraction semigroup in L^2. The equation $iW_t = W_{xx}+G(W,t)$ is converted to an intergral equation

$$W(t) = N(t)W_0 + \int_0^t N(t-s)G(W,s)ds = N(t)W_0 + \int_0^t N(s)G(t-s)ds \tag{2.9}$$

Again, we have $\|w\|_{1,2}\le\|u\|_{1,2}+\|v\|_{1,2}\le 2\tilde{\lambda}$ and $\|w\|_\infty \le \|u\|_\infty+\|v\|_\infty \le 2\lambda$, hence $\|w\|_{1,2}$ and $\|w\|_\infty$ are also bounded. Meanwhile, $\|\Delta Q\|_{C^2[0,T]}\le \|Q_1\|_{C^2[0,T]} + \|Q\|_{C^2[0,T]}\le 2M$. Since $W_0\in D(A)$, one has $(N(t)W_0)_t = N(t)AW_0$. By (2.9), we have

$$\begin{aligned} W_t(t) &= (N(t)W_0)_t + N(t)G(W,0) + \int_0^t N(t)G'(W,t-s)ds \\ &= N(t)(-iD_x^2W_0) + N(t)G(W,0) + \int_0^t N(t-s)G_s(W,s)ds \end{aligned} \tag{2.10}$$

and $\|G(W,0)\|_2 \le c_1\|\Delta Q\|_{C^1[0,T]} + c_2\|W_0\|_2$. Since $N(t)$ is a contraction semigroup on L^2, one has $\|N(t)(-iD_x^2W_0)\|_2 \le c_3\|W_0\|_{2,2}$ which implies that

$$\|G(W,0)\|_2 \le c_0(\|\Delta Q\|_{C^1[0,T]} + \|W_0\|_{2,2}) \tag{2.11}$$

and

$$\|G_t(W,t)\|_2 \le c_4\|\Delta Q\|_{C^2[0,T]} + c_5(\|W\|_2 + \|W_t\|_2) \tag{2.12}$$

Now put (2.11) and (2.12) in (2.10) to get

$$\begin{aligned} \|W_t\|_2 \le& \|N(t)(-iD_x^2W_0)\|_2 + \|G(W,0)\|_2 + \int_0^t \|N(t-s)G_s(W,s)\|_2 ds \\ \le& c_3\|W_0\|_{2,2} + c_0(\|\Delta Q\|_{C^1[0,T]} + \|W_0\|_{2,2}) \\ &+ \int_0^t c_7(\|\Delta Q\|_{C^2[0,T]} + \|W\|_2 + \|W_t\|_2)ds \\ \le& c_8\|W_0\|_{2,2} + \tilde{c}\|\Delta Q\|_{C^2[0,T]} + \int_0^t \|W\|_2 ds + c_7\int_0^t \|W_t\|_2 ds \end{aligned} \tag{2.13}$$

By Gronwall's lemma

$$\|W_t\|_2 \le c'(\|\Delta Q\|_{C^2[0,T]} + \|W_0\|_{2,2}) + \bar{c}\int_0^t \|W\|_2 ds \tag{2.14}$$

Since $w = W + \Delta Q(t)e^{-x}, w_t = W_t + \Delta Q'(t)e^{-x}$, (2.14) implies that

$$\|w_t\|_2 \le \|W_t\|_2 + \|\Delta Q\|_{C^1[0,T]} \le c(\|\Delta Q\|_{C^2[0,T]} + \|w_0\|_{2,2}) + \bar{c}\int_0^t \|w\|_2 ds \tag{2.15}$$

By Lemma 3,

$$\begin{aligned}\|v - u\|_{Y_2} &= \sup_{0\le t\le T} (\|w_t\|_2 + \|w\|_2)\\ &\le \sup_{0\le t\le T} (c(\|\Delta Q\|_{C^2[0,T]} + \|w_0\|_{2,2})\\ &\quad + \bar{c}\int_0^t m\|\Delta z\|_{X_1}^{\frac{1}{2}} ds + m\|\Delta z\|_{X_1}^{\frac{1}{2}}) \le \hat{c}\|\Delta z\|_X^{\frac{1}{2}}\end{aligned} \tag{2.16}$$

Therefore, there exists a positive constant $\tilde{c}$ such that $\|v - u\|_{Y_2} \le \tilde{c}\|z_1 - z\|_X^{\frac{1}{2}}$. To prove the well-posedness, it suffices now to show that there exists $M > 0$ such that $\|v - u\|_{Y_3} \le M\|z_1 - z\|_X^{\frac{1}{2}}$, where $Y_3 = C^0(H^2, [0,T])$. From (2.2) and the fact that w, u both bounded under H^1 and L^∞ norms, we obtain

$$\|w_{xx}\|_2 \le \|w_t\|_2 + \|G(w,t)\|_2 \le \|w\|_{Y_2} + c_0\|w\|_2 \tag{2.17}$$

Put (2.8) and (2.16) in (2.17) to get

$$\|w_{xx}\|_2 \le \hat{c}\|\Delta z\|_X^{\frac{1}{2}} + c'\|w\|_2 \le \hat{c}\|\Delta z\|_X^2 + m'\|\Delta z\|_{X_0}^{\frac{1}{2}} \le \tilde{c}\|\Delta z\|_X^{\frac{1}{2}} \tag{2.18}$$

The above two estimates show that

$$\|v - u\|_{Y_3} = \sup_{0\le t\le T} (\|w_{xx}\|_2 + \|w\|_2) \le \tilde{c}\|\Delta z\|_X^{\frac{1}{2}} + m\|\Delta z\|_{X_0}^{\frac{1}{2}} \le M\|\Delta z\|_X^{\frac{1}{2}} \tag{2.19}$$

We conclude that $f : X \to Y = Y_2 \cap Y_3$ is continuous at z and have the following theorem.

Theorem 1 *Assume that conditions for global existence theorems in [3] are satisfied, then NLS (1.2) with Dirichlet boundary condition is well-posed.*

For the NLS (1.2) with Robin boundary condition, a proof similar to [5] yields the following theorem.

Theorem 2 *Assume that conditions for global existence theorems in [3] are satisfied, then NLS (1.2) with Robin boundary condition is well-posed.*

3. Decay Estimates and Blow-up Theorem

Under the assumption that $\int_0^\infty x\bar{u}u_x dx$ makes sense then a number of interesting equations and estimates can be obtained. In particular, we are interested in decay estimates of the solution to the forced NLS with both types of boundary data. Thus multiply equation (1.2) by $2x\bar{u}_x$ and integrate the real parts from 0 to ∞. The result can be expressed as

$$\begin{aligned} I = 2\mathrm{Re}i\int_0^\infty x\bar{u}_x u_t dx &= 2\mathrm{Re}\int_0^\infty x\bar{u}_x u_{xx} dx - \frac{2g}{p+1}\int_0^\infty x\partial_x(|u|^{p+1})dx \\ &= -\int_0^\infty |u_x|^2 dx + \frac{2g}{p+1}\int_0^\infty |u|^{p+1}dx \end{aligned} \tag{3.1}$$

The left-hand side of (3.1) can be rewritten as

$$\begin{aligned} I &= \mathrm{Re}\int_0^\infty (ix(u_t\bar{u}_x - \bar{u}_t u_x)dx) = \mathrm{Re}\int_0^\infty (ix(\partial_t(u\bar{u}_x) - \partial_x(u\bar{u}_t))dx) \\ &= \partial_t \mathrm{Re}\int_0^\infty (ix\bar{u}_x u dx) + \mathrm{Re}\int_0^\infty (iu\bar{u}_t dx) \end{aligned} \tag{3.2}$$

Substitute for $i\bar{u}_t$ from (1.2) to get

$$I = \partial_t \mathrm{Im}\int_0^\infty x u_x \bar{u} dx + \int_0^\infty |u_x|^2 dx + g\int_0^\infty |u|^{p+1}dx + \mathrm{Re}P\bar{Q} \tag{3.3}$$

Write $y = \mathrm{Im}\int_0^\infty x u_x \bar{u} dx$ and combine (3.2) with (3.3) to obtain

$$y' = \partial_t \mathrm{Im}\int_0^\infty x u_x \bar{u} dx = -2\int_0^\infty |u_x|^2 dx - \frac{g(p-1)}{p+1}\int_0^\infty |u|^{p+1}dx - \mathrm{Re}P\bar{Q} \tag{3.4}$$

We can obtain a "pseudo-conformal" identity and decay estimate for $\|u\|_{p+1}^{p+1}$ similar to the one proved in [6]. For the Dirichlet boundary condition, integrate the equation $\partial_t|u|^2 = -i\partial_x(u_x\bar{u} - \bar{u}_x u)$ in t to obtain

$$\partial_t \int_0^\infty x^2|u|^2 dx = -4y \tag{3.5}$$

Multiply (3.4) by $4t$ and rewrite it as

$$\begin{aligned} \partial_t(4ty) - 4y = 4ty' &= -4t\mathrm{Re}P\bar{Q} - 8t\int_0^\infty |u_x|^2 dx - \frac{4g(p-1)}{p+1}t\int_0^\infty |u|^{p+1}dx \\ &= -4t\mathrm{Re}P\bar{Q} - \partial_t\left(4t^2\int_0^\infty |u_x|^2 dx\right) \\ &\quad + 4t^2\partial_t\int_0^\infty |u_x|^2 dx - \frac{4g(p-1)}{p+1}t\int_0^\infty |u|^{p+1}dx \end{aligned} \tag{3.6}$$

To derive the pseudoconformal identity, one starts with

$$\begin{aligned} |xu - 2itu_x|^2 &= (xu - 2itu_x)(x\bar{u} + 2it\bar{u}_x) \\ &= x^2|u|^2 + 4t^2|u_x|^2 + 2ixt(\bar{u}_x u - u_x\bar{u}) \\ &= x^2|u|^2 + 4t^2|u_x|^2 + 4t\mathrm{Im}x\bar{u}u_x \end{aligned} \tag{3.7}$$

Differentiate (3.7) in t, integrate in x and apply (3.6), (1.4) to obtain

$$\begin{aligned}
\partial_t \int_0^\infty |xu - 2itu_x|^2 dx =& \partial_t \int_0^\infty x^2|u|^2 dx + \partial_t \int_0^\infty 4t^2|u_x|^2 dx + \partial_t \int_0^\infty 4t\mathrm{Im} u\bar{u}_x dx \\
=& -4y + \partial_t \left(4t^2 \int_0^\infty |u_x|^2 dx\right) + \partial_t(4ty) \\
=& -4t\mathrm{Re}P\bar{Q} + 4t^2\partial_t \int_0^\infty |u_x|^2 dx - \frac{4g(p-1)}{p+1} t \int_0^\infty |u|^{p+1} dx \\
=& -4t\mathrm{Re}P\bar{Q} - 8t^2\mathrm{Re}P\bar{Q}' - \partial_t \left(\frac{8gt^2}{p+1}\int_0^\infty |u|^{p+1}dx\right) \\
&+ \frac{4g(5-p)}{p+1} t \int_0^\infty |u|^{p+1} dx
\end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned}
&\partial_t \left(\int_0^\infty |xu - 2itu_x|^2 dx + \frac{8gt^2}{p+1}\int_0^\infty |u|^{p+1} dx\right) \\
&= -4t\mathrm{Re}P\bar{Q} - 8t^2\mathrm{Re}P\bar{Q}' + \frac{4g(5-p)}{p+1} t \int_0^\infty |u|^{p+1} dx
\end{aligned} \tag{3.9}$$

(3.9) agrees with [6] for $p = 3$ and no P, Q terms. We integrate both sides of (3.9) in t variable to get

$$\frac{8gt^2}{p+1}\|u\|_{p+1}^{p+1} \le \|xh\|_2^2 + \frac{4g(5-p)}{p+1}\int_0^t \tau\|u\|_{p+1}^{p+1} d\tau + \int_0^t |F(P,Q,\tau)| d\tau \tag{3.10}$$

where $F = -4t\mathrm{Re}P\bar{Q} - 8t^2\mathrm{Re}P\bar{Q}'$. Evidently on $[0,T]$,

$$\int_0^t |F| d\tau \le \left(\int_0^t |P(\tau)|^2 d\tau\right)^{\frac{1}{2}} \left(4\left(\int_0^t \tau^2|Q|^2 d\tau\right)^{\frac{1}{2}} + 8\left(\int_0^t \tau^4|Q'|^2 d\tau\right)\right) \tag{3.11}$$

If we assume that $Q \in C^1(\mathbf{R}^+), tQ, t^2Q' \in L^2(\mathbf{R}^+)$, then $Q \in H^1(\mathbf{R}^+)$ and $Q \in L^{p+1}(\mathbf{R}^+)$. Also, if we assume that $xh \in L^2(\mathbf{R}^+)$, then by standard arguments in semigroup theory (e.g. [7]), for each solution u, one has $xu \in L^2(\mathbf{R}^+)$. Thus $xuu_x \in L^1(\mathbf{R}^+)$ and $\int_0^\infty xu\bar{u}_x dx$ makes sense. Direct calculations on (1.3)–(1.5) yield

$$\|u\|_2^2 \le \|h\|_2^2 + 2\int_0^t |P(\tau)Q(\tau)| d\tau \tag{3.12}$$

$$\|u_x\|^2 \le \|h'\|_2^2 + \frac{2g}{p+1}\|h\|_{p+1}^{p+1} + 2\int_0^t |P(\tau)Q'(\tau)| d\tau \tag{3.13}$$

and

$$\begin{aligned}\int_0^t |P(\tau)|^2 d\tau \leq& \int_0^\infty |u\bar{u}_x| dx + \left(\int_0^\infty |Q|^2 dt\right)^{\frac{1}{2}} \left(\int_0^\infty |Q'|^2 dt\right)^{\frac{1}{2}} \\ &+ \frac{2g}{p+1}\int_0^\infty |Q|^{p+1} dt + \int_0^\infty |h(x)\bar{h}'(x)| dx \\ \leq& \frac{1}{2}(\|u\|_2^2 + \|u_x\|^2) + \|Q\|_{H^1}^2 + \frac{2g}{p+1}\|Q\|_{p+1}^{p+1} + c' \\ \leq& \bar{c} + c_0\left(\left(\int_0^t |P(\tau)|^2 d\tau\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.14}$$

Since the constants in (3.14) do not depend on T, it follows that

$$\int_0^t |P(t)|^2 dt \leq c_1 \tag{3.15}$$

for all $t > 0$ uniformly. In particular, (3.15) holds when $t \to \infty$. By substituting (3.11) and (3.15) in (3.10) and using the assumption that $tQ, t^2Q' \in L^2(\mathbf{R}^+)$ we get

$$\begin{aligned}\frac{8gt^2}{p+1}\|u\|_{p+1}^{p+1} \leq& c_2 + \frac{4g(5-p)}{p+1}\int_0^t \tau\|u\|_{p+1}^{p+1} d\tau \\ =& c_2 + \frac{4g(5-p)}{p+1}\int_0^1 \tau\|u\|_{p+1}^{p+1} d\tau + \frac{4g(5-p)}{p+1}\int_1^t \tau\|u\|_{p+1}^{p+1} d\tau \end{aligned} \tag{3.16}$$

Clearly $\frac{4g(5-p)}{p+1}\int_0^1 \tau\|u\|_{p+1}^{p+1} d\tau$ is bounded due to global existence theorem. Therefore,

$$t^2\|u\|_{p+1}^{p+1} \leq c + \frac{5-p}{2}\int_1^t \tau\|u\|_{p+1}^{p+1} d\tau \tag{3.17}$$

and $(H(t) = t^2\|u\|_{p+1}^{p+1})$

$$H(t) \leq c + \int_1^t \frac{5-p}{2\tau} H(\tau) d\tau \tag{3.18}$$

Since $F(t)$ and $\frac{5-p}{2t}$ are continuous on $[1, \infty)$, Gronwall's lemma implies that

$$H(t) \leq c\exp\left(\int_1^t \frac{5-p}{\tau} d\tau\right) = ct^{\frac{5-p}{2}} \tag{3.19}$$

and

$$\|u\|_{p+1}^{p+1} \leq ct^{-(p-1)/2} \tag{3.20}$$

for $t > 1$. This gives the decay estimate for the solution.

Theorem 3 *Assume that $xh \in L^2(\mathbf{R}^+), tQ, t^2Q' \in L^2(\mathbf{R}^+)$ and u is the global solution to (1.2) with Dirichlet boundary condition. Then for $t \geq 1$ one has $\|u\|_{p+1}^{p+1} \leq ct^{-(p-1)/2}$ for some $c > 0$.*

Similar result is also available in the case of Robin boundary condition and we omit the proof here.

Theorem 4 *Assume that* $xh \in L^2(\mathbf{R}^+), tR, t^2R' \in L^2(\mathbf{R}^+)$ *and* u *is the global solution to* (1.2) *with Robin boundary condition. Then for* $t \geq 1$ *one has* $\|u\|_{p+1}^{p+1} \leq ct^{-(p-1)/2}$ *for some* $c > 0$.

Finally we address the issue of blow-up of solutions to the forced NLS.

Definition *A solution to the forced NLS* (1.2) *blows up at* T *if there exists* $t_n \to T$ *such that* $\left\|\frac{\partial u}{\partial x}(t_n)\right\|_2 \to \infty$ *as* $n \to \infty$.

For the cubic NLS with Dirichlet boundary condition, a conjecture for necessary and sufficient conditon for the solution to blow-up $\left(\int_0^T |Q'(t)|^2dt = \infty\right)$ was proposed in [8] and subsequently proved by B. Guo and Y. Wu [9]. It appears that similar conclusion holds for the forced NLS with higher order of nonlinearity and the same proof goes through for $p > 4$ by changing values of the constants in various estimates. Similar necessary and sufficient condition for blow-up of solution to the Robin boundary value problem for NLS (1.2) at T has not been examined yet. Here we state the theorem.

Theorem 5 *A necessary and sufficient conditon for the solution to the forced NLS* (1.2) *to blow up at* T *is* $\int_0^T |Q'(t)|^2dt = \infty$.

We conclude our discussion on the forced NLS by presenting a numerical result in the case of $g < 0$. We feel that there is a significant difference between the pure initial value problem and forced problem for the NLS. This is caused by the presence of the inhomogeneous boundary $u(0,t) = Q(t)$ or $u_x(0,t) + \alpha u(0,t) = R(t)$ which destroys Hamiltonian properties and conserved quantities. For the free Schrödinger equation in $\mathbf{R}^n$ with the "bad" sign for the nonlinear term, a global solution is available provided that $p < 1 + \frac{4}{n}$. However, our conjecture is that for the NLS in $\mathbf{R}^n$ with inhomogeneous boundary condition, a global existence probably requires that $p \leq 1 + \frac{4}{n+1}$. For $n = 1$, these numbers correspond to $p = 5$ (free NLS) and $p = 3$ (forced NLS), respectively. Therefore, we suspect that for the forced NLS when $p > 3, g < 0$, the solution does not exist for all times. We obtained some numerical results for the forced NLS when $p = 4, g = -5$ on a rectangular domain with a particular set of initial-boundary data. PDE2D, a general purpose finite element program which solves systems of nonlinear time-dependent, nonlinear steady-state and linear eigenvalue partial differential equations, is utilized in the calculation for the 1D situation. This software has been shown fairly effective in dealing with boundary value problems for nonlinear evolution equations [10]. Let $u = U + iV$, then the corresponding NLS equation in (1.2) becomes

$$\begin{cases} U_t = V_{xx} + 5V(U^2 + V^2)^{\frac{3}{2}} \\ V_t = -U_{xx} - 5U(U^2 + V^2)^{\frac{3}{2}} \end{cases} \tag{3.21}$$

for $0 \leq x \leq 20, t \geq 0$. The initial and boundary conditions are set as follows:

$$\begin{cases} u(x,0) = \dfrac{21}{20(1+x)} - \dfrac{1}{20} \\ u(0,t) = 1 + t^3, \quad u(20,t) = 0 \end{cases} \tag{3.22}$$

It appears that the solution with the above initial-boundary data grows explosively large while remains reasonably stable for $p = 3$. The numerical outputs for approximate values of the spacial derivatives for u near the boundary are given by the following table.

T=0.000000E+00	Derivative Estimate=−1
T=1.000000E+00	Derivative Estimate=−1999
T=2.000000E+00	Derivative Estimate=−8994
T=3.000000E+00	Derivative Estimate=−27484
T=4.000000E+00	Derivative Estimate=−48641
T=5.000000E+00	Derivative Estimate=−87273117
T=6.000000E+00	Derivative Estimate=−93815767
T=7.000000E+00	Derivative Estimate=−102884920
T=8.000000E+00	Derivative Estimate=−114856860
T=9.000000E+00	Derivative Estimate=−130093570
T=1.000000E+01	Derivative Estimate=−489468100

Table 1. Numerical Outputs of Re $\dfrac{\partial u(0,t)}{\partial x}$ for $p = 4, g = -5$

References

[1] Sheerin J. et al., *J. Atmos. Terr. Phys.*, **44** (1982), 1043–1046.

[2] Zakharov V. and Shabat A., Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Soviet Phys.-JETP,* **34**(1972), 62–69.

[3] Bu C., Nonlinear Schrödinger equation on the semi-infinite line, *Chinese Ann. Math.*, to appear.

[4] Pazy A., Semigroups of Linear Operators and Applications to PDE, Springer, New York, 1983.

[5] Bu C., Forced cubic Schrödinger equation with Robin boundary data: continuous dependency result, *J. Austral. Math. Soc., Series* **B**, **41**(2000), 301–311.

[6] Barab J., Nonexistence of asmptotically free solutions for a nonlinear Schrödinger equation, *J. Math. Phys.*, **25**(1984), 3270–3273.

[7] Segal I., Nonlinear semigroups, *Ann. Math.*, **78** (1963), 339–364.

[8] Bu C., On well-posedness of the forced nonlinear Schrödinger equation, *Appl. Anal.*, **46** (1992), 219–239.

[9] Guo B. and Wu Y., Global existence and nonexistence of a forced nonlinear Schrödinger equation, *J. Math. Phys.*, **36** (1995), 3479–3484.

[10] Sewell G., PDE2D: Easy -to-use software for general two-dimensional partial differential equations, *Advances in Eng. Software,* **17** (1993), 105–112.

[11] Carroll R. and Bu C., Solutions of the forced nonlinear Schrödinger equation (NLS) using PDE techniques, *Appl. Anal.*, **41** (1991), 33–51.