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INITIAL VALUE PROBLEM FOR A NONLINEAR EVOLUTION SYSTEM WITH SINGULAR INTEGRAL DIFFERENTIAL TERMS

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Abstract The initial value problem for a nonlinear evolution system with singular integral differential terms is studied. By means of *a priori* estimates of the solutions and Leray-Schauder's fixed point theorem, we demonstrate the existence and uniqueness theorems of the generalized and classical global solutions to the problem.

Key Words Initial value problem; integral estimate; nonlinear evolution system; singular integral differential term.

Classification 35Q20.

1. Introduction

In this paper, we study the initial value problem (IVP) for the following nonlinear evolution system with singular integral differential terms (NES with SIDT) [5-7]

$$U_t + U_{x^{2p+1}} + [\text{grad}\ \Phi(U)]_x + \alpha H U_{x^{2r}} + (-1)^s \beta H U_{x^{2s-1}} + \gamma H U = A(x,t)U + g(x,t) \tag{1}$$

$$U(x,0) = U_0(x) \tag{2}$$

in the unbounded domain $Q_T = \{(x,t) : -\infty < x < \infty,\ 0 \le t \le T\}$, where H is the Hilbert singular integral operator

$$HU(x,t) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{U(y,t)}{y-x} dy \tag{3}$$

In system (1), $U(x,t) = (U_1(x,t), \cdots, U_N(x,t))$ is a N-dimensional vector valued unknown function of the two real variables $-\infty < x < \infty$ and $t \ge 0$; $\Phi(U)$ is a scalar function of the vector variable U; "grad" denotes the gradient operator with respect to the vector variable U; $A(x,t)$ is a $N \times N$ matrix of functions $a_{i,j}(x,t)(1 \le i,j \le N)$, $g(x,t)$ is a N-dimensional vector valued function of functions $g_i(x,t)(1 \le i \le N)$; α, β, and γ are real constants; $p \ge 1$, $1 \le r, s \le p$ are integers.

System (1) is a much generalized NES. In fact, if $\alpha = \beta = \gamma = 0$, then it is the generalized Korteweg-de Vries (KdV) system of higher order [1]

$$U_t + U_{x^{2p+1}} + [\text{grad}\ \Phi(U)]_x = A(x,t)U + g(x,t) \tag{4}$$

In order to study the IVP (2) for the NES with SIDT (1), we need to investigate the IVP (2) for the corresponding NES with dissipative term

$$U_t + (-1)^{p+1}\varepsilon U_{x^{2p+2}} + U_{x^{2p+1}} + [\text{grad }\Phi(U)]_x + \alpha H U_{x^{2r}}$$
$$+(-1)^s \beta H U_{x^{2s-1}} + \gamma H U = A(x,t)U + g(x,t) \tag{5}$$

where $0 < \varepsilon < 1$. The solution of problem (1,2) will be obtained by the limiting procedure of approaching to zero of the dissipative coefficient ε for the solution of problem (5, 2).

If $f(x) \in L^1(\boldsymbol{R}) \cap L^2(\boldsymbol{R})$, define its Fourier transform as follows:

$$F[f](\zeta) \equiv \hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)\exp(-ix\zeta)dx \tag{6}$$

Let the $L^p(1 \le p \le \infty)$ norm on $\boldsymbol{R}$ be denoted by $\|U\|_{L^p}$, and define the Sobolev spaces H^m and H_0^m by means of the norms

$$\|U\|_{H^m} = \Big[\frac{1}{2\pi}\int_{-\infty}^{\infty}(1+|\zeta|^2)^m|\hat{U}(\zeta)|^2 d\zeta\Big]^{1/2} \tag{7}$$

$$\|U\|_{H_0^m} = \Big[\frac{1}{2\pi}\int_{-\infty}^{\infty}|\zeta|^{2m}|\hat{U}(\zeta)|^2 d\zeta\Big]^{1/2} \tag{8}$$

where $m \ge 0$.

For simplicity, we will denote by C any positive constant appeared in our paper, which depends only on the coefficients α, β, and γ, the norms of the initial function $U_0(x)$, and the norms of $A(x,t)$ and $g(x,t)$. Furthermore, we regard

$$\|U(t)\| = \|U(t)\|_{L^2(\boldsymbol{R})}, \ \|U(t)\|_\infty = \|U(t)\|_{L^\infty(\boldsymbol{R})}$$
$$\|U(t)\|_m = \|U(t)\|_{H^m(\boldsymbol{R})}, \ |U(t)|_m = \|U(t)\|_{H_0^m(\boldsymbol{R})}$$

Now let us introduce some functional spaces.

$$B = L^\infty(0,T;H^1(\boldsymbol{R})),\ \overline{B} = \{U = (U_1,\cdots,U_N) \in \boldsymbol{R}^N : U_i \in B,\ 1 \le i \le N\}$$
$$Z = L^\infty(0,T;H^{p+1}(\boldsymbol{R})) \cap L^2(0,T;H^{2p+2}(\boldsymbol{R})) \cap H^1(0,T;L^2(\boldsymbol{R}))$$
$$\overline{Z} = \{U = (U_1,\cdots,U_N) \in \boldsymbol{R}^N : U_i \in Z,\ 1 \le i \le N\}$$

If $U(x,t) \in B$ (or Z), define its norm as follows:

$$\|U\|_B^2 = \sup_{0\le t\le T}\|U(t)\|_1^2$$

$$\|U\|_Z^2 = \sup_{0\le t\le T}\|U(t)\|_{p+1}^2 + \int_0^T \|U(t)\|_{2p+2}^2 dt + \int_0^T \|U_t(t)\|^2 dt$$

2. Preliminary Lemmas

Lemma 1 *For any $U(x),\ V(x) \in L^2(R)$, the following relations hold:*

$$(1)\quad \widehat{HU}(\zeta) = \text{sign}\,(\zeta)\hat{U}(\zeta),\ H^2U = -U,$$

$$H(UV) = H(HUHV) + UHV + VHU;$$

$$(2)\quad \int_{-\infty}^{\infty} U(x)HV(x)dx = -\int_{-\infty}^{\infty} V(x)HU(x)dx, \quad \int_{-\infty}^{\infty} U(x)HU(x)dx = 0;$$

$$\int_{-\infty}^{\infty} HU(x)HV(x)dx = \int_{-\infty}^{\infty} U(x)V(x)dx,$$

$$\int_{-\infty}^{\infty} |HU(x)|^2 dx = \int_{-\infty}^{\infty} |U(x)|^2 dx;$$

(3) $(HU)_x = H(U_x)$, *for any* $U(x) \in H^1(R)$. see [2].

Lemma 2 *For any $U(x) \in H^2(R)$, we have the following inequalities:*

$$(1)\quad \int_{-\infty}^{\infty} U_x HU dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\zeta||\hat{U}(\zeta)|^2 d\zeta \geq 0;$$

$$(2)\quad \int_{-\infty}^{\infty} |U_x|^2 dx \leq \Big[\int_{-\infty}^{\infty} U_x HU dx\Big]^{\frac{1}{2}}\Big[\int_{-\infty}^{\infty} U_{xx} HU_x dx\Big]^{\frac{1}{2}};$$

$$(3)\quad \int_{-\infty}^{\infty} |U_{x^k}|^2 dx \leq \Big[\int_{-\infty}^{\infty} |U|^2 dx\Big]^{\frac{1}{2k+1}}\Big[\int_{-\infty}^{\infty} U_{x^{k+1}} HU_{x^k} dx\Big]^{\frac{2k}{2k+1}},$$

where $k \geq 1$ is a integer.

Proof By means of the Parseval's equality we get

$$(1)\quad \int_{-\infty}^{\infty} U_x HU dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{i\zeta\hat{U}(\zeta)}\ \text{sign}\,(\zeta)\hat{U}(\zeta)d\zeta = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\zeta||\hat{U}(\zeta)|^2 d\zeta;$$

$$(2)\quad \int_{-\infty}^{\infty} |U_x|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\zeta|^2|\hat{U}(\zeta)|^2 d\zeta$$

$$\leq \frac{1}{2\pi}\Big[\int_{-\infty}^{\infty} |\zeta||\hat{U}(\zeta)|^2 d\zeta\Big]^{\frac{1}{2}}\Big[\int_{-\infty}^{\infty} |\zeta|^3|\hat{U}(\zeta)|^2 d\zeta\Big]^{\frac{1}{2}}$$

$$= \Big[\int_{-\infty}^{\infty} U_x HU dx\Big]^{\frac{1}{2}}\Big[\int_{-\infty}^{\infty} U_{xx} HU_x dx\Big]^{\frac{1}{2}};$$

$$(3)\quad \int_{-\infty}^{\infty} |U_{x^k}|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\zeta|^{2k}|\hat{U}(\zeta)|^2 d\zeta$$

$$\leq \frac{1}{2\pi}\Big[\int_{-\infty}^{\infty} |\hat{U}(\zeta)|^2 d\zeta\Big]^{\frac{1}{2k+1}}\Big[\int_{-\infty}^{\infty} |\zeta|^{2k+1}|\hat{U}(\zeta)|^2 d\zeta\Big]^{\frac{2k}{2k+1}}$$

$$= \Big[\int_{-\infty}^{\infty} |U|^2 dx\Big]^{\frac{1}{2k+1}}\Big[\int_{-\infty}^{\infty} U_{x^{k+1}} HU_{x^k} dx\Big]^{\frac{2k}{2k+1}}.$$

In the light of the notation listed above, the following formula will appear in our later discussions:

$$\int_{-\infty}^{\infty}(U_{x^{k+1}},\ HU_{x^k})dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\zeta|^{2k+1}|\hat{U}(\zeta)|^2d\zeta=|U|^2_{k+1/2} \tag{9}$$

where $U(x)\in H^{k+1}(\boldsymbol{R})$ is a vector valued function, $k\geq 0$ is an integer.

Lemma 3 (Gronwall's inequality)

Suppose that the functions $g(t)$, $h(t)\geq 0$ satisfy the inequality

$$g(t)\leq C\int_0^t g(s)h(s)ds \quad \text{for any } 0\leq t<\infty$$

where $C>0$ is a constant, $h(t)$ satisfies $\int_0^\infty h(t)dt<\infty$. Then we obtain the estimate

$$g(t)\leq C\exp\Big[\int_0^t h(s)ds\Big] \quad \text{for any } 0\leq t<\infty$$

Lemma 4 *Suppose that $\varepsilon>0$, $f(x,t)\in L^2(Q_T)$, $U_0(x)\in H^{p+1}(\boldsymbol{R})$, $p\geq 1$. Then the IVP (2) for the linear parabolic equation*

$$U_t+(-1)^{p+1}\varepsilon U_{x^{2p+2}}+U_{x^{2p+1}}=f(x,t)$$

has a unique global solution $U(x,t)\in Z$. See [1].

Lemma 5 *Suppose that $\varepsilon>0$, $b(x,t)\in L^\infty\ (Q_T)$, $f(x,t)\in L^2(Q_T)$, $U_0(x)\in H^{p+1}(\boldsymbol{R})$, $p\geq 1$. Then the* IVP (2) *for the linear parabolic equation with* SIDT

$$\begin{aligned}U_t+&(-1)^{p+1}\varepsilon U_{x^{2p+2}}+U_{x^{2p+1}}+\alpha HU_{x^{2r}}\\&+(-1)^s\beta HU_{x^{2s-1}}+\gamma HU+b(x,t)U_x=f(x,t)\end{aligned}$$

has a unique generalized global solution $U(x,t)\in Z$.

Proof With the technique of continuation of parameter, we can prove the lemma.

3. Generalized Global Solution of Problem (5, 2)

In the present section, we are going to establish the existence and uniqueness theorems of the IVP (2) for the NES with SIDT (5).

Theorem 1 *Suppose that $\alpha,\beta\geq 0$, γ, and $0<\varepsilon<1$ are real constants; $p\geq 1$; $1\leq r,s\leq p$ are integers; $U_0(x)\in H^{p+1}(\boldsymbol{R})$, $\Phi(U)\in C^3(\boldsymbol{R}^N)$, and there exists a constant $C>0$, such that*

$$\begin{aligned}&(-1)^{p-1}\Phi(U)\leq C|U|^2+C|U|^{4p+2-\delta},\ |\text{grad }\Phi(U)|\leq C|U|^{4p+1}+C|U|\\&\Big|\frac{\partial^2\Phi(U)}{\partial U_i\partial U_j}\Big|\leq \quad C|U|^{4p}+C, \quad \text{where } 0<\delta\leq 4p,\ U\in \boldsymbol{R}^N\end{aligned} \tag{10}$$

Then the IVP (2) *for the* NES (5) *has a unique generalized global solution* $U(x,t) \in \overline{Z}$.

Proof For any $V(x,t) = (V_1(x,t), \cdots, V_N(x,t)) \in \overline{B}$, consider the IVP (2) for the linear parabolic system with SIDT

$$U_t + (-1)^{p+1}\varepsilon U_{x^{2p+2}} + U_{x^{2p+1}} + \alpha H U_{x^{2r}} + (-1)^s \beta H U_{x^{2s-1}} + \gamma H U + F(V) U_x = \lambda A(x,t) U + \lambda g(x,t) \tag{11}$$

$$U(x,0) = \lambda U_0(x) \tag{12}$$

where $F(U) = \left(\dfrac{\partial^2 \Phi(U)}{\partial U_i \partial U_j}\right)_{1 \le i,j \le N}$ is the Hessian matrix of the nonlinear function $\Phi(U)$.

Since $\Phi(U) \in C^3(R^N)$, $V(x,t) \in \overline{B}$, from Lemma 5 we see that problem (11, 12) has a unique global solution $U(x,t) \in \overline{Z} \subset \overline{B}$. Therefore we can define a map $A_\lambda : \overline{B} \times [0,1] \to \overline{B}$: for any $V(x,t) \in \overline{B}$, and $0 \le \lambda \le 1$, let $U_\lambda(x,t) = A_\lambda V(x,t)$ be the unique generalized global solution of the IVP (12) for system (11). Since the imbedding map $\overline{Z} \hookrightarrow \overline{B}$ is completely continuous, the operator A_λ is completely continuous for every $0 \le \lambda \le 1$. It can be easily seen that for any bounded subset $E \subset \overline{B}$, the continuity of A_λ with respect to $0 \le \lambda \le 1$ is uniform. Furthermore, it is obvious that $A_0 B = \{0\}$.

In order to obtain the existence of the generalized global solutions of problem (11, 12), it is sufficient to verify the uniform boundedness of all possible fixed points of the map A_λ with respect to the parameter $0 \le \lambda \le 1$, namely, it needs to establish some *a priori* estimates of the solutions of the IVP (12) for the system

$$U_t + (-1)^{p+1}\varepsilon U_{x^{2p+2}} + U_{x^{2p+1}} + [\text{grad } \Phi(U)]_x + \alpha H U_{x^{2r}} + (-1)^s \beta H U_{x^{2s-1}} + \gamma H U = \lambda A(x,t) U + \lambda g(x,t) \tag{13}$$

with respect to the parameter $0 \le \lambda \le 1$.

Taking the scalar product of system (13) and the vector U, integrating the product in the unbounded domain $Q_t (0 \le t \le T)$ with respect to (x,s), regarding Lemmas 1 and 2, we get the integral inequality

$$\|U(t)\|^2 + 2\varepsilon \int_0^t \|U_{x^{p+1}}(s)\|^2 ds + 2\beta \int_0^t |U(s)|^2_{s-1/2} ds \le \|U_0\|^2 + C \int_0^t \|U(s)\|^2 ds + C \int_0^t \|g(s)\|^2 ds$$

By means of Lemma 3, we get the integral estimates

$$\sup_{0 \le t \le T} \|U(t)\|^2,\ 2\varepsilon \int_0^T \|U_{x^{p+1}}(t)\|^2 dt,\ 2\beta \int_0^T |U(t)|^2_{s-1/2} dt \le C\|U_0\|^2 + C \int_0^T \|g(t)\|^2 dt \tag{14}$$

where C is a constant independent of $0 \le \lambda \le 1$ and ε.

Making the scalar product of system (13) and $U_{x^{2p}} + \alpha H U_{x^{2r-1}} + \text{grad}\ \Phi(U)$ and integrating in Q_t with respect to (x,s), after proper integration by parts, we get

$$
\begin{aligned}
&\|U_{x^p}(t)\|^2 + 2\varepsilon \int_0^t \|U_{x^{2p+1}}(s)\|^2 ds + 2\beta \int_0^t |U(s)|^2_{p+s-1/2} ds \\
&\quad + 2\varepsilon \int_0^t \int_{-\infty}^{\infty} (U_{x^{2p+1}}, [\text{grad}\ \Phi(U)]_x) dx ds + 2\alpha\varepsilon \int_0^t \int_{-\infty}^{\infty} (H U_{x^{2r}}, U_{x^{2p+1}}) dx ds \\
&\quad + 2(-1)^{p-s-1}\beta \int_0^t \int_{-\infty}^{\infty} (H U_{x^{2s-2}}, [\text{grad}\ \Phi(U)]_x) dx ds \\
&\quad + 2(-1)^{p-r}\alpha\beta \int_0^t \|U_{x^{r+s-1}}(s)\|^2 ds \\
&= \lambda^2 \|U_{0x^p}\|^2 + 2(-1)^p \Big[\int_{-\infty}^{\infty} \Phi(\lambda U_0) dx - \int_{-\infty}^{\infty} \Phi(U) dx \Big] \\
&\quad + (-1)^{p-r}\alpha [\lambda^2 |U_0|^2_{r-1/2} - |U(t)|^2_{r-1/2}] \\
&\quad - 2(-1)^p \gamma \int_0^t \int_{-\infty}^{\infty} (HU, \text{grad}\ \Phi(U)) dx ds + 2\lambda \int_0^t \int_{-\infty}^{\infty} (U_{x^p}, (AU+g)_{x^p}) dx ds \\
&\quad + 2(-1)^p \lambda \int_0^t \int_{-\infty}^{\infty} (AU+g, \text{grad}\ \Phi(U)) dx ds \\
&\quad + 2(-1)^{p-r} \lambda\alpha \int_0^t \int_{-\infty}^{\infty} ((AU+g)_{x^r}, H U_{x^{r-1}}) dx ds
\end{aligned} \tag{15}
$$

We rewrite the last identity as a simple form:

$$L_1 + \cdots + L_7 = R_1 + \cdots + R_9$$

where $L_i(R_i)$ is the i-th term of the left (right) hand side of the last identity.

The following simplifications hold.

$$|L_4| \le \varepsilon \int_0^t \|U_{x^{2p+1}}(s)\|^2 ds + \varepsilon \int_0^t \int_{-\infty}^{\infty} |[\text{grad}\ \Phi(U)]_x|^2 dx ds$$

By using (10), we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} |[\text{grad}\ \Phi(U)]_x|^2 dx &\le C\|U_x(t)\|^2 + C\|U(t)\|^{8p}_{\infty} \|U_x(t)\|^2 \\
&\le C\|U_{x^{p+1}}(t)\|^{\frac{2}{p+1}} \|U(t)\|^{\frac{2p}{p+1}} + C\|U(t)\|^{8p-2} \|U_{x^p}(t)\|^2 \|U_{x^{p+1}}(t)\|^2 \\
&\le C\|U(t)\|^2 + C\|U_{x^{p+1}}(t)\|^2 + C\|U_{x^p}(t)\|^2 \|U_{x^{p+1}}(t)\|^2
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
|L_4| &\le \varepsilon \int_0^t \|U_{x^{2p+1}}(s)\|^2 ds + C\varepsilon \int_0^t \|U_{x^{p+1}}(s)\|^2 ds \\
&\quad + C \int_0^t \varepsilon \|U_{x^p}(s)\|^2 \|U_{x^{p+1}}(s)\|^2 ds + CT\varepsilon
\end{aligned}
$$

Since

$$\left|\int_{-\infty}^{\infty}(HU_{x^{2r}}, U_{x^{2p+1}})dx\right| \leq \|U_{x^{2r}}(t)\|\|U_{x^{2p+1}}(t)\|$$

$$\leq \|U_{x^{2p+1}}(t)\|^{\frac{2p+2r+1}{2p+1}}\|U(t)\|^{\frac{2p-2r+1}{2p+1}} \leq \frac{1}{2\alpha}\|U_{x^{2p+1}}(t)\|^2 + C\|U(t)\|^2$$

Thus we have

$$|L_5| \leq \varepsilon\int_0^t \|U_{x^{2p+1}}(s)\|^2 ds + CT\varepsilon$$

As for L_6, regarding Lemmas 1 and 2, we have

$$|L_6| \leq 2\beta\int_0^t\int_{-\infty}^{\infty}|HU_{x^{2s-2}}\|[\text{grad }\Phi(U)]_x|dxds$$

$$\leq 2\beta\int_0^t \|U_{x^{2s-2}}(s)\|\|[\text{grad }\Phi(U(s))]_x\|ds$$

$$\leq C\beta\int_0^t \|U_{x^{2s-2}}(s)\|\|U(s)\|_{\infty}^{4p}\|U_x(s)\|ds + C\beta\int_0^t \|U_{x^{2s-2}}(s)\|\|U_x(s)\|ds$$

We will employ the interpolation inequality presented in the Appendix to estimate L_6.

If $s=1$, then we have

$$|L_6| \leq C\beta\int_0^t \|U(s)\|\|U(s)\|_{\infty}^{4p}\|U_x(s)\|ds + C\beta\int_0^t \|U(s)\|\|U_x(s)\|ds$$

$$\leq C\beta\int_0^t \|U(s)\|[\|U(s)\|^{2p-1}\|U_{x^p}(s)\|][\|U(s)\|^{2p-3+1/p}\|U_{x^p}(s)\|^{1-1/p}]$$

$$\times[\|U(s)\|\|U_x(s)\|^2]ds + C\beta\int_0^t \|U(s)\||U(s)|_{1/2}^{1/2}|U(s)|_{3/2}^{1/2}ds$$

$$\leq C\beta\int_0^t \|U(s)\|^{4p-1}\|U_{x^p}(s)\||U(s)|_{1/2}|U(s)|_{p+1/2}ds$$

$$+C\beta\int_0^t \|U(s)\|^{3p/(2p+1)}|U(s)|_{1/2}^{1/2}|U(s)|_{p+1/2}^{3/(4p+2)}ds$$

$$\leq \beta\int_0^t |U(s)|_{p+1/2}^2 ds + C\beta\int_0^t \|U_{x^p}(s)\|^2|U(s)|_{1/2}^2 ds$$

$$+C\beta\int_0^t |U(s)|_{1/2}^2 ds + C\beta T$$

If $2 \leq s \leq p$, then we get

$$|L_6| \leq C\beta\int_0^t \|U_{x^{2s-2}}(s)\|\|U(s)\|_{\infty}^{4p}\|U_x(s)\|ds + C\beta\int_0^t \|U_{x^{2s-2}}(s)\|\|U_x(s)\|ds$$

$$\leq C\beta\int_0^t [\|U_{x^{p+s-1}}(s)\|^{\frac{2s-2}{p+s-1}}\|U(s)\|^{\frac{p-s+1}{p+s-1}}][\|U_{x^p}(s)\|\|U(s)\|^{2p-1}]$$

$$\times[\|U_{x^{s-1}}(s)\|^{\frac{2s-1}{2s-2}}\|U(s)\|^{\frac{(2s-1)(2s-3)}{2s-2}}][\|U_{x^{p+s-1}}(s)\|^{\frac{2p-2s+1}{2(p+s-1)}}$$

$$\times\|U(s)\|^{\frac{(2p+2s-3)(2p-2s+1)}{2(p+s-1)}}][\|U_{x^{p+s-1}}(s)\|^{\frac{1}{p+s-1}}\|U(s)\|^{\frac{p+s-2}{p+s-1}}]ds$$

$$+C\beta\int_0^t |U(s)|_{s-1/2}^{(2p-2s+3)/(2p)+2/(2s-1)}|U(s)|_{p+s-1/2}^{(2s-3)/(2p)}\|U(s)\|^{(2s-3)/(2s-1)}ds$$

$$\leq C\beta\int_0^t \|U(s)\|^{4p-1}\|U_{x^p}(s)\||U(s)|_{s-1/2}|U(s)|_{p+s-1/2}ds$$

$$+C\beta\int_0^t |U(s)|_{s-1/2}^{(2p-2s+3)/(2p)+2/(2s-1)}|U(s)|_{p+s-1/2}^{(2s-3)/(2p)}\|U(s)\|^{(2s-3)/(2s-1)}ds$$

$$\leq \beta\int_0^t |U(s)|_{p+s-1/2}^2 ds + C\beta\int_0^t \|U_{x^p}(s)\|^2|U(s)|_{s-1/2}^2 ds$$

$$+C\beta\int_0^t |U(s)|_{s-1/2}^2 ds + C\beta T$$

Moreover, we have

$$\|U_{x^{r+s-1}}(t)\|^2 \leq C\|U_{x^{p+s-1}}(t)\|^{\frac{2r+2s-2}{p+s-1}}\|U(t)\|^{\frac{2p-2r}{p+s-1}}$$

$$\leq C|U(t)|_{p+s-1/2}^{\frac{4r+4s-4}{2p+2s-1}}\|U(t)\|^{\frac{4p-4r+2}{2p+2s-1}} \leq C\|U(t)\|^2 + \frac{1}{4\alpha}|U(t)|_{p+s-1/2}^2$$

$$|L_7| \leq \frac{\beta}{2}\int_0^t |U(s)|_{p+s-1/2}^2 ds + CT$$

$$|R_3| \leq C\int_{-\infty}^{\infty}|U|^{4p+2-\delta}dx \leq \frac{1}{4}\|U_{x^p}(t)\|^2 + C$$

$$|R_5| \leq \frac{1}{4}\|U_{x^p}(t)\|^2 + C$$

We deal with R_6 in this way

$$\Big|\int_{-\infty}^{\infty}(HU, \text{grad }\Phi(U))dx\Big| \leq C\|U(t)\|^2 + C\|U_{x^p}(t)\|^2$$

$$|R_6| \leq C\int_0^t \|U_{x^p}(s)\|^2 ds + CT$$

$$|R_7| \leq C\int_0^t \|U_{x^p}(s)\|^2 ds + C\int_0^t \|g_{x^p}(s)\|^2 ds + CT$$

$$|R_8| \leq C\int_0^t \|U_{x^p}(s)\|^2 ds + C\int_0^t \|U_{x^p}(s)\|^2\|g(s)\|^2 ds + C\int_0^t \|g(s)\|^2 ds + CT$$

$$|R_9| \leq C\int_0^t \|U_{x^p}(s)\|^2 ds + C\int_0^t \|g(s)\|^2 ds + C\int_0^t \|g_{x^p}(s)\|^2 ds + CT$$

Identity (15) now yields the following inequality

$$\frac{1}{2}\|U_{x^p}(t)\|^2+\beta\int_0^t|U(s)|^2_{p+s-1/2}ds$$
$$\leq C+C\int_0^t\|g(s)\|_p^2ds+C\int_0^t\varepsilon\|U_{x^{p+1}}(s)\|^2ds+C\beta\int_0^t|U(s)|^2_{s-1/2}ds$$
$$+C\int_0^t\|U_{x^p}(s)\|^2[1+\varepsilon\|U_{x^{p+1}}(s)\|^2+\beta|U(s)|^2_{s-1/2}+\|g(s)\|^2]ds$$

By using Gronwall's Lemma, we get

$$\sup_{0\leq t\leq T}\|U(t)\|_p^2\leq C \tag{16}$$

where C is a constant independent of $0\leq\lambda\leq 1$ and ε.

Estimates (14, 16) show that all possible solutions of problem (13, 12) are uniformly bounded in the space $\overline{B}$ with respect to the parameter $0\leq\lambda\leq 1$. Thus by using Leray-Schauder's fixed point principle, we obtain the existence of the generalized global solutions of problem (5, 2).

Suppose that $U(x,t)$ and $V(x,t)\in Z$ are the solutions of problem (5, 2). Then $W(x,t)=U(x,t)-V(x,t)$ satisfies the linear parabolic equation with SIDT

$$W_t+(-1)^{p+1}\varepsilon W_{x^{2p+2}}+W_{x^{2p+1}}+\alpha HW_{x^{2r}}+(-1)^s\beta HW_{x^{2s-1}}+\gamma HW+F(U)W_x$$
$$+\Big(\sum_{k=1}^N\int_0^1\frac{\partial^3\Phi(\tau U+(1-\tau)V)}{\partial U_i\partial U_j\partial U_k}d\tau W_k\Big)_{1\leq k\leq N}V_x=0 \tag{17}$$

and the homogeneous initial condition

$$W(x,0)=0 \tag{18}$$

Making the scalar product of the LES with SIDT (17) and W and integrating the product in the domain $Q_t(t>0)$ with respect to (x,t), we get

$$\|W(t)\|^2+2\varepsilon\int_0^t\|W_{x^{p+1}}(s)\|^2ds+2\beta\int_0^t|W(s)|^2_{s-1/2}ds$$
$$=\int_0^t\int_{-\infty}^{\infty}(F(U)_xW,W)dxds$$
$$-\int_0^t\int_{-\infty}^{\infty}\Big(\Big(\sum_{k=1}^N\int_0^1\frac{\partial^3\Phi(\tau U+(1-\tau)V)}{\partial U_i\partial U_j\partial U_k}d\tau W_k\Big)_{1\leq k\leq N}V_x,W\Big)dxds$$

Because $U(x,t)$ and $V(x,t)\in Z$, the right hand side of the last formula can be dominated by $\int_0^t\|W(s)\|^2ds$, and the left hand side of the equality can be treated as before. By means of Gronwall's inequality, we see that problem (17, 18) has one solution $W(x,t)=0$.

4. IVP (2) for NES with SIDT (1)

Lemma 6. *The following estimates hold for problem* (5,2):

$$\sup_{0\le t\le T}\|U(t)\|_p\le C,\ \sup_{0\le t\le T}\|U(t)\|_\infty\le C$$

where C is some constant independent of ε.

Proof By means of Lemma 2 and the estimates obtained in Theorem 1, we know that Lemma 6 is correct.

Lemma 7 *Suppose that the conditions in Theorem 1 hold. Suppose also that $\Phi(U)\in C^4(\boldsymbol{R}^N)$, $a_{i,j}(x,t)\in L^\infty(0,T;W^{2,\infty}(\boldsymbol{R}))$, $g_i(x,t)\in L^2(0,T;H^2(\boldsymbol{R}))$. Then, if $p=1$, we have the estimate*

$$\sup_{0\le t\le T}\|U_{xx}(t)\|\le C \tag{19}$$

where constant C does not depend on $0<\varepsilon<1$.

Proof By using system (5) and some calculations, we obtain the identities

$$\frac{1}{2}(U_{xx},U_{xx})_t+\varepsilon(U_{xxxx},U_{xxxx})$$
$$+\Big[\varepsilon(U_{xxxxx},U_{xx})-\varepsilon(U_{xxxx},U_{xxx})+(U_{xxxx},U_{xx})-\frac{1}{2}(U_{xxx},U_{xxx})\Big]_x$$
$$=-([\text{grad }\Phi(U)]_{xxx},U_{xx})-([\alpha HU_{xx}-\beta HU_x+\gamma HU-AU-g]_{xx},U_{xx}) \tag{I}$$

$$-([\text{grad }\Phi(U)]_x,U_x)_t=2([\text{grad }\Phi(U)]_x,\varepsilon U_{xxxxx}+U_{xxxx}+[\text{grad }\Phi(U)]_{xx}$$
$$+[\alpha HU_{xx}-\beta HU_x+\gamma HU-AU-g]_x)+(\varepsilon U_{xxxx}+U_{xxx}$$
$$+[\text{grad }\Phi(U)]_x+\alpha HU_{xx}-\beta HU_x+\gamma HU-AU-g,F(U)_xU_x) \tag{II}$$

$$(U_{xx},F(U)U_{xx})_x=2(U_{xxx},F(U)U_{xx})+(U_{xx},F(U)_xU_x) \tag{III}$$

$$([\text{grad }\Phi(U)]_x,U_{xxx})_x=([\text{grad }\Phi(U)]_x,U_{xxxx})+F(U)U_{xx},U_{xxx})$$
$$+(F(U)_xU_x,U_{xxx}) \tag{IV}$$

$$(U_{xx},F(U)_xU_x)_x=(U_{xxx},F(U)_xU_x)+(U_{xx},F(U)_xU_{xx})+(U_{xx},F(U)_{xx}U_x) \tag{V}$$

In order to eliminate the nonlinear terms

$$([\text{grad }\Phi(U)]_x,U_{xxxx}),(F(U)_xU_x,U_{xxx})$$

Make the linear combination of (I–V): 6 (I) +5 (II) +8 (III) −10 (IV)+5 (V), we get

$$3(U_{xx},U_{xx})_t-5([\text{grad }\Phi(U)]_x,U_x)_t+6\varepsilon(U_{xxxx},U_{xxxx})$$
$$+[6\varepsilon(U_{xxxxx},U_{xx})-6\varepsilon(U_{xxxx},U_{xxx})+6(U_{xxxx},U_{xx})-3(U_{xxx},U_{xxx})$$

$$+8(U_{xx}, F(U)U_{xx}) - 10([\text{grad }\Phi(U)]_x, U_{xxx}) + 5(U_{xx}, F(U)_x U_x)]_x$$

$$= -6(\alpha H U_{xx} - \beta H U_x + \gamma H U - AU - g)_{xx}, U_{xx})$$

$$+10([\text{grad }\Phi(U)]_x, \varepsilon U_{xxxxx} + [\text{grad }\Phi(U)]_{xx} + [\alpha H U_{xx} - \beta H U_x + \gamma H U$$

$$-AU - g]_x) + 5(\varepsilon U_{xxxx} + [\text{grad }\Phi(U)]_x + \alpha H U_{xx} - \beta H U_x + \gamma H U$$

$$-AU - g, F(U)_x U_x) + (U_{xx}, F(u)_x U_{xx} - F(U)_{xx} U_x) \tag{20}$$

After integrating the above identity in the domain $Q_t(0 \le t \le T)$ with respect to (x, s), we have the following estimates.

$$5\Big|\int_{-\infty}^{\infty}([\text{grad }\Phi(U)]_x, U_x)dx\Big| \le C$$

$$6\Big|\int_0^t\int_{-\infty}^{\infty}([\alpha H U_{xx} - \beta H U_x + \gamma H U - AU - g]_{xx}, U_{xx})dxds\Big|$$

$$\le -6\beta\int_0^t |U(s)|^2_{s+1+3/2} + C\int_0^t \|U_{xx}(s)\|^2 ds + C\int_0^t \|U_{xx}(s)\|^2 ds + CT$$

$$10\Big|\int_0^t\int_{-\infty}^{\infty}([\text{grad }\Phi(U)]_x, \varepsilon U_{xxxxx} + [\text{grad }\Phi(U)]_x$$

$$+[\alpha H U_{xx} - \beta H U_x + \gamma H U - AU - g]_x)dxds\Big|$$

$$\le \varepsilon\int_0^t \|U_{xxxx}(s)\|^2 + C\int_0^t \|U_{xx}(s)\|^2 ds + 2\beta\int_0^t |U(s)|^2_{s+1+1/2}ds$$

$$+C\int_0^t \|U(s)\|^2 + CT$$

$$5\Big|\int_0^t\int_{-\infty}^{\infty}(\varepsilon U_{xxxx} + [\text{grad }\Phi(U)]_x + \alpha H U_{xx} - \beta H U_x + \gamma H U - AU - g, F(U)_x U_x)dxds\Big|$$

$$\le \varepsilon\int_0^t \|U_{xxxx}(s)\|^2 ds + C\int_0^t \|U_{xx}(s)\|^2 ds + 2\beta\int_0^t |U(s)|^2_{s+1+1/2}ds$$

$$+C\int_0^t \|U(s)\|^2 + CT$$

since

$$F(U)_x U_{xx} - F(U)_{xx} U_x = -\left(\sum_{l=1}^N\sum_{j=1}^N\sum_{k=1}^N \frac{\partial^4 \Phi(U)}{\partial U_i \partial U_j \partial U_k \partial U_l}\frac{\partial U}{\partial x_j}\frac{\partial U}{\partial x_k}\frac{\partial U}{\partial x_l}\right)_{1\le i\le N}$$

we have

$$\Big|\int_0^t\int_{-\infty}^{\infty}(U_{xx}, F(U)_x U_{xx} - F(U)_{xx} U_x)dxds\Big| \le C\int_0^t \|U_{xx}(s)\|^2 ds$$

Finally, we get

$$\|U_{xx}(s)\|^2 \le C\int_0^t \|U_{xx}(s)\|^2 ds + C\int_0^t \|U(s)\|^2 ds + C\int_0^t \|U(s)\|^2 ds + \|U_{xx}(s)\|^2 + CT$$

By means of Gronwall's inequality, we can verify estimate (19).

Lemma 8 *Under the conditions of Theorem 1 and Lemma 7, there is the estimate*

$$\sup_{0\le t\le T} \|U_x(t)\|_\infty \le C$$

Lemma 9 *Under the conditions of Theorem 1 and Lemma 7, if $U_0(x) \in H^{2p+2}(\mathbf{R})$, there are the estimations*

$$\sup_{0\le t<T} \|U_{x^h}(t)\| \le C,\ \sup_{0\le t<T} \|U_{x^k t}(t)\| \le C,\ 0 \le h \le 2p+2,\ 0 \le k \le p$$

Theorem 2 *Suppose that $\alpha, \beta \ge 0$, γ, and $0 < \varepsilon < 1$ are real constants; $p \ge 1$; $1 \le r, s \le p$ are integers; $U_0(x) \in H^{3p+2}(\mathbf{R})$, $\Phi(U) \in C^{p+4}(\mathbf{R}^N)$, and there exists a constant $C > 0$, such that*

$$(-1)^{p-1}\Phi(U) \le C|U|^2 + C|U|^{4p+2-\delta}, \quad \text{where } 0 < \delta \le 4p$$

$$|\mathrm{grad}\ \Phi(U)| \le C|U|^{4p+1} + C|U|, \left|\frac{\partial^2 \Phi(U)}{\partial U_i \partial U_j}\right| \le C|U|^{4p} + C, \ \text{forall } U \in \mathbf{R}^N$$

Suppose also that $a_{i,j}(x,t) \in L^\infty(0,T; W^{2p+2,\infty}(\mathbf{R})) \cap W^{1,\infty}(0,T; W^{q,\infty}(\mathbf{R}))$ $(1 \le i,j \le N)$, $g_i(x,t) \in L^2(0,T; H^{2p+2}(\mathbf{R})) \cap W^{1,2}(0,T; H^q(\mathbf{R}))$ $(1 \le i \le N)$. Then the IVP (2) *for the* NES *with* SIDT (5) *has a unique generalized global solution*

$$U(x,t) = (U_1(x,t), \cdots, U_N(x,t)) \in L^\infty(0,T; H^{2p+2}(\mathbf{R})) \cap W^{1,\infty}(0,T; H^q(\mathbf{R}))$$

Therefore the functions $U_{x^k}(x,t)$ $(0 \le k \le 2p+1)$ are Hölder continuous in Q_T, where $q = \max\{2, p\}$.

Theorem 3 *Under the conditions of Theorem 2, as $\varepsilon \to 0$, the global solution $U_\varepsilon(x,t)$ of problem (5,2) tends to the global solution $U(x,t)$ of problem (1,2) in the functional space $L^\infty(0,T; W^{2p+1,\infty}(\mathbf{R}))$.*

The Proofs of these two Theorems are similar to those of the corresponding Theorems in reference [1].

Theorem 4 *Suppose that the following conditions hold as well as the conditions in Theorem 2: $U_0(x) \in H^{4p+4}(\mathbf{R})$, $\Phi(U) \in C^{p+4}(\mathbf{R}^N)$, $a_{i,j}(x,t) \in W^{2,\infty}(0,T; L^\infty(\mathbf{R}))$ $(1 \le i,j \le N)$, $g_i(x,t) \in W^{2,2}(0,T; L^2(\mathbf{R}))(1 \le i \le N)$. Then the* IVP (2) *for the* NES *with* SIDT (5) *has a unique classical global solution $U(x,t) = U_1(x,t), \cdots, U_N(x,t)) \in L^\infty(0,T; H^{2p+2}(\mathbf{R})) \cap W^{1,\infty}(0,T; H^q(\mathbf{R}))$. See* [1].

Theorem 5 *Under the conditions of Theorem 4, as $\varepsilon \to 0$, the unique classical global solution $U_\varepsilon(x,t)$ of problem (5,2) tends to the unique classical global solution $U(x,t)$ of problem (1,2) in the space $L^\infty(0,T; W^{2p+1,\infty}(\mathbf{R})) \cap W^{1,\infty}(0,T; W^{q-1,\infty}(\mathbf{R}))$. See* [1].

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Appendix

Suppose that $m \geq 0$, $n \geq 1$ and $p \geq 1$ are integers, $m < p < m+n$. Then for any $U(x) \in H^{(m+n)/2}(\boldsymbol{R})$, there are the relations

$$|U|_{p/2} \leq |U|_{m/2}^{1-\alpha} |U|_{(m+n)/2}^{\alpha}, \quad \alpha = (p-m)/n$$

Proof By means of the Fourier transform and Parseval's equality, we have

$$|U|_{p-2}^2 = \int_{-\infty}^{\infty} |\zeta|^p |\hat{U}(\zeta)|^2 d\zeta$$

$$= \int_{-\infty}^{\infty} |\zeta|^{m(m+n-p)/n+(m+n)(p-m)/n} |\hat{U}(\zeta)|^{2(m+n-p)/n+2(p-m)/n} d\zeta$$

$$\leq \Big(\int_{-\infty}^{\infty} |\zeta|^m |\hat{U}(\zeta)|^2 d\zeta \Big)^{(m+n-p)/n} \Big(\int_{-\infty}^{\infty} |\zeta|^{m+n} |\hat{U}(\zeta)|^2 d\zeta \Big)^{(p-m)/n}$$

$$= |U|_{m/2}^{2-2\alpha} |U|_{(m+n)/2}^{2\alpha}, \qquad \alpha = (p-m)/n$$