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EVERYWHERE REGULARITY FOR WEAK SOLUTIONS TO VARIATIONAL INEQUALITIES OF TRIANGULAR FORM

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Abstract In this paper, we obtain two results of weak solutions to variational inequalities of triangular form under controllable growth and a class of natural growth conditions, i.e. 1°. L^p-estimate for the gradient; 2°. $C^{1,\beta}_{\text{loc}}(\Omega, R^N)$ regularity.
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1. Introduction

Lewy[1], Lewy-Stampacchia[2], Brezis-Stampacchia[3] and Giaquinta M.[4] et al. have made a systematic study on the convex obstacle problem. But for vector-valued functions, especially for the case $n \geq 3$, there are a few papers dealing with the regularity. Under the quadratic growth condition, Hildebrandt S. and Widman K-O.[5] have proved the regularity to variational inequalities of diagonal form with general obstacle. This paper proves the regularities for variational inequalities of triangular form with some special obstacle. We emphasize that the diagonal ellipticity condition we introduced here is weaker than the strong ellipticity condition and under natural growth condition (I), we seek solutions in $\mathcal{C} \cap L^{n(r-1)/(2-r)}(\Omega, R^N)$ which is larger than $\mathcal{C} \cap L^\infty(\Omega, R^N)$.

We finding $u \in \mathcal{C}$ satisfying the variational inequality

$$\int_\Omega [A^{\alpha\beta}_{ij}(x,u)D_\beta u^j + a_i(x,u)]D_\alpha(u^i - v^i)dx$$
$$\leq \int_\Omega B_i(x,u,Du)(u^i - v^i)dx, \quad \forall v \in \mathcal{C} \tag{1.1}$$

where $\mathcal{C} = \{v \in H^1(\Omega, R^N) : v^k \geq \psi^k,\ k = 1,2,\cdots,N \text{ in } \Omega \subset R^n, n \geq 3;\ v - v_0 \in H^1_0(\Omega, R^N)\}$, is a closed convex set, v_0 and ψ are prescribed functions with $\psi^k|_{\partial\Omega} \leq v^k_0|_{\partial\Omega}$

We assume that

(i) $A^{\alpha\beta}_{ij}(x,u) = 0$, when $j > i$, (Triangular condition)

$$A_{kk}^{\alpha\beta}(x,u)\xi_\alpha\xi_\beta \geq \lambda|\xi|^2,\ \forall\xi \in R^n,\ \lambda > 0;\ \text{(Diagonal elliptic condition)}$$

(ii) Controllable growth conditions:

$$\begin{aligned} &|a_k^\alpha(x,u)| \leq C(|u|^{n/(n-2)} + f_k^\alpha),\ f_k^\alpha \in L^\sigma(\Omega), \sigma > n \\ &|B_k(x,u,p)| \leq C(|p|^{(n+2)/n} + |u|^{(n+2)/(n-2)} + g_k),\ g_k \in L^s(\Omega), s > n/2 \end{aligned} \tag{1.2}$$

$(\text{iii})_1$ Natural growth conditions:

$$u \in \mathcal{C} \cap L^{n(r-1)/(2-r)}(\Omega, R^N) \text{and satisfies (2) and}$$

$$|B_k(x,u,p)| \leq C(|\, p\,|^r + |\, u\,|^{(n+2)/(n-2)} + g_k)$$

$$g_k \in L^s(\Omega), s > n/2, (1 + 2/n) < r < 2$$

or $(\text{iii})_2$ $u \in \mathcal{C} \cap L^\infty(\Omega, R^N)$. $M = \sup\limits_\Omega |u|$ and satisfy $a_k^\alpha(x,u) \in L^\sigma(\Omega \times R^N), \sigma \geq 3$,

$$|B_k(x,u,p)| \leq a\sum_{j=1}^k |p_j|^2 + b_k(x), b_k \in L^s(\Omega), s \geq 2,\ \text{a} - \text{const.}$$

The repeated indices i, j are to be summed from 1 to N, while the repeated indices α, β are to be summed from 1 to n; but the repeated index k doesn't. C represents the constant at various cases. $2^* = 2n/(n-2), q = 2n/(n+2), Du = \{D_\alpha u^i : \alpha = 1,2,\cdots,n; i = 1,2,\cdots,N\}, u_R^k = \fint_{B_R} u^k dx = \frac{1}{|B_R|}\int_{B_R} u^k dx.$

2. L^p-estimates

Proposition 2.1 *Suppose that* (i) *and* (ii) *are satisfied and* $\psi \in H^{1,\sigma}(\Omega, R^N), \sigma > n$, *if* u *is a weak solution to* (1), *then there exists a constant* $p > 2$, *such that* $u \in H_{\text{loc}}^{1,p}(\Omega, R^N)$, *and for any* $x_0 \in \Omega$ *and* $R_0 < \text{dist}(x_0, \partial\Omega)$, *when* $R < R_0$, *the following estimate holds*

$$\begin{aligned} \Big(\fint_{B_{R/2}} (|Du|^2 + |u|^{2^*})^{p/2} dx\Big)^{1/p} &\leq C\Big\{\Big(\fint_{B_R} (|Du|^2 + |u|^{2^*}) dx\Big)^{1/2} \\ &+ \Big(\fint_{B_R} (\sum_{\alpha,i} |f_i|^2 + |D\psi|^2)^{p/2} dx\Big)^{1/p} + R\Big(\fint_{B_R} \sum_i |g_i|^{pq/2} dx\Big)^{2/pq}\Big\} \end{aligned} \tag{2.1}$$

Proof Take $\eta \in C_0^\infty(B_R), 0 \leq \eta \leq 1, \eta \equiv 1$ in $B_{R/2}, |D\eta| \leq C/R$. Since $u^k \geq \psi^k$, then $u_R^k \geq \psi_R^k$. Setting $v^k = \psi^k + (1-\eta^2)(u^k - \psi^k) + \eta^2(u_R^k - \psi_R^k), v^i = u^i, i \neq k$, and substituting v into (1), we get

$$\int_{B_R} |Du^k|^2\eta^2 dx \le C\Big\{ \int_{B_R} \sum_{j=1}^{k-1} |Du^j|^2\eta^2 dx + \int_{B_R} |D\psi^k|^2 dx$$
$$+R^{-2}\int_{B_R} |u^k - u_R^k|^2 dx + \int_{B_R} |u|^{2^*} dx + \int_{B_R} \sum_\alpha |f_k^\alpha|^2 dx$$
$$+\int_{B_R} |B_k||u^k - v^k| dx\Big\}$$

$$\int_{B_R} |B_k||u^k - v^k| dx$$
$$\le C\int_{B_R} (|Du|^{2/q} + |u|^{2^*-1} + |g_k|)|(u^k - u_R^k) - (\psi^k - \psi_R^k)|\eta^2 dx \tag{2.2}$$

$$\int_{B_R} |Du|^{2/q}|\, u^k - u_R^k|\eta^2 dx$$
$$\le \varepsilon\int_{B_R} |Du^k|^2 dx + C\Big(\int_{B_R} |Du|^2 dx\Big)^{2/n}\int_{B_R} |Du|^2 dx \tag{2.3}$$

$$\int_{B_R} |u|^{2^*-1}|u^k - u_R^k|\eta^2 dx$$
$$\le C\Big(\int_{B_R} |Du^k|^2 dx\Big)^{2^*/n}\int_{B_R} |Du^k|^2 dx + \int_{B_R} |u|^{2^*} dx \tag{2.4}$$

$$\int_{B_R} |g_k||u^k - u_R^k|\eta^2 dx \le \varepsilon\int_{B_R} |Du^k|^2 dx + C\Big(\int_{B_R} |g_k|^q dx\Big)^{2/q} \tag{2.5}$$

In (2.3)–(2.5), substituting $|u^k - u_R^k|$ for $|\psi^k - \psi_R^k|$, we have the similar estimates. Note that

$$\int_{B_R} |u|^{2^*} dx \le C\Big(\int_{B_R} |Du|^2 dx\Big)^{2^*/n}\int_{B_R} |Du|^2 dx + C|B_R|\Big(\fint_{B_R} |u|^{2^* q/2} dx\Big)^{2/q} \tag{2.6}$$

$$R^{-n-2}\int_{B_R} |u^k - u_R^k|^2 dx \le C\Big(\fint_{B_R} |Du^k|^q dx\Big)^{2/q} \tag{2.7}$$

Because of the absolute continuity, there exists $R_0 < \operatorname{dist}(x_0, \partial\Omega)$, such that when $R < R_0$, we have

$$\int_{B_R} (|Du^k|^2 + |u|^{2^*})\eta^2 dx \le C\Big\{ \int_{B_R} \sum_{j=1}^{k-1} |Du^j|^2\eta^2 dx + \Big(\int_{B_R} |g_k|^q dx\Big)^{2/q}$$
$$+|B_R|\Big(\fint_{B_R} (|Du^k|^2 + |u|^{2^*})^{q/2} dx\Big)^{2/q} + \int_{B_R} (|D\psi|^2 + \sum_\alpha |f_k^\alpha|^2) dx$$
$$+\varepsilon\int_{B_R} |Du|^2 dx\Big\} \tag*{$(2.8)_k$}$$

Substituting $(2.8)_{k-1}$ into $(2.8)_k$, and summing up these inequalities gained through iteration, we have

$$\int_{B_{R/2}}(|Du|^2+|u|^{2^*})dx \le C\Big\{\Big(\int_{B_R}(|Du|^2+|u|^{2^*})^{q/2}dx\Big)^{2/q}+\int_{B_R}(|D\psi|^2 + \sum_{i,\alpha}|f_i^\alpha|^2+\sum_i|G_i|^2)dx\Big\}+\frac{1}{2}\int_{B_R}|Du|^2dx \tag{2.9}$$

$$G_i=\Big(\int_{B_R}|g_i|^q dx\Big)^{1/n} g_i^{q/2},\quad i=1,2,\cdots,N$$

Then we get (2.1) immediately by Prop.1.1 of Chap.V in [6].

Proposition 2.2 *Suppose* (i) *and* $(\text{iii})_1$ *are satisfied,* $\psi\in H^{1,\sigma}\cap L^{n(r-1)/(2-r)}(\Omega,\boldsymbol{R}^N)$, $\sigma>n$, *if* u *is a weak solution to* (1), *then there exists* $p>2$, *such that* $u\in H^{1,p}_{\text{loc}}(\Omega,\boldsymbol{R}^N)$ *and for any* $x_0\in\Omega$, *and* $R_0<\operatorname{dist}(x_0,\partial\Omega)$, *when* $R<R_0$, (2.1) *holds.*

Proof

$$\int_{B_R}|Du|^r|u^k-u_R^k|\eta^2dx\le\Big(\int_{B_R}|u|^{n\frac{r-1}{2-r}}dx\Big)^{(2-r)/n}\int_{B_R}|Du|^2dx \tag{2.10}$$

$$\int_{B_R}|Du|^r|\psi^k-\psi_R^k|\eta^2dx\le\frac{r}{2}\,\varepsilon\int_{B_R}|Du|^2dx+\frac{2-r}{2}\varepsilon^{-\frac{r}{2-r}}\Big(\int_{B_R}|\psi|^{n\frac{r-1}{2-r}}dx\Big)^{2/r}\times\int_{B_R}|D\psi|^2dx \tag{2.11}$$

Substitute (2.4)–(2.7) and (2.10) (2.11) into (2.2) and repeat the proof as in Proposition 2.1, the proof is complete.

Proposition 2.3 *Suppose* (i) and $(\text{iii})_2$ *hold, and* $\psi\in H^{1,\sigma}\cap L^\infty(\Omega,\boldsymbol{R}^N)$, $\sigma>n$, $|\psi|\le M$, $2aM<\lambda$, *if* u *is a weak solution to* (1), *then there exists* $p>2$, *such that* $u\in H^{1,p}_{\text{loc}}(\Omega,\boldsymbol{R}^N)$ *and for sufficiently small* $R<\operatorname{dist}(x_0,\partial\Omega)\wedge R_0\wedge 1$, *the following estimate holds*

$$\Big(\int_{B_{R/2}}\sum_{j=1}^k|Du^j|^p dx\Big)^{1/p}\le C\Big\{\Big(\int_{B_R}\sum_{j=1}^k|Du^j|^2dx\Big)^{1/2}+\Big(\int_{B_R}\sum_{j=1}^k(|b_j|+|D\psi^j|^2+\sum_\alpha|a_j^\alpha|^2)^{p/2}dx\Big)^{1/p}\Big\}$$

$$k=1,2,\cdots,N \tag{2.12}$$

Proof As in the proof of Proposition 2.1, now we have

$$(\lambda-2aM)\int_{B_R}|Du^k|^2\eta^2dx\le C\Big\{\int_{B_R}\sum_{j=1}^{k-1}|Du^j|^2\eta^2dx+|B_R|\Big(\int_{B_R}|Du^k|^q dx\Big)^{2/q}+\int_{B_R}\big(\sum_\alpha|a_k^\alpha|^2+|D\psi^k|^2+|b_k|\big)dx+\varepsilon\int_{B_R}|Du^k|^2dx\Big\} \tag*{$(2.13)_k$}$$

By the iteration as stated before, it follows from $(2.13)_k$ that

$$\int_{B_{R/2}}\sum_{j=1}^{k}|Du^j|^2dx \le C\Big\{\Big(\int_{B_R}\sum_{j=1}^{k}|Du^j|^q dx\Big)^{2/q}+\int_{B_R}\sum_{j=1}^{k}\big(|b_j|+|D\psi^j|^2$$

$$+\sum_{\alpha}|a_j^\alpha|^2\big)dx+\frac{1}{2}\int_{B_R}\sum_{j=1}^{k}|Du^j|^2dx\Big\}$$

Therefore by Prop.1.1 of Chap.V in [6], (2.12) holds.

3. $C^{0,\alpha}$ - regularity

Theorem 3.1 *Suppose* (i) *and* (ii) *hold,* $\psi \in H^{1,\sigma}(\Omega, R^N), \sigma > n$ *and* u *is a weak solution to* (1), *then* $u \in C^{0,\alpha}_{\rm loc}(\Omega, R^N), \alpha \in (0,1)$.

Proof We split $u^k = U^k + (u^k - U^k)$, here U^k is a weak solution to the Dirichlet problem

$$\begin{cases}\displaystyle\int_{B_R}A_{kk}^{\alpha\beta}(x,u(x))D_\beta U^k D_\alpha\varphi^k dx = 0, \quad \forall\varphi^k \in H_0^1(B_R)\\ U^k - u^k \in H_0^1(B_R)\end{cases} \tag{3.1}$$

We know that for some $\mu \in (0,1)$ and for all $\rho < R < \mathrm{dist}(x_0,\partial\Omega)\wedge 1$

$$\int_{B_\rho}|DU^k|^2dx \le C(\frac{\rho}{R})^{n-2+2\mu}\int_{B_R}|Du^k|^2dx$$

$$\int_{B_\rho}|Du^k|^2dx \le C\Big[\Big(\frac{\rho}{R}\Big)^{n-2+2\mu}\int_{B_R}|Du^k|^2dx+\int_{B_R}|D(u^k-U^k)|^2dx\Big] \tag{3.2}$$

By (1), $u^k - U^k$ satisfies

$$\int_{B_R}[A_{kk}^{\alpha\beta}(x,u)D_\beta(u^k-U^k)+\sum_{j=1}^{k-1}A_{kj}^{\alpha\beta}(x,u)D_\beta u^j + a_k^\alpha(x,u)]D_\alpha(u^k-v^k)dx$$

$$\le \int_{B_R}B_k(x,u,Du)(u^k-v^k)dx, \quad \forall v^k \ge \psi^k, u^k - v^k \in H_0^1(B_R) \tag{3.3}$$

Choosing $v^k = \max(U^k,\psi^k) = U^k \vee \psi^k$, we get,

$$\int_{B_R}|D(u^k-U^k)|^2dx \le C\Big[\int_{B_R}|D(U^k-U^k\vee\psi^k)|^2dx+\int_{B_R}\sum_{j=1}^{k-1}|Du^j|^2dx$$

$$+\int_{B_R}|u|^{2^*}dx+\Big(\int_{B_R}|Du|^2dx\Big)^{2/q}+R^{n-2+2\nu}\Big] \tag{3.4}$$

where $\nu = \min\Big(1-\dfrac{n}{\sigma}, 1-\dfrac{n}{2s}\Big)$. But $U^k - U^k\vee\psi^k \in H_0^1(B_R)$ is a weak solution to

$$\int_{B_R}A_{kk}^{\alpha\beta}(x,u)D_\beta(U^k-U^k\vee\psi^k)D_\alpha\varphi^k dx = -\int_{B_R}A_{kk}^{\alpha\beta}(x,u)D_\beta(U^k\vee\psi^k)D_\alpha\psi^k dx$$

$$\forall\varphi^k \in H_0^1(B_R) \tag{3.5}$$

Choosing $\varphi^k = U^k - U^k \vee \psi^k$, since $U^k \vee \psi^k = \psi^k$ for $x \in \operatorname{supp}(U^k - U^k \vee \psi^k)$, we obtain

$$\int_{B_R} |D(U^k - U^k \vee \psi^k)|^2 dx \le C \int_{B_R} |D\psi^k|^2 dx \le C R^{n-2+2\nu} \tag{3.6}$$

so

$$\int_{B_R} |Du^k|^2 dx \le C\Big[\Big(\frac{\rho}{R}\Big)^{n-2+2\mu} \int_{B_R} |Du^k|^2 dx + \int_{B_R} \sum_{j=1}^{k-1} |Du^j|^2 dx + \int_{B_R} |u|^{2^*} dx$$

$$+ \Big(\int_{B_R} |Du|^2 dx\Big)^{2/q} + R^{n-2+2\alpha}\Big], \ \alpha < \min(u, \nu) \tag{3.7$_k$}$$

By iteration, we find (c.f.[8]).

$$\int_{B_\rho} |Du|^2 dx \le c\rho^{n-2+2\alpha} \tag{3.8}$$

By Morrey Lemma, we have $u \in C^{0,\alpha}_{\text{loc}}(\Omega, R^N)$.

Theorem 3.2 *Suppose* (i) *and* $(\text{iii})_1$ *hold,* $\psi \in H^{1,\sigma} \cap L^{n\frac{r-1}{2-r}}(\Omega, R^N), \sigma > n$, *and* u *is a weak solution to (1), then* $u \in C^{0,\alpha}_{\text{loc}}(\Omega, R^N), \alpha \in (0,1)$.

Proof Let U^k be a weak solution to the Dirichlet problem

$$\begin{cases} \int_{B_R} A^{\alpha\beta}_{kk}(x, u(x)) D_\beta U^k D_\alpha \varphi^k dx = 0, & \forall \varphi^k \in H^1_0 \cap L^{n\frac{r-1}{2-r}}(B_R) \\ U^k - u^k \in H^1_0 \cap L^{n\frac{r-1}{2-r}}(B_R) & \end{cases} \tag{3.9}$$

by using $(\text{iii})_1$, we have

$$\int_{B_R} |Du|^r |u^k - v^k| dx \le C\Big(\int_R |Du|^2 dx\Big)^r + \varepsilon\Big[\int_{B_R} |D(u^k - U^k)|^2 dx$$

$$+ \int_{B_R} |D(U^k - U^k \vee \psi^k)|^2 dx + \Big(\int_{B_R} |u^k - U^k \vee \psi^k|^{n\frac{r-1}{2-r}} dx\Big)^{\frac{2(2-r)}{n(r-1)}}\Big] \tag{3.10}$$

$$\int_{B_R} |u|^{2^*-1} |u^k - v^k| dx \le \varepsilon \int_{B_R} |D(u^k - U^k)|^2 dx + C\int_{B_R} |D(U^k - U^k \vee \psi^k)|^2 dx$$

$$+ C\Big(\int_{B_R} |u|^{2^*} dx\Big)^{2/q} \tag{3.11}$$

$$\int_{B_R} |g_k||u^k - v^k| dx \le \varepsilon \int_{B_R} |D(u^k - U^k)|^2 dx + C\int_{B_R} |D(U^k - U^k \vee \psi^k)|^2 dx$$

$$+ C\Big(\int_{B_R} |g_k|^s dx\Big)^{2/s} R^{n+2-2n/s} \tag{3.12}$$

By (3.10)–(3.12), (3.6) and choosing $\varepsilon < \min(\lambda/4, R^{n-2+2\nu}, \nu = \min(1 - n/\sigma, 1 - n/2s)$, we have

$$\int_{B_R}|D(u^k-U^k)|^2dx\le C\Big\{\int_{B_R}\sum_{j=1}^{k-1}|Du^j|^2dx+\int_{B_R}|u|^{2^*}dx+\Big(\int_{B_R}|Du|^2dx\Big)^r$$
$$+R^{n-2+2\nu}\Big\}$$

$$\int_{B_\rho}|Du^k|^2dx\le C\Big\{\Big(\frac{\rho}{R}\Big)^{n-2+2\mu}\int_{B_R}|Du^k|^2dx+\int_{B_R}\sum_{j=1}^{k-1}|Du^j|^2dx+\int_{B_R}|u|^{2^*}dx$$
$$+\Big(\int_{B_R}|Du|^2dx\Big)^r+R^{n-2+2\alpha}\Big\},\alpha<\min(\mu,\nu) \qquad (3.13)_k$$

By the iteration as that in the proof of Theorem 3.1, we finally get (3.8).

4. $C^{1,\beta}$-regularity

In this section, we assume that $A_{kk}^{\alpha\beta}$ and a_k^α are Hölder continuous functions with exponent β, then there exists a nonnegative bounded function $\omega(t)$ which is increasing and concave continuous, $\omega(0)=0$ such that for $x,y\in\Omega$, and $u,v\in \boldsymbol{R}^N$

$$|A_{kk}^{\alpha\beta}(x,u)-A_{kk}^{\alpha\beta}(y,v)|\le\omega(|x-y|^2+|u-v|^2) \qquad (4.1)$$

$$|a_k^\alpha(x,u)-a_k^\alpha(y,v)|\le\omega(|x-y|^2+|u-v|^2) \qquad (4.2)$$

$$\omega(t)\le Ct^\beta,\ 0<\beta<1 \qquad (4.3)$$

$$\psi\in C^{1,\beta}(\Omega,\boldsymbol{R}^N) \qquad (4.4)$$

Theorem 4.1 *Suppose* (i) *and* (ii) (*or* (i) *and* (iii)$_1$) *hold* ($g_k\in L^\infty(\Omega)$) *and that* (4.1)–(4.4) *hold, if u is a weak solution to (1), then* $u\in C^{1,\beta}_{\mathrm{loc}}(\Omega,\boldsymbol{R}^N)$.

Lemma 4.2 Under the assumptions of Theorem 4.1, $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega,\boldsymbol{R}^N)$, for all $\alpha\in(0,1)$.

Proof Because of Theorem 3.1 and 3.2, u is locally bounded. Therefore in $B_R\subset\subset\Omega$,

$$|B_k(x,u,p)|\le C(|p|^r+1),\quad 2/q\le r<2 \qquad (4.5)$$

$$\Big(\int_{B_{R/2}}(1+|Du|^2)^{p/2}dx\Big)^{1/p}\le C\Big(\int_{B_R}(1+|Du|^2)dx\Big)^{1/2} \qquad (2.1)$$

Let U^k be a weak solution to the Dirichlet problem

$$\begin{cases}\displaystyle\int_{B_{R/2}}A_{kk}^{\alpha\beta}(x_0,u_R)D_\beta U^kD_\alpha\psi^kdx=0, & \forall\varphi^k\in H_0^1(B_{R/2})\\ U^k-u^k\in H_0^1(B_{R/2})\end{cases}$$

Because of $(0.1)'$ and ω being bounded concave, we get

$$\int_{B_{R/2}} \omega^2 dx \leq CR^n \tag{4.6}$$

$$\int_{B_{R/2}} \omega^2 |Du^k|^2 dx \leq CR^{2\alpha\beta} \int_{B_R} (1+|Du|^2)dx \tag{4.7}$$

$$\int_{B_{R/2}} (1+|Du|^r)|u^k - v^k| dx \leq C\Big(\int_{B_R}(1+|Du|^2)dx\Big)$$
$$\times [R^{2-n}\int_{B_R}|Du|^2 dx + R^2]^{1-2/p} \tag{4.8}$$

So

$$\int_{B_{R/2}} |D(u^k - U^k)|^2 dx \leq C\Big\{\int_{B_{R/2}} \sum_{j=1}^{k-1} |Du^j|^2 dx + \chi(x_0, R)\int_{B_R}(1+|Du|^2)dx + R^n\Big\}$$

$$\chi(x_0, R) = R^{2\alpha\beta} + \Big(R^{2-n}\int_{B_R}|Du|^2 dx + R^2\Big)^{1-2/p}$$

From Theorem 2.1 of Chapter III in [6], we have, for $\rho < R/2$

$$\int_{B_\rho} |Du^k|^2 dx \leq C\Big\{\Big(\frac{\rho}{R}\Big)^n \int_{B_{R/2}} |Du^k|^2 dx + \int_{B_{R/2}} \sum_{j=1}^{k-1} |Du^j|^2 dx$$
$$+\chi(x_0,R)\int_{B_R}(1+|Du|^2)dx + R^n\Big\} \tag{4.9$_k$}$$

By iteration, we deduce from $(4.9)_k$

$$\int_{B_\rho}(1+|Du|^2)dx \leq C_1\Big[\Big(\frac{\rho}{R}\Big)^n + \chi(x_0,R)\Big]\int_{B_R}(1+|Du|^2)dx + C_2R^n \tag{4.10}$$

For $R/2 \leq \rho < R$, (4.10) is trivial.

For any $\varepsilon_0 > 0$, we have $\chi(x_0, R) < \varepsilon_0$, when R is sufficiently small. Using Lemma 2.1 of Chapter III in [6], we get

$$\int_{B_\rho} |Du|^2 dx \leq C\rho^{n-2+2\alpha}, \quad \forall \alpha \in (0,1) \tag{4.11}$$

Proof of Theorem 4.1

Let m be an integer such that $2^{-m(n-2+2\alpha)} \leq R^{2\beta}$, we have

$$\int_{B_{R/2^m}} |Du|^2 dx \leq CR^{n-2+2\alpha+2\beta} \tag{4.11$'$}$$

when $\rho < R/2^m$, here we use $R/2^m$ to replace $R/2$ in the proof of Lemma 4.2.

By (4.3) and (4.11), we deduce from (4.6)–(4.8)

$$\int_{B_R} \omega^2 dx \le CR^{n+2\alpha\beta} \tag{4.6$'$}$$

$$\int_{B_R} \omega^2 |Du^k|^2 dx \le CR^{n-2+2\alpha+\alpha\beta} \tag{4.7$'$}$$

$$\int_{B_{R/2}} (1+|Du|^r)|u^k - v^k| dx \le CR^{n-2+2\alpha+\alpha} \tag{4.8$'$}$$

From Theorem 2.1 of Chapter III in [6], we obtain

$$\int_{B_\rho} |Du - (Du)_\rho|^2 dx = C\Big\{\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R} |Du - (Du)_R|^2 dx + R^{n-2+2\alpha+2\beta}\Big\} \tag{4.12}$$

Take α so close to 1 such that $\alpha(2+\beta) > 2$, then we have

$$\int_{B_\rho} |Du - (Du)_\rho|^2 dx \le C\Big[\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R} |Du - (Du)_R|^2 dx + R^{n+2\varepsilon}\Big], \quad 0 < \varepsilon < 1$$

Using Lemma 2.1 and Theorem 1.2 of Chapter III in [6], we may conclude that Du is locally bounded. Now (4.6)$'$–(4.8)$'$ can be improved again

$$\int_{B_R} \omega^2 dx \le CR^{n+2} \tag{4.6$''$}$$

$$\int_{B_R} \omega^2 |Du^k|^2 dx \le CR^{n+2} \tag{4.7$''$}$$

$$\begin{aligned}\int_{B_{R/2}} (1+|Du|^r)|u^k - v^k| dx &\le \varepsilon \int_{B_{R/2}} |D(u^k - U^k)|^2 dx \\ &+ C\Big\{\int_{B_{R/2}} |D(U^k - U^k \vee \psi^k)^2| dx + R^{n+2}\Big\}\end{aligned} \tag{4.8$''$}$$

Just as (4.12) was deduced, we have, for all $\rho < R$

$$\int_{B_\rho} |Du - (Du)_\rho|^2 dx \le C\Big\{\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R} |Du - (Du)_R|^2 dx + R^{n+2\beta}\Big\}$$

Therefore $Du \in C^{0,\beta}\text{loc}(\Omega, R^{nN})$.

Theorem 4.3 *Suppose* (i) *and* $(\text{iii})_2$ *hold (in* $(\text{iii})_2$ $b_k \in L^\infty(\Omega)$*), and* (4.1)–(4.4) *are satisfied,* $|\psi| \le M, 2aM < \lambda$*, if* u *a weak solution to* (1)*, then* $u \in C^{1,\beta}_{\text{loc}}(\Omega, R^N)$.

Let us divide the proof in three steps:

1. As in the proof of Lemma 4.2, under the assumptions of Theorem 4.3, we can prove that for any $\rho < R$, (4.10) holds. Set

$$\phi(x_0, R) = R^{2-n} \int_{B_R} (1+|Du|^2) dx$$

From (4.10) we get, for $0 < \tau < 1$

$$\phi(x_0, \tau R) = C_1[1 + \chi(x_0, R)\tau^{-n}]\tau^2\phi(x_0, R) + C_2\tau^{2-n}R^{2\alpha}, \quad \forall \alpha \in (0,1) \tag{4.13}$$

Let $\alpha < \beta < 1$ and choose τ in such a way that $2C_1\tau^{2-2\delta} \leq 1$. Since we have $\chi(x_0, R) < \tau^n$, provided $R < R_1 \leq \operatorname{dist}(x_0, \partial\Omega) \wedge 1$ and $\phi(x_0, R) < \varepsilon_1$, setting $H_0 = C_2\tau^{2-n}$, we get

$$\phi(x_0, \tau R) = \tau^{2\delta}\phi(x_0, R) + H_0 R^{2\alpha} \tag{4.14}$$

Therefore by iteration we obtain

$$\phi(x_0, \tau^k R) \leq \Big[\phi(x_0, R) + H_0\frac{R^{2\alpha}}{\tau^{2\alpha} - \tau^{2\delta}}\Big]\tau^{2k\delta}$$

Provided $\phi(x_0, R) < \varepsilon_0, 2\varepsilon_0 < \varepsilon_1$, and $R < R_0 < R_1$ such that $H_0R^{2\alpha}/(\tau^{2\alpha} - \tau^{2\delta}) < \varepsilon_0$, we can get,

$$\phi(x_0, \rho) \leq C\Big(\frac{\rho}{R}\Big)^{2\alpha}, \quad \forall \rho < R_0 \tag{4.15}$$

Therefore $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_0, R^N)$, for any $\alpha \in (0,1)$, where

$$\Omega_0 = \Big\{x_0 \in \Omega : \liminf_{R\to 0}\ R^{2-n}\int_{B_R(x_0)} |Du|^2 dx = 0\Big\}$$

2. Let us consider restrictively in Ω_0, then by the same reason as that in the proof of Theorem 4.1, we can get $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega_0, R^N)$.

3. Use inductive method, when $k = 1$, (1) implies

$$\int_{B_{2R}} [A^{\alpha\beta}_{11}(x,u)D_\beta u^1 + a^\alpha_1(x,u)]D_\alpha(u^1 - v^1)dx \leq \int_{B_{2R}} B_1(u^1 - v^1)dx$$
$$\forall v^1 \geq \psi^1, u^1 - v^1 \in H^1_0 \cap L^\infty(B_{2R})$$

Choosing $v^1 = u^1 - \varepsilon\eta(u^1 - \psi^1)e^{t|u^1-\psi^1|^2}, 0 \leq \eta \in C^\infty_0(B_{2R}), \varepsilon > 0$ then we obtain

$$\begin{aligned}\int_{B_{2R}} |D(u^1 - \psi^1)^2|e^{t|u^1-\psi^1|^2}\eta dx + t\int_{B_{2R}} |D|u^1 - \psi^1|^2|^2 e^{t|u^1-\psi^1|^2}\eta dx \\ \leq -C_0\int_{B_{2R}} A^{\alpha\beta}_{11}(x,u)e^{t|u'-\psi'|^2}D_\beta|u' - \psi'|^2 D\alpha\eta dx \\ -C_1\int_{B_{2R}} a^\alpha_1(x,u)e^{t|u^1-\psi^1|^2}|u^1 - \psi^1|D_\alpha\eta dx \\ +C_2\int_{B_{2R}} |D(u^1-\psi^1)|^2|u^1 - \psi^1|^2 e^{t|u^1-\psi^1|^2}\eta_1 dx + C_3\int_{B_{2R}}\eta dx\end{aligned}$$

$$\begin{aligned}2|u^1 - \psi^1||D(u^1 - \psi^1)^2 = |D(u^1 - \psi^1)||D|u^1 - \psi^1|^2 \\ \delta|D(u^1 - \psi^1)|^2 + C|D|u^1 - \psi^1|^2|^2\end{aligned}$$

Choose t large enough and δ small enough, we get

$$\int_{B_{2R}} |D(u^1-\psi^1)|^2 e^{t|u^1-\psi^1|^2}\eta dx \le -C_0\int_{B_{2R}} A_{11}^{\alpha\beta}(x,u)e^{t|u^1-\psi^1|^2}D_\beta|u^1-\psi^1|^2 - D_\alpha\eta dx$$
$$-C_1\int_{B_{2R}} a_1^\alpha(x,u)e^{t|u^1-\psi^1|^2}|u^1-\psi^1|D_\alpha\eta dx + C_3\int_{B_{2R}}\eta dx$$

Note that the nonnegative function $W^1 = \max\limits_{B_{2R}}|u^1-\psi^1|^2 - |\overset{*}{u}{}^1-\psi^1|^2$ is a supersolution for an elliptic operator

$$\int_{B_{2R}} A_{11}^{\alpha\beta}(x,u)e^{t|u^1-\psi^1|^2}D_\beta W^1 D_\alpha\eta dx$$
$$\ge \frac{C}{C_0}\int_{B_{2R}} a_1^\alpha(x,u)e^{t|u^1-\psi^1|^2}|u^1-\psi^1|D_\alpha\eta dx - \frac{C_3}{C_0}\int_{B_{2R}}\eta dx$$

Because $a_1 e^{t|u^1-\psi^1|^2}|u^1-\psi^1| \in L^\infty(B_{2R})$, hence by weak Harnack inequality (see [7], Theorem 8.18) we get

$$R^{-n}\int_{B_{2R}} W^1 dx \le C\Big\{\inf_{B_R} W^1 + R\Big\} \tag{4.16$_1$}$$

If $W^1 \not\equiv 0$ in B_{2R}, we can and do choose η in such a way that $\eta \le C_4$ in $B_{2R;\eta} \ge C_5 > 0$ in B_R, and

$$\int_{B_{2R}} A_{11}^{\alpha\beta}(x,u)e^{t|u^1-\psi^1|^2}D_\beta W^1 D_\alpha\eta dx$$
$$-\frac{C_1}{C_0}\int_{B_{2R}} a_1^\alpha(x,u)e^{t|u^1-\psi^1|^2}|u^1-\psi^1|D_\alpha\eta dx$$
$$=\frac{C_6}{R^2}\int_{B_{2R}} W^1 dx + CR^{n-1}$$

Confer Theorem 4 in [9], therefore we conclude

$$\int_{B_R}|D(u^1-\psi^1)|^2 dx \le CR^{n-2}\Big\{\max_{B_{2R}}|u^1-\psi^1|^2 - \max_{B_R}|u^1-\psi^1|^2 + R\Big\} \tag{4.17}$$

which implies that for fixed $x_0\in\Omega, \varepsilon>0$, there exists R_1, such that for all $\rho < R_1$

$$\rho^{2-n}\int_{B_\rho}|Du^1|^2 dx < \varepsilon$$

Now assuming that $u^1,\cdots,u^{k-1}\in C^{1,\beta}_{\rm loc}(\Omega)$, by (1), we have

$$\int_{B_{2R}}\Big[A_{kk}^{\alpha\beta}(x,u)D_\beta u^k + \sum_{j=1}^{k-1}A_{kj}^{\alpha\beta}(x,u)D_\beta u^j + a_k^\alpha(x,u)\Big]D_\alpha(u^k-v^k)dx$$
$$\le \int_{B_{2R}} B_k(u^k-v^k)dx,\ \forall v^k\ge\psi^k, u^k-v^k\in H_0^1\cap L^\infty(B_{2R})$$

Choosing $v^k = u^k - \varepsilon\eta(u^k - \psi^k)e^{t|u^k-\psi^k|^2}$, and noting that Du^j is locally bounded, when $j < k$, we can obtain

$$\int_{B_R} |D(u^k - \psi^k)|^2 dx \le CR^{n-2}\Big\{ \max_{B_{2R}} |u^k - \psi^k|^2 - \max_{B_R} |u^k - \psi^k|^2 + R\Big\}$$

which implies that there exists R_k such that for all $\rho < R_k$

$$\rho^{2-n} \int_{B_\rho} |Du^k|^2 dx < \varepsilon$$

Take $R_0 = \min(R_1, \cdot, R_N)$, hence when $\rho < R_0$, we have

$$\rho^{2-n} \int_{B_\rho} |Du|^2 dx < \varepsilon$$

The proof is complete.

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