

Opinion Dynamics of Sznajd Model on Small-World Network

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Abstract. The society structure plays an important role in shaping the attitudes, beliefs and public opinion. For studying the role of the society structure in opinion dynamics, we analyze the Sznajd model on small-world network formed by adding shortcuts in a lattice consisting of N nodes arranged in a ring and on two-dimensional (2-D) regular lattice. Through computer simulation, we find that there exists a pseudo-phase transition from the coexistence state for $\phi < \phi_c$ to the consensus state for $\phi > \phi_c$, where ϕ_c is some threshold for the shortcut density ϕ , which is dependent of the complex network topology and the dimensionality of complex networks. Our observations indicate the dependence of the opinion dynamics on the complex system topology.

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Key words: Opinion formation, small-world networks, shortcut density, pseudo-phase transition.

1 Introduction

Our local society, which can be well modeled as complex network, has its own structure depending on the geography, culture and history. Recently it has also been realized that many real social networks arising in society, such as networks of collaborations between actors [1, 2] and scientists [3, 4], web-based social networks [5], peer-to-peer social network [6], and the social networks of a bulletin board system in a university [7] all share the small-world effects, including the shortest path length and higher clustering coefficient, which probably caused by the shortcuts in society systems. Those features will affect the dynamics in society systems, especially the opinion dynamics. Many natural

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and man-made networks have been successfully studied as a framework of several celebrated opinion models. Nevertheless, the understanding of the opinion dynamics on complex networks remains a challenge.

The Ising model, one of the fundamental models of Statistical Mechanics, has been modified to model the problem of opinion formation [8–11]. Since many real questions have only two possible answers (yes or no), the Ising model with two possible spins (up or down) is suitable in describing such systems [12]. Following the Ising model and the old principle “united we stand, divided we fall”, a celebrated consensus model [8] (Sznajd model) is proposed by K. Sznajd-Weron and J. Sznajd to describe a simple mechanism of making up decisions in a closed community. In the Sznajd model, the opinion S_i of individual i is a binary variable assuming the value +1 (\uparrow) and -1 (\downarrow) that referring to two opposite opinions on a certain thing. Assume that each pair of adjacent individuals can affect the state of their nearest neighbors using the following updating rule:

$$\begin{aligned} \text{if } S_i S_{i+1} = +1, \text{ then } S_{i-1} = S_{i+2} = S_i, \\ \text{if } S_i S_{i+1} = -1, \text{ then } S_{i-1} = S_{i+1}, S_{i+2} = S_i. \end{aligned} \quad (1.1)$$

Simulating the model for the long time, where at each time step the individual i is chosen randomly, one finally obtains one of the three fixed states: $\uparrow\uparrow\uparrow\uparrow$, $\downarrow\downarrow\downarrow\downarrow$ and $\uparrow\downarrow\uparrow\downarrow$, with probabilities, refer to an initially random distribution, 0.25, 0.25 and 0.5, respectively [8, 12].

During those years, many physicists have been studied the Sznajd model on two-dimension lattices [9], deterministic pseudo-fractal networks [13], small-world networks [12, 14] and scale-free networks [15] through numerical simulation and got some interesting results. However, all those works can not show the important role of the structure of complex systems in opinion formation completely that we will do in our present work. On the other hand, the Sznajd model has been applied in marketing [16, 17] and politics [18–20], and investigated also from the theoretical point of view [21, 22].

The main goal of this paper is to analyze the crucial role of complex network topology in opinion dynamics. we find that, by numerical simulation, there exists a long-range opinion-opinion correlation due to the shortcuts added randomly. On the other hand, the probability that there exists a phase transition from the coexistence state to the complete consensus state is one when the shortcut density $\phi > \phi_c$, where ϕ_c is dependent of the complex network size N , its first neighbor parameter (FNP) K and the dimensionality of complex network. Maybe, our present work can explain why a phase transition was not found in [8] and [10], and is helpful for studying the interaction between dynamics and complex system topology.

2 The Sznajd model on complex networks

Many real society systems can be mapped to undirected complex networks, which is a set of agents with relationships of different kinds among them, such as friendship, col-

laboration, business, sexual and other interactions [23]. The undirected complex network can be described as a graph consisting of N nodes (agents) and L edges. The connectivity is represented by the $N \times N$ adjacency matrix A whose element a_{ij} is equal to one when agent i and agent j can affect each other and zero otherwise. There are no self-connections or multiple edges.

Next, we generate the underlying complex network using the algorithm of the Newman-Watts small-world network model [24]. The SWN is defined on a lattice consisting of N nodes arranged in a ring. Initially each node is connected to all of its neighbors up to some fixed range K to make the network with average coordination number $z=2K$, randomness is then introduced by adding edges between two randomly chosen nodes with probability ϕ , so that there are again ϕN shortcuts on average. For convenience, we call K the first neighbor parameter (FNP) and ϕ the shortcut density. Tuning K and ϕ , we can get a series of complex networks with different structural properties. This model is equivalent to the Watts-Strogatz model for small ϕ , whilst being better behaved when ϕ becomes comparable to 1 [24].

Then the Sznajd model is generalized to complex networks [12]. The opinions of individuals can be described by the spins of nodes in complex network. For studying the role of structure of underlying complex network in opinion formation simply, we consider a one-dimensional lattice with periodic boundary conditions and the tunable shortcut density ϕ . These shortcutting edges are fixed beforehand. The updating rule is generalized to include the shortcutting neighbors, if exist, as follow:

$$\begin{aligned} \text{if } S_i S_{i+1} = +1, \text{ then } S_{n(i, \bar{i}+1)} = S_{n(i+1, \bar{i})} = S_{cn(i, i+1)} = \\ S_{sc(i, \bar{i}+1)} \text{ (if exists)} = S_{sc(i+1, \bar{i})} \text{ (if exists)} = S_i, \\ \text{if } S_i S_{i+1} = -1, \text{ then } S_{n(i, \bar{i}+1)} = S_{sc(i, \bar{i}+1)} \text{ (if exists)} = S_{i+1}, \\ S_{n(i+1, \bar{i})} = S_{sc(i+1, \bar{i})} \text{ (if exists)} = S_i, \text{ and } S_{cn(i, i+1)} \text{ fixed,} \end{aligned} \quad (2.1)$$

where $sc(i, \bar{j})$ is the shortcutting neighbor of the i th node and not of the j th node, if exists; $n(i, \bar{i}+1)$ is the first neighbor of the i th node and not of the $(i+1)$ th node in the regular lattice before the randomness is introduced. For example, the $(i-K)$ th node is the first neighbor $n(i, \bar{i}+1)$ of the i th node and not of the $(i+1)$ th node, and on the other hand, the $(i+1+K)$ th node is the first neighbor $n(i+1, \bar{i})$ of the $(i+1)$ th node and not of the i th node in the one dimensional regular lattice with $z=2K$. And $cn(i, i+1)$ is the common neighbor of both the i th and $(i+1)$ th nodes.

3 Numerical results and discussions

We realize our model on complex network with 1000 nodes and the results are averaged over 100 independent samples. Initially the opinions of nodes are distributed randomly, +1 with the probability p and -1 with the probability $(1-p)$. Here, we call p the initial opinion probability. For simplicity and studying the role of complex network topology, we set $p=0.5$.

The primary interest is the dynamical evolution of the average magnetization of the system,

$$m(t) = \frac{1}{N} \sum_{i=1}^N S_i(t) \quad (3.1)$$

which is the difference in the number of opinions with +1 and -1. In Fig. 1 we represent the evolution of the average magnetization $m(t)$ as a function of time t on the one dimensional small-world networks. We find that there exists a phase transition from the coexistence state to the complete consensus state (+1 consensus state or -1 consensus state) when the shortcut density $\phi > \phi_c = 0.14$ for $N = 1000, K = 4$. Here, the coexistent state is defined as that the individuals can be divided into two or more camps with +1 or -1 opinion. The consensus state is defined as that all the individuals share the same opinion. On the other hand, the ϕ_c is dependent on the FNP K and the complex system size N , which will be shown below.

What's more, we also mimic the opinion dynamics of the Sznajd model on 2-D regular lattice with periodic boundary condition and each node having four neighbors. The boundary condition here is different from that in [9]. In Fig. 2 (a), we represent the evolution of the average magnetization $m(t)$ as a function of time t on the 2-D regular lattice. Interestingly, We find that there exists a phase transition from the coexistence state to the consensus state, which also be found in [9], on the 2-D regular lattice. The larger the regular lattice is, the more difficult the system reaches the consensus state. On the other hand, the randomness is introduced in the 2-D regular lattice with the shortcut density ϕ , following the algorithm of Newman-Watts small-world network model. In Fig. 2 (b), we plot the evolution of $m(t)$ as a function of time t on 2-D small-world networks with various ϕ and the same size $N = 20 \times 20$. We find that the larger the shortcut density ϕ , the easier the system reaches the consensus state. Our results on 2-D regular lattice, along with that of [9], indicate that the dimensionality of complex system also plays an important role in the opinion dynamics of the Sznajd model.

In order to study the spatial correlation of opinion dynamics on small-world network, we study the opinion-opinion correlation on small-world network and on regular lattice (i.e., $\phi=0$). Just as the spin-spin correlation in Ising model, we define the opinion-opinion correlation function as follows:

$$g(r) = |\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle|, \text{ for } r = d_{ij}, \quad (3.2)$$

where $\langle \cdot \rangle$ means the ensemble average over at least 100 different networks — and in each of them selected all node pairs — in this paper and d_{ij} is the shortest path length from node i to node j in the underlying small-world networks. The opinion-opinion correlation, also called spatial correlation, describes the correlations in the fluctuations of the opinions s_i and s_j around their average values $\langle s_i \rangle$ and $\langle s_j \rangle$. If the average of the product $\langle s_i s_j \rangle$ is equal to the product of the averages $\langle s_i \rangle$ and $\langle s_j \rangle$, the opinion-opinion correlation function is zero. In the zero shortcut density $\phi = 0$, this is the case for the parameter $K > K_c \neq 0$, where all individuals share the same opinion, i.e., $|m| = 1$, and

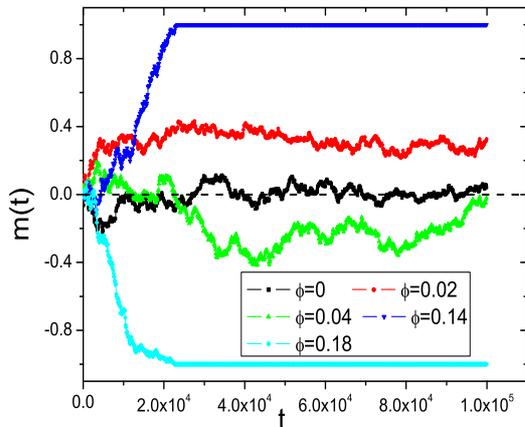


Figure 1: The plots for the evolution of magnetization $m(t)$ as a function of time t , with different shortcut density ϕ and the same initial opinion probability $p=0.5$. The five curves correspond to $\phi=0.14, 0.02, 0, 0.04$ and 0.18 , from top to bottom. The parameters of the complex network are $N=1000, K=4$.

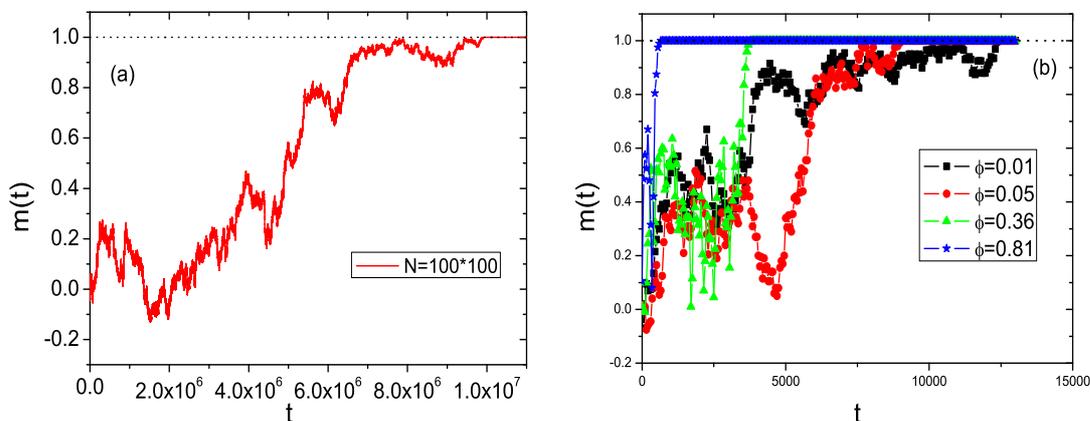


Figure 2: The plots for the evolution of magnetization $m(t)$ as a function of time t on 2-D small-world network with various shortcut density ϕ . The parameters are: (a) $\phi=0$ and $N=100 \times 100$, i.e., the 2-D regular lattice with the size $N=100 \times 100$ and each node having four neighbors; (b) $\phi=0.01, 0.05, 0.36$ and 0.81 , and the size $N=20 \times 20$.

at $K=0$, where all individuals do not exchange their opinions at all and the society is an isolated one, where individuals share the +1 opinion and -1 opinion randomly, i.e., $|m|=0$. Hence, the change between $g(r) > 0$ and $g(r) = 0$ is the fingerprint of phase transition of opinion formation on the small-world networks.

In Fig. 3 we represent the evolution of the spatial correlation $g(r)$ as a function of the shortest path length r on small-world network with various shortcut density ϕ . Comparing the evolution of the correlation function with the shortcut density $\phi=0$ and $\phi=0.01$, we find that there exists both the short- and long-range spatial correlation simultaneously on complex network with shortcuts. Namely, the nearest neighbors and the farthest neighbors share the same opinion easily on small-world network, probability caused by the small-world effects of shortest path length and larger clustering coefficient. Interest-

ingly, for intermediate r and a given K , the correlation function decreases with increasing the shortcut density ϕ as more and more individuals share the same opinion. The correlation function $g(r) = 0$ and the average magnetization $|m| = 1$ for $\phi > \phi_c$, i.e., there exists a phase transition from the coexistence state to the consensus state when $\phi > \phi_c$. The most important thing is that the result of the spatial correlation of opinion formation indicates the crucial role of the local topology and the shortcuts in complex system in opinion dynamics. What's more, comparing the evolution of $m(t)$ as a function of t on one dimensional regular lattice and on 2-D regular lattice, we find that the dimensionality of complex system also plays an important role in the opinion dynamics.

Furthermore, we also focus on the effects of the small-world network topology on opinion dynamics using the finite-size effect [25]. In order to do this, we define the probability, denoted by P_c , as the probability that the consensus state occurs on small-world network with various size N and the FNP K . During simulation, we find that the probability P_c is equal to one when the shortcut density $\phi > \phi_c$, where ϕ_c is defined as the critical shortcut density that is related to the FNP K and the small-world network size N , see the insets in Fig. 4 and Fig. 5 respectively. The larger the FNP K is and the smaller the small-world network size N is, the smaller the critical shortcut density ϕ_c will be. Namely, the small-world network reaches the consensus state easily for the smaller size N and larger FNP K .

In Fig. 4 we represent the evolution of the critical shortcut density ϕ_c as a function of the FNP K on small-world network with the same size $N = 1000$. We find that $\phi_c = 0$ when $K > K_c$, where K_c is the critical FNP that is dependent of the system size N and $K_c = 20$ for $N = 1000$. The larger the system size N is, the larger the K_c will be. In the thermodynamical limit, i.e., $N \rightarrow \infty$, the critical FNP K_c also tends to infinity. Namely, in $\phi = 0$ (i.e., the regular lattice), the system also produces a phase transition from the coexistence state to the consensus state for the FNP $K > K_c$. On the other hand, we find that there also exists a phase transition from the coexistence state when $\phi < \phi_c$ to the consensus state when the shortcut density $\phi > \phi_c$ for the FNP $3 \leq K < 20$, which shows the role of shortcuts in small-world network in opinion formation. Surprisingly, $\phi_c = 0.135(5)$ is a constant for $3 \leq K < 12$ and ϕ_c decreases with increasing K for $12 \leq K < 20$.

Fig. 5 represents the evolution of the critical shortcut density ϕ_c as a function of the network size N . For a given FNP K , the critical shortcut density ϕ_c increases as the complex size N increases. Namely, the larger the complex network is, the more difficult the consensus phase is reached. In the thermodynamic limit, i.e., $N \rightarrow \infty$, the consensus phase does not emerge at all. In a strict sense, the behavior from the coexistence state to the consensus state can not be called phase transitions since they disappear in the thermodynamic limit [25]. This behavior may be called the pseudo-phase transition. Hence, the emergence of the phase transition of opinion dynamics is dependent of the number of individuals or agents considered in a society. Furthermore, the complex system topology parameters are $K = 2, N = 1000$, and $\phi = 0$ for [8] and $K = 3, N = 61 \times 61$, and $\phi = 0$ for [10]. Their FNP K is too small to see the phase transition. Hence, our present work can explain why a phase transition was not found in [8, 10].

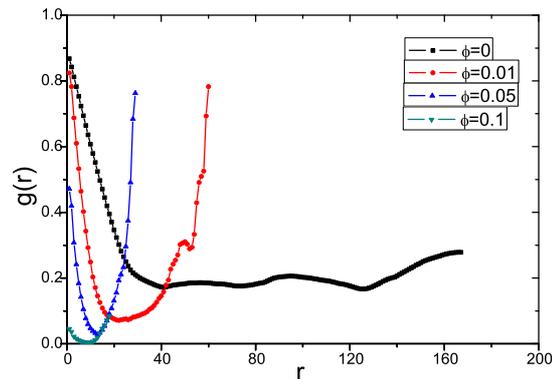


Figure 3: The plots for the evolution of spatial correlation $g(r)$ as a function of r with various shortcut density ϕ and the same initial opinion probability $p=0.5$. The four curves correspond to $\phi=0, 0.01, 0.05$ and 0.1 , from top to bottom. The parameters of the complex network are $N=1000, K=4$.

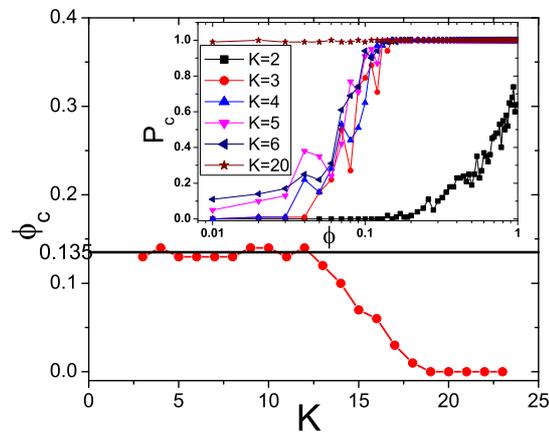


Figure 4: The plots for the critical shortcut density ϕ_c as a function of the first neighbor parameter K on complex network of the same size $N=1000$. Inset: the probability P_c of phase transition occurring versus the shortcut density ϕ with various K and the same size $N=1000$.

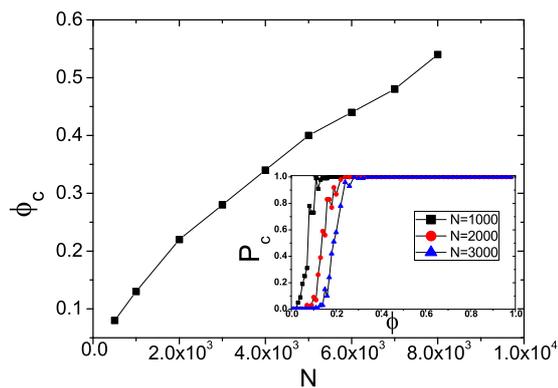


Figure 5: The plots for the critical shortcut density ϕ_c as a function of the complex network size N . The lines are guides to the eye. Inset: the probability P_c of phase transition occurring versus the shortcut density ϕ with various size $N=1000, 2000$, and 3000 , from left to right, and the same first neighbors $K=4$.

4 Conclusions

In this paper, we study the dependence of opinion formation on small-world network topology. By large-scale numerical simulations, we find that there exists a pseudo-phase transition from the coexistence state to the consensus state for the FNP $K > K_c$ ($K_c \neq 0$) in zero shortcut density (i.e., in the regular lattice). On the other hand, for a given FNP K , the spatial correlation $g(r)$ decreases with increasing shortcut density as more and more individuals share the same opinion, just as that in Ising model. For $\phi > \phi_c$, $g(r) = 0$ and $|m| = 1$, i.e., all individuals share the same opinion. Finally, we analyze the emergence of phase transition by using the finite-size effect, and find that the critical shortcut density ϕ_c is related to the small-world network size N and the FNP K . Hence, there exists the behavior of opinion dynamics from the coexistence state to the consensus state only in finite systems. The changes of behavior is called as pseudo-phase transition. Interestingly, our results can explain why a phase transition of opinion dynamics was not found in [8,10]. Our present work provides a new perspective to understand the opinion dynamics in our society.

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