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EFFECTS OF APPROXIMATE DECONVOLUTION MODELS ON THE SOLUTION OF THE STOCHASTIC NAVIER-STOKES EQUATIONS*

M. Gunzburger and A. Labovsky
Department of Scientific Computing, Florida State University, Tallahassee, FL 32306
Email: gunzburg@fsu.edu, alabovsky@fsu.edu

Abstract

The direct numerical simulation of Navier-Stokes equations in the turbulent regime is not computationally feasible either in the deterministic or (especially) in the stochastic case. Therefore, turbulent modeling must be employed. We consider the family of approximate deconvolution models (ADM) for the simulation of the turbulent stochastic Navier-Stokes equations (NSE). For moderate values of the Reynolds number, we investigate the effect stochastic forcing (through the boundary conditions) has on the accuracy of solutions of the ADM equations compared to direct numerical simulations. Although the existence, uniqueness and verifiability of the ADM solutions has already been proven in the deterministic setting, the analyticity of a solution of the stochastic NSE is difficult to prove. Hence, we approach the problem from the computational point of view. A Smolyak-type sparse grid stochastic collocation method is employed for the approximation of first two statistical moments of the solution − the expected value and variance. We show that for different test problems, the modeling error in the stochastic case is the same as predicted for the deterministic setting. Although the ADMs are arguably only applicable for certain boundary conditions (zero or periodic), we test the model on a problem with a boundary layer and recirculation region and demonstrate that the model correctly predicts the solution of the stochastic NSE with the noise in the boundary data.

Mathematics subject classification: 65M70, 76F65.
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1. Introduction

Realistic simulations of complex systems governed by nonlinear partial differential equations (in this paper we consider the case of fluid flow, described by Navier-Stokes equations (NSE)) must account for "noisy" features of modeled phenomena, such as material properties, coefficients, domain geometry, excitations and boundary data. "Noise" can be understood as uncertainties in the specification of the physical model. In an attempt to capture the noisy aspects of the system, we describe the input data on the boundary as random fields. In this work we consider the boundary data that can be described by a finite number of random variables.

Direct numerical simulation of a turbulent flow is often not computationally economical or even feasible. The problem is magnified many times over in the case of the stochastic NSE because deterministic sampling must be employed; one solves the discrete turbulent Navier-Stokes system with random coefficients many times, once for each sampling of the random data. Although these computations can be parallelized, the direct simulation in the turbulent

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case is not feasible in the foreseeable future. One has to introduce a turbulent model, as is the case for the deterministic turbulent NSE.

The largest structures in the flow (containing most of the energy) are responsible for much of the mixing and most of the momentum transport. This observation led to the development of various numerical regularizations; one of these is Large Eddy Simulation (LES) that is based on the idea that the flow can be represented by a collection of scales with different sizes and, instead of trying to approximate all of them down to the smallest one, one defines a filter width $\delta > 0$ and computes only the scales of size bigger than δ (large scales) whereas the effect of the small scales on the large scales is modeled. This reduces the number of degrees of freedom in a simulation and represents accurately the large structures in the flow.[1] In this report we consider one particular LES model, the Approximate Deconvolution Model (ADM), introduced in [1]. The model has been extensively studied in the deterministic setting; see, e.g., [11,13,15,18] and the references therein.

The ADM for the stochastic NSE with the noise on the boundary is given by

$$w_t - \frac{1}{\mathrm{Re}}\Delta w + \nabla \cdot \overline{(G_N w)(G_N w)}^{\delta} + \nabla q = \overline{f}^{\delta}, \tag{1.1a}$$

$$\nabla \cdot w = 0, \tag{1.1b}$$

subject to

$$w(0, x, \omega) = \overline{u}_0^{\delta}(x)$$

and noisy boundary conditions

$$w(t, x, \omega)|_{\partial\Omega} = \overline{u}^{\delta}(t, x)|_{\partial\Omega} + \sum_{i=1}^{K} \omega_i \Phi_i.$$

Here G_N is an approximate deconvolution operator, defined in Section 2. For the computational tests we consider the zeroth order ADM with $N = 0$.

The solution of (1.1) therefore depends on K random variables. In the computational tests we use $K = 2$ to reduce the computational cost. We assume $\Gamma_k = [-1, 1]$, $\forall k = 1, \cdots, K$, where Γ_k denotes the image of k-th random variable. We let $\Gamma^K = \prod_{k=1}^{K} \Gamma_k$; assume also that the random variables have a joint probability density function

$$\rho : \Gamma^K \to \mathbb{R}_+, \text{ with } \rho \in L^\infty(\Gamma^K).$$

For all the test problems we will assume that the given probability density functions are uniform.

Even though the ADM solution of the deterministic NSE is computationally feasible, the most popular approach to solving a partial differential equation in a probabilistic setting (the Monte Carlo method) is too costly due to a large number of sampling. Hence, in order to obtain a solution to a stochastic turbulent NSE, we need to combine the ADM turbulence model with a probabilistic method which has higher convergence rate than the Monte Carlo method.

Different methods have been proposed for solving probabilistic partial differential equations with (in certain cases) a much higher convergence rate than the Monte Carlo method. We

1) One should notice that there is always a dilemma when choosing the filtering width δ. The larger δ is, the less costly the computations are (less degrees of freedom left), but the larger the modeling error is. On the other hand, choosing δ too small makes the problem too computationally costly. Usually δ is taken to be of order h, the diameter of the mesh employed. This guarantees that the computations are feasible and at the same time we only need to model the eddies of the size smaller than the mesh diameter.

mention the Spectral Galerkin Method [5, 6], the method of Neumann expansions [2, 10], and the Stochastic Collocation Method [4, 14]. In this paper we will combine the ADM turbulence model with the stochastic collocation method to treat the finite number of random parameters in the boundary data. See [4] for the detailed description of the stochastic collocation techniques. We employ the Smolyak isotropic formulas [3, 19] for the sparse grid stochastic collocation.

For the test problem we choose the circular flow (Chorin's model) and flow past an obstacle. We compare the expected values of the (spacially averaged) velocity obtained by direct numerical simulation (with sufficiently low Reynolds number) of the noisy NSE, and the velocity obtained by 1.1. Next, we consider the case of flows with $Re = 500$ - the near turbulent regime, where the direct numerical simulation is still feasible.

Finally, we apply the model to a problem with a boundary layer and recirculation region − flow past a forward-facing step. Again, the computational results demonstrate that for flow with $Re = 100$ the convergence rates of expected value and variance are exactly as predicted by theory for the deterministic setting (for the problem with boundary layer at $Re = 500$ and with reasonably small mesh size the direct numerical simulation is no longer feasible).

2. Preliminaries

Definition 2.1 (Approximate Deconvolution Operator) *For a fixed finite N, define the Nth approximate deconvolution operator G_N by*

$$G_N\phi = \sum_{n=0}^{N}(I - A_\delta^{-1})^n\phi,$$

where the averaging operator A_δ^{-1} is the differential filter: given $\phi \in L_0^2(\Omega)$, $\overline{\phi}^\delta \in H^2(\Omega) \cap L_0^2(\Omega)$ is the unique solution of

$$A_\delta\overline{\phi}^\delta := -\delta^2\Delta\overline{\phi}^\delta + \overline{\phi}^\delta = \phi \quad \text{in } \Omega, \tag{2.1}$$

subject to periodic boundary conditions. Under periodic boundary conditions, this averaging operator commutes with differentiation.

Lemma 2.1. *The operator G_N^i is compact, positive, and is an asymptotic inverse to the filter $A_{\delta_i}^{-1}$, i.e., for very smooth ϕ and as $\delta_i \to 0$ satisfies*

$$\begin{aligned} \phi &= G_N^1\overline{\phi}^{\delta_1} + (-1)^{N+1}\delta_1^{2N+2}\Delta^{N+1}A_{\delta_1}^{-(N+1)}\phi, \\ \phi &= G_N^2\overline{\phi}^{\delta_2} + (-1)^{N+1}\delta_2^{2N+2}\Delta^{N+1}A_{\delta_2}^{-(N+1)}\phi. \end{aligned} \tag{2.2}$$

The proof of Lemma 2.1 can be found in [8].

2.1. Stochastic collocation. Smolyak formula

The detailed explanation of stochastic collocation idea and the Smolyak approximation can be viewed in [3, 16]. The idea of the collocation method is to approximate the function $w = w(y, x)$, $\forall y \in \Gamma^K, \forall x \in \Omega$.

Introduce the span of tensor product polynomials with degree at most $\mathbf{p} = (p_1, \cdots, p_K)$:

$$\mathcal{P}_{\mathbf{p}}(\Gamma^K) \subset L_\rho^2(\Gamma^K),\ \mathcal{P}_{\mathbf{p}}(\Gamma^K) = \bigotimes_{k=1}^{K}\mathcal{P}_{p_k}(\Gamma_k),$$

where $\mathcal{P}_{p_k}(\Gamma_k) = span(y_k^l, l = 0, \cdots, p_k), k = 1, \cdots, K$. Here

$$L^2_\rho(\Gamma^K) = \left\{ v \;\middle|\; \int_{\Gamma^K} \|v(\omega,\cdot)\|^2_{L^2(\Omega)} d\rho(\omega) < \infty \right\}.$$

Stochastic collocation provides the approximation to the solution $w(y)$ on a suitable set of points $y_k \in \Gamma^K$. The fully discrete solution $w_{h,\mathbf{p}}$ is then a global interpolation

$$w_{h,\mathbf{p}}(y,\cdot) = \sum_{k\in\mathcal{K}} w_h(y_k,\cdot) l_k^{\mathbf{p}}(y),$$

where the functions $l_k^{\mathbf{p}}$ can be Lagrange polynomials.

Also, we will use this formulation to compute the first moments (mean value and variance) of w, since the expected value of functionals $\psi(w)$ could be approximated (see [4], [16]) as

$$E[\psi(w)] \approx E[\psi(w_{h,\mathbf{p}})] \approx \sum_{k\in\mathcal{K}} \psi(w_h(y_k)) E[l_k^{\mathbf{p}}].$$

Let $i \in \mathbb{N}_+$ be an index. For each value of i, let $\{y_1^i, \cdots, y_{m_i}^i\} \subset [-1,1]$ be a sequence of abscissas for Lagrange interpolation in $[-1,1]$.

For any Banach space $W(\Omega)$ of functions $v : \Omega \to \mathbb{R}$ we introduce a sequence of one-dimensional Lagrange interpolation operators $\mathcal{U}^i$ in a corresponding stochastic Banach space $\mathcal{C}^0(\Gamma^1; W(\Omega))$ as follows:

$$\mathcal{U}^i(w)(y) = \sum_{j=1}^{m_i} w(y_j^i) l_j^i(y), \qquad \forall w \in \mathcal{C}^0(\Gamma^1; W(\Omega)),$$

where $l_j^i \in \mathcal{P}_{m_i-1}(\Gamma^1)$ are the Lagrange polynomials of degree $m_i - 1$:

$$l_j^i(y) = \prod_{k=1, k\neq j}^{m_i} \frac{(y - y_k^i)}{(y_j^i - y_k^i)}.$$

In the multivariate case $K > 1$, for each $w \in \mathcal{C}^0(\Gamma^1; W(\Omega))$ and the multi-index $\mathbf{i} = (i_1, \cdots, i_K)$ define the full tensor product interpolation formulas

$$\mathcal{I}_{\mathbf{i}}^K w(y) = (\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_K})(w)(y) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_K=1}^{m_{i_K}} w(y_{j_1}^{i_1}, \cdots, y_{j_K}^{i_K})(l_{j_1}^{i_1} \otimes \cdots \otimes l_{j_K}^{i_K}). \tag{2.3}$$

The above product needs $\prod_{k=1}^K m_{i_k}$ function evaluations, i.e., solutions of deterministic Navier-Stokes equations. The Smolyak isotropic formulas $\mathcal{A}(m,K)$ are linear combinations of (2.3), but only the products with a relatively small number of points are used. Set $\mathcal{U}^0 = 0$; for $i \in \mathbb{N}_+$ define

$$\Delta^i = \mathcal{U}^i - \mathcal{U}^{i-1}.$$

Given an integer level $m \in \mathbb{N}_+$ and for $\mathbf{i} \in \mathbb{N}_+$ with $|\mathbf{i}| = i_1 + \cdots + i_K$ define the sets

$$X(m,K) = \left\{ \mathbf{i} \in \mathbb{N}_+^K, \mathbf{i} \geq \mathbf{1} : \sum_{k=1}^K (i_k - 1) \leq m \right\}, \tag{2.4}$$

$$\tilde{X}(m,K) = \left\{ \mathbf{i} \in \mathbb{N}_+^K, \mathbf{i} \geq \mathbf{1} : \sum_{k=1}^K (i_k - 1) = m \right\}, \tag{2.5}$$

$$Y(m,K) = \left\{ \mathbf{i} \in \mathbb{N}_+^K, \mathbf{i} \geq \mathbf{1} : m - K + 1 \leq \sum_{k=1}^K (i_k - 1) \leq m \right\}. \tag{2.6}$$

The isotropic Smolyak formula is given by

$$\mathcal{A}(m,K) = \sum_{\mathbf{i}\in X(m,K)} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_K}).$$

To compute $\mathcal{A}(m,K)(w)$, one only needs to know function values on the sparse grid

$$\mathcal{H}(m,K) = \bigcup_{\mathbf{i}\in Y(m,K)} (\nu^{i_1} \times \cdots \times \nu^{i_K}) \subset [-1,1]^K,$$

where $\nu^i = \{y_1^i, \cdots, y_{m_i}^i\} \subset [-1,1]$ denotes the set of abscissas used by $\mathcal{U}^i$. We will use the nested sets (see Section 2.2), therefore $\mathcal{H}(m,K) \subset \mathcal{H}(m+1,K)$ and

$$\mathcal{H}(m,K) = \bigcup_{\mathbf{i}\in \tilde{X}(m,K)} (\nu^{i_1} \times \cdots \times \nu^{i_K}).$$

The Smolyak approximation (especially with nested a bscissas) requires much less function evaluations than the tensor product formula (2.3), which is very important in the case of turbulent NSE. In the computational tests we consider the problem with two random variables ($K=2$) and we employ stochastic collocation method with Smolyak approximation at the level $m=2$.

2.2. Interpolation abscissas

We use Clenshaw-Curtis abscissas, which are the extrema of Chebyshev polynomials. The number of abscissas on the i-th level is m_i. One sets $m_1 = 1$ and $m_i = 2^{i-1}+1$ for $i>1$. Then the abscissas are given by

$$y_j^1 = 0;\ y_j^i = -\cos\left(\frac{\pi(j-1)}{m_i-1}\right), \qquad \forall j = 1,\cdots,m_i,\ i>1.$$

With this choice the sets of abscissas are nested (see [7] for details).

3. Computational Results

In this section we test the verifiability of the approximate deconvolution model for the stochastic Navier-Stokes equations by looking into the first moments (expected value and variance) of the quantity $w_{ADM} - \bar{u}_{DNS}$, where w_{ADM} is the velocity field computed by the model, and $\bar{u}_{DNS}$ is the average of the DNS solution of the NSE. The solution obtained by the ADM in the deterministic setting, was proven to be second order accurate (see, e.g., [15] and references therein). Hence, we anticipate the expected value $E(w_{ADM} - \bar{u}_{DNS})$ at the final time T to be second order accurate

$$\left\| E(w_{ADM}(T) - \bar{u}_{DNS}(T)) \right\|_{L_2(\Omega)} \le C(Re,f)h^2, \tag{3.1}$$

and therefore the variance to be fourth order accurate:

$$\left\| var(w_{ADM} - \bar{u}_{DNS}, \mathbf{x}, T) \right\|_{L_2(\Omega)} \le C(Re,f)h^4. \tag{3.2}$$

Note that the expressions in the right hand sides of (3.1) – (3.2) should be $C\delta^2$ and $C\delta^4$ respectively, but we have chosen the filtering width $\delta = h$, as was discussed in the footnote in the introduction.

The computations are made for the two-dimensional problem. The results presented are obtained by using the software $FreeFEM++$ [17]. The velocity field is sought in the finite element space of piecewise quadratic polynomials, and the pressure in the space of piecewise linears.

First, consider the fluid flow in $\Omega = (0.5, 1.5) \times (0.5, 1.5)$. The flow is laminar with Reynolds number $Re = 100$, the final time is $T = 1$, and the filtering radius is set to be the mesh size $\delta = h$.

For the Chorin's model (circular motion in a square) we take

$$f = \begin{pmatrix} \frac{1}{2}\pi \sin(2\pi x) e^{-4\pi^2 t/Re} \\ \frac{1}{2}\pi \sin(2\pi y) e^{-4\pi^2 t/Re} \end{pmatrix}.$$

The noisy boundary conditions are $w_{\partial\Omega_i} = u_{\partial\Omega_i} + \alpha_i \Phi_i$, $i = 1, \cdots, 4$ where

$$\begin{aligned} \partial\Omega_1 &= \big\{(x, y) \bigm| 0.5 \le x \le 1.5, y = 0.5\big\}, \\ \partial\Omega_2 &= \big\{(x, y) \bigm| x = 1.5, 0.5 \le y \le 1.5\big\}, \\ \partial\Omega_3 &= \big\{(x, y) \bigm| 0.5 \le x \le 1.5, y = 1.5\big\}, \\ \partial\Omega_4 &= \big\{(x, y) \bigm| x = 0.5, 0.5 \le y \le 1.5\big\}. \end{aligned}$$

The value $u_{\partial\Omega_i}$ is obtained from the value of the known true solution of the deterministic problem with $\alpha_i = 0, \forall i$:

$$u(x, y, t) = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) e^{-2\pi^2 t/Re} \\ \sin(\pi x)\cos(\pi y) e^{-2\pi^2 t/Re} \end{pmatrix}.$$

We let $\alpha_2 = \alpha_4 = 0$, so that the parameter space is $\Gamma^2 = \{(\alpha_1, \alpha_3)\} \subset [-1, 1] \times [-1, 1]$. This way we still need to implement the stochastic collocation method (the Monte-Carlo method is computationally expensive, since solving each deterministic NSE is costly), but the number of sample deterministic problems is moderate. We introduce noise on the boundary by choosing

$$\Phi_1(x, y) = \begin{pmatrix} 0 \\ (x - \frac{1}{2})(\frac{3}{2} - x) \end{pmatrix}, \qquad \Phi_3(x, y) = \begin{pmatrix} 0 \\ -(x - \frac{1}{2})(\frac{3}{2} - x) \end{pmatrix}.$$

We compute the Smolyak approximation with Clenshaw-Curtis nested abscissas at the level $\omega = 2$ (see, e.g., [16], [4]). Using the stochastic collocation method, we obtain

$$E[w - \bar{u}_{DNS}] \approx \sum_{k \in K} (w(y_k) - \bar{u}_{DNS}(y_k)) E[\ell_k],$$

where the sum is taken over the Smolyak points $y_k \in \Gamma^2$. Finally, we use Lagrange polynomials as functions ℓ_k and we approximate the integrals in $E[\ell_k]$ by three-point Gaussian quadrature rule. Similarly, the variance $var(w - \bar{u}_{DNS}, \mathbf{x}, T)$ is computed as

$$var\big((w - \bar{u}_{DNS}), \mathbf{x}, T\big) = E\Big(\big(\big(w(T) - \bar{u}_{DNS}(T)\big) - E\big(w(T) - \bar{u}_{DNS}(T)\big)\big)^2\Big).$$

The convergence rates of the error in the case of laminar flow ($Re = 100$) are

This verifies our prediction of the expected convergence rates (3.1),(3.2). Now we test the convergence rates on the near-turbulent flow, with $Re = 500$. The obtained results are shown in Table 3.2.

Table 3.1: L_2 norm of expected value and variance, $\mathbf{Re} = \mathbf{100}$.

h	$\|E[w(T) - \bar{u}_{DNS}(T)]\|_{L^2(\Omega)}$	$rate(E)$	$\|var[(w - \bar{u}_{DNS}), \mathbf{x}, T]\|_{L^2(\Omega)}$	$rate(var)$
1/4	0.238434		0.120709	
1/8	0.104785	1.186	0.0185519	2.702
1/16	0.0327235	1.679	0.00190266	3.285
1/32	0.00856659	1.934	0.000137505	3.79
1/64	0.00216618	**1.984**	0.00000894536	**3.942**

Table 3.2: L_2 norm of expected value and variance, $\mathbf{Re} = \mathbf{500}$.

h	$\|E[w(T) - \bar{u}_{DNS}(T)]\|_{L^2(\Omega)}$	$rate(E)$	$\|var[(w - \bar{u}_{DNS}), \mathbf{x}, T]\|_{L^2(\Omega)}$	$rate(var)$
1/4	0.461581		0.3470543	
1/8	0.16789	1.459	0.052419	2.727
1/16	0.0578959	1.536	0.00617369	3.086
1/32	0.0162982	1.829	0.00055579	3.474
1/64	0.0042095	**1.953**	0.000037463	**3.891**

Thus, as the Reynolds number increases the convergence rates increase slower, but we still obtain the desired rates.

Next we want to verify these results for a two dimensional flow past the obstacle. Consider $\Omega = [0,1] \times [0,1]$ with the obstacle $(0.25 + 0.05\cos(t), 0.5 + 0.05\sin(t))$, $t \in [0, 2\pi)$, and the parabolic inflow at the left boundary $\partial\Omega_1 = \{(x,y) | x = 0, 0 \le y \le 1\}$. We introduce the two-parameter noise on the left (inflow) boundary

$$\begin{aligned} w_{\partial\Omega_1} &= u_{\partial\Omega_1} + \alpha_1\Phi_1 + \alpha_2\Phi_2, \\ w_{\partial\Omega_i} &= u_{\partial\Omega_i}, \qquad i = 2,3,4. \end{aligned}$$

Thus, we work in a parameter space $\Gamma^2 = (\alpha_1, \alpha_2) \subset [-1,1] \times [-1,1]$, and we choose

$$\Phi_1(x,y) = \begin{pmatrix} \epsilon y(1-y) \\ 0 \end{pmatrix}, \qquad \Phi_2(x,y) = \begin{pmatrix} \epsilon y(1-y)^2 \\ 0 \end{pmatrix},$$

with $\epsilon = 0.001$. The right hand side is

$$f(x,y) = \begin{pmatrix} 0 \\ 2xy - x + 1 \end{pmatrix},$$

the pressure is $p = xy^2 - xy - 2\nu t x + y$.

As in the previous example, the value $u_{\partial\Omega_i}$ is obtained from the value of the known true solution of the deterministic problem with $\alpha_i = 0, \forall i$:

$$u(x,y,t) = \begin{pmatrix} y(1-y)t \\ 0 \end{pmatrix}.$$

The expected value and variance of the modeling error $w(T) - \bar{u}_{DNS}(T)$ are computed for flow past the obstacle at the Reynolds numbers $Re = 100$ and $Re = 500$. As before, we conclude that convergence rates verify those predicted by the theory.

Table 3.3: Flow past the obstacle, $\mathbf{Re} = \mathbf{100}$.

h	$\\|E[w(T) - \bar{u}_{DNS}(T)]\\|_{L^2(\Omega)}$	$rate(E)$	$\\|var[(w - \bar{u}_{DNS}), \mathbf{x}, T]\\|_{L^2(\Omega)}$	$rate(var)$
1/4	0.0245412		0.000408932	
1/8	0.0175452	0.513	0.000212643	1.041
1/16	0.00780092	1.198	0.0000422653	2.128
1/32	0.0024091	1.659	0.00000394668	3.401
1/64	0.000458815	**2.015**	0.00000016644	**3.989**

Table 3.4: Flow past the obstacle, $\mathbf{Re} = \mathbf{500}$.

h	$\\|E[w(T) - \bar{u}_{DNS}(T)]\\|_{L^2(\Omega)}$	$rate(E)$	$\\|var[(w - \bar{u}_{DNS}), \mathbf{x}, T]\\|_{L^2(\Omega)}$	$rate(var)$
1/4	0.055516		0.00232432	
1/8	0.0452287	0.296	0.00178408	0.382
1/16	0.0276592	0.709	0.000754306	1.242
1/32	0.0111982	1.304	0.000142444	2.408
1/64	0.0030934	**1.856**	0.0000108547	**3.714**

At $Re = 500$ the convergence rates increase slower, similar to the case of Chorin's model.

Finally, consider the parabolic flow in the domain with the step (Fig. 3.1). The size of the domain is $[0, 1] \times [0, 1]$ and the height of the step is 0.2. We consider the same right hand side and the parabolic inflow, as in the previous test problem (flow past the obstacle); however, the step guarantees the presence of a boundary layer.

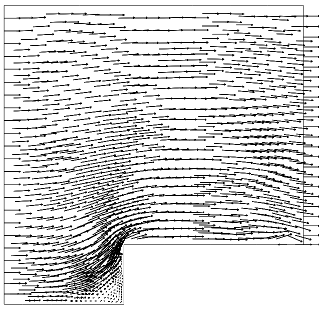

Fig. 3.1. Flow past the step.

We set the zero boundary conditions except the left boundary (parabolic inflow with noise) and the right boundary (outflow). Thus, on the left boundary $\partial\Omega_1 = \{(x, y)|x = 0, 0 \leq y \leq 1\}$ we introduce the same inflow as in the case of the flow past the obstacle, with two random parameters:

$$w_{\partial\Omega_1} = u_{\partial\Omega_1} + \alpha_1\Phi_1 + \alpha_2\Phi_2,$$

where

$$\Phi_1(x, y) = \begin{pmatrix} \epsilon y(1-y) \\ 0 \end{pmatrix}, \qquad \Phi_2(x, y) = \begin{pmatrix} \epsilon y(1-y)^2 \\ 0 \end{pmatrix},$$

with $\epsilon = 0.001$.

The direct numerical simulation at $Re = 500$ for flow past the step is no longer feasible with the iterative solver we employ; at $h = 1/32$ the solver fails to converge within the time constraints imposed (although the Approximate Deconvolution Model still works and provides approximate solutions even in the case of Reynolds numbers of the order $Re = 10^6$).

At the Reynolds number $Re = 100$ the direct numerical computation is still feasible. The results demonstrate faster increase (compare with Table 3.3) in the convergence rates to the values predicted by the theory.

Table 3.5: Flow past the step, $\mathbf{Re} = \mathbf{100}$.

h	$\Vert E[w(T) - \bar{u}_{DNS}(T)]\Vert_{L^2(\Omega)}$	$rate(E)$	$\Vert var[(w - \bar{u}_{DNS}), \mathbf{x}, T]\Vert_{L^2(\Omega)}$	$rate(var)$
1/4	0.0724682		0.00319198	
1/8	0.0297043	1.287	0.000543116	2.555
1/16	0.00947135	1.649	0.0000580259	3.226
1/32	0.00259538	1.868	0.00000437377	3.956
1/64	0.00065426	**1.988**	0.0000002741196	**3.996**

Hence, the computational results verify the claimed accuracy of the model, even in the case of probabilistic setting with the noise in boundary data.

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