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A FINITE ELEMENT METHOD WITH RECTANGULAR PERFECTLY MATCHED LAYERS FOR THE SCATTERING FROM CAVITIES*

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Abstract

We develop a finite element method with rectangular perfectly matched layers (PMLs) for the wave scattering from two-dimensional cavities. The unbounded computational domain is truncated to a bounded one by using of a rectangular perfectly matched layer at the open aperture. The PML parameters such as the thickness of the layer and the fictitious medium property are determined through sharp a posteriori error estimates. Numerical experiments are carried out to illustrate the competitive behavior of the proposed method.

Mathematics subject classification: 35Q60, 65L60, 78A45.
Key words: Cavity, Perfectly matched layers, Finite element method.

1. Introduction

Consider a time-harmonic electromagnetic plane wave incident on a shaped open cavity embedded in an infinite ground plane. The ground plane and the walls of the open cavity are perfect electric conductors (PEC), and the interior of the open cavity is filled with a non-magnetic inhomogeneous material. The half-space above the ground plane is filled with a homogeneous, linear, isotropic medium characterized by its permittivity ε_0 and permeability μ_0. In the TM and TE polarization, we study the diffraction problem by a finite element method with rectangular perfectly matched absorbing layers. Several computational experiments indicate that the method is efficient.

The study of the wave scattering by a 2-D cavity-backed aperture in the infinite ground plane has been of great importance in aircraft industries. There has been a considerable interest in computation and design of cavities, see, e.g., [4, 12, 16]. However, there has not been much studied on the analysis of the problem. Recently, in [1, 2], Ammari and Bao developed a variational approach for solving the cavity problems in two- and three-dimensional media, and studied the well-posedness of the problem. They also investigated the problem by an integral equation method in [3]. In [18], we introduced a perfectly matched layer in curvilinear coordinates to study the locally perturbed half plane problems (including the cavity problems), and presented several numerical results.

The purpose of this paper is to develop efficient numerical methods for solving the cavity scattering problems. The main difficulty is to truncate the infinite domain into a bounded computational domain. The method studied in [1, 2] is based on a variational formulation in the cavity with a transparent boundary condition at the open aperture. The boundary operators are nonlocal, which yields some difficulties in practical computations. In [18], we overcome the

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difficulty by introducing the PML in curvilinear coordinates. However, for wide open cavities it will lead to large computational costs.

The purpose of this paper is twofold: First we explore the possibility of introducing a rectangular perfectly matched layer to deal with the difficulty in truncating the unbounded domain. Second we explore the possibility of using an error analysis to determine the PML parameters such as the thickness of the PML region and the medium property inside the region. We hope the ideas developed in this paper will be useful for solving other locally perturbed half plane problems.

The basic idea of the PML technique is to surround the computational domain by a finite thickness layer of the specially designed model medium that would attenuate outgoing waves propagating from the computational domain. Since Berenger proposed the PML method for the time dependent Maxwell equations in [6], various constructions of PML absorbing layers have been proposed and studied in the literature. In [8], for the wave scattering by bounded obstacles, Collino and Monk derived the perfectly matched layer in curvilinear coordinates. Subsequently, Chen and Liu [7] established the convergence theory of the PML method to the solution of the original problem. We refer to Turkel and Yefet [13] for a review on various proposed models, and Lassas and Somersalo [10] for some study of mathematical properties of the PML equations.

The layout of the paper is as follows. In the next section, we state the model problem and derive the variational formulations. The well-posedness of the variational problems is also studied. In Section 3, we introduce our PML formulations, and establish the existence, uniqueness and convergence of the PML formulations. In Section 4, we present several numerical examples to illustrate the competitive behavior of the method.

2. Two-dimensional Cavity Problem

Consider a two-dimensional cavity D of arbitrary cross section embedded in a perfectly conducting medium (see Fig 2.1). Above the line $\{x_2 = 0\}$, the medium is homogeneous with a positive dielectric coefficient ε_0. The medium inside D is inhomogeneous with dielectric coefficient $\varepsilon(x_1, x_2)$. We assume that $\mathrm{Re}\,\varepsilon(x_1, x_2) > 0$ and $\mathrm{Im}\,\varepsilon(x_1, x_2) \geq 0$. In this paper, the media are assumed to be non-magnetic, and the magnetic permeability μ_0 is constant. We are interested in the scattering of an incident plane wave by the cavity.

We denote by Γ the cavity aperture, and S the cavity walls. Let $\mathrm{R}^2_+ = \{x \in \mathrm{R}^2 : x = (x_1, x_2),\ x_2 > 0\}$ be the region above the ground plane, and $\Gamma^c = \partial\mathrm{R}^2_+ \backslash \Gamma$. Let n be the unit outward normal to ∂D. Denote in the whole space

$$k^2(x) = \begin{cases} \omega^2 \varepsilon_0 \mu_0 & \text{in } \mathrm{R}^2_+, \\ \omega^2 \varepsilon(x) \mu_0 & \text{in } D. \end{cases}$$

With the perfectly electric conducting boundary condition in mind, we investigate the TM and TE cases separately.

2.1. TM Polarization

In this case, the incident electric field and the total electric field are parallel to the invariant dimension, i.e., $E_I = (0, 0, u^i)$ and $E = (0, 0, u)$. By the field continuity conditions, u vanishes

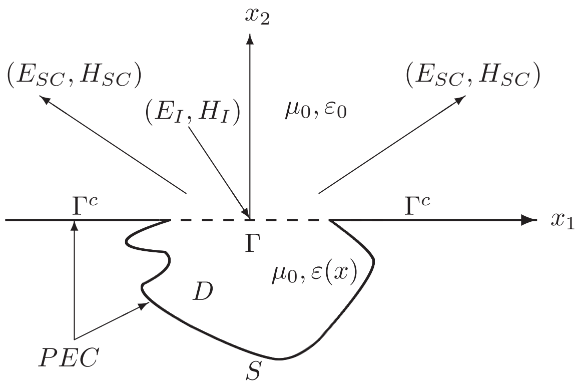

Fig. 2.1. The geometry for the cavity problem

on S and Γ^c, and is continuous over Γ. Moreover, since the media is nonmagnetic, $\frac{\partial u}{\partial n}$ is also continuous over Γ. Therefore, u satisfies

$$(\Delta + k^2)u = 0 \qquad \text{in } \; D \cup \mathrm{R}^2_+, \tag{2.1a}$$

$$u = 0 \qquad \text{on } \; \Gamma^c \cup S, \tag{2.1b}$$

$$u, \frac{\partial u}{\partial n} \qquad \text{are continuous on } \Gamma. \tag{2.1c}$$

Along with the radiation condition,

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial \eta} - \mathbf{i}k_0 u^s\right) = 0, \tag{2.2}$$

where u^s is the scattered field, $r = |x|, \eta = x/|x|$ and $k_0 = \omega^2\varepsilon_0\mu_0$.

Assume the incident field $u^i = e^{\mathbf{i}\alpha x_1 - \mathbf{i}\beta x_2}$. Here $\alpha = k_0 \sin\theta$, $\beta = k_0 \cos\theta$, and $-\pi/2 < \theta < \pi/2$ is the incident angle. Denote $v = u - u^i + u^\rho$, where $u^\rho = e^{\mathbf{i}\alpha x_1 + \mathbf{i}\beta x_2}$. Hence, the scattering problem is to find v such that

$$(\Delta + k^2)v = g \qquad \text{in } D \cup \mathrm{R}^2_+, \tag{2.3a}$$

$$v = h \qquad \text{on } \Gamma^c \cup S, \tag{2.3b}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\tfrac{\partial v}{\partial \eta} - \mathbf{i}k_0 v\right) = 0, \tag{2.3c}$$

where $g = (k^2 - k_0^2)(u^\rho - u^i)$, $h = u^\rho - u^i$. It is clear that $g = 0$ in R^2_+, and $h = 0$ on Γ^c.

By the radiation condition and the boundary condition, we see that in R^2_+ the field v can be expressed as

$$v(x) = 2\int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} v(y) ds(y),$$

where

$$G(x,y) = \frac{\mathbf{i}}{4} H_0^1\left(k_0\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}\right)$$

is the fundamental solution of the Helmholtz equation with wavenumber k_0, and $H_0^{(1)}$ is the Hankel function of the first kind with order zero. Therefore, we know that

$$\left.\frac{\partial v}{\partial n}\right|_{x_2=0^+} = 2\frac{\partial}{\partial n(x)} \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} v(y) ds(y) \qquad x \in \Gamma. \tag{2.4}$$

We introduce the space

$$\widetilde{H}^{1/2}(\Gamma) = \left\{ w \in H^{1/2}(\mathrm{R});\ \mathrm{supp} w \subset \Gamma \right\}.$$

From (2.4), we define the Dirichlet-Neumann mapping $T : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ by

$$T(\phi)(x) = 2\frac{\partial}{\partial n(x)} \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y) ds(y).$$

Hence, we have the following boundary condition:

$$\frac{\partial v}{\partial n} = T(v) \qquad \text{on } \Gamma. \tag{2.5}$$

We present some important properties of the boundary operator T (see [15]).

Lemma 2.1. *(i) The mapping $T : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is continuous. (ii) There exists a constant $\gamma > 0$ and a compact mapping K_0 from $\widetilde{H}^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$, such that for all $\phi \in \widetilde{H}^{1/2}(\Gamma)$*

$$Re((-T + K_0)\phi, \phi)_{L^2(\Gamma)} \geq \gamma \parallel \phi \parallel^2_{\widetilde{H}^{1/2}(\Gamma)}.$$

By combining (2.3a) and (2.5), the scattering problem (2.1a)-(2.2) can be reformulated as follows: to find v such that

$$(\Delta + k^2)v = g \qquad \text{in } D, \tag{2.6a}$$

$$v = h \qquad \text{on } S, \tag{2.6b}$$

$$\frac{\partial v}{\partial n} - T(v) = 0 \qquad \text{on } \Gamma. \tag{2.6c}$$

Now, we introduce the following equivalent variational formulation of (2.6a): Given g and h as above, find $v \in H^1(D)$ such that $v = h$ on S, $v \in \widetilde{H}^{1/2}(\Gamma)$, and

$$a(v, \psi) = -\int_D g\,\bar{\psi}\,dx \qquad \forall \psi \in \widetilde{H}^1_0(D), \tag{2.7}$$

where

$$a(\varphi, \psi) = \int_D \left(\nabla\varphi \cdot \nabla\bar{\psi} - k^2 \varphi\bar{\psi}\right) dx - \int_\Gamma (T\varphi)\bar{\psi}\,ds,$$

$$\widetilde{H}^1_0(D) = \left\{ w \in H^1(D) : w = 0 \text{ on } S, w \in \widetilde{H}^{1/2}(\Gamma) \right\}.$$

The following theorem is our main result on the existence and uniqueness of a solution to the variational problem (2.7).

Theorem 2.1. *The variational problem* (2.7) *admits a unique solution v in $H^1(D)$.*

Proof. From Lemma 2.1, we see that the Fredholm Alternative Theorem can be applied to (2.7).

Next, we prove the uniqueness. It is sufficient to show that if $g = 0$ and $h = 0$, then the solution v must vanish in D. Suppose $g = 0$ and $h = 0$. Let v be a solution of (2.7), i.e., $a(v, \psi) = 0$ for all $\psi \in \widetilde{H}^1_0(D)$. Hence, Im $a(v, v) = 0$. Further, from Im$\varepsilon \geq 0$, we get

$$\mathrm{Im} \int_\Gamma (Tv)\bar{v}\,ds \leq 0. \tag{2.8}$$

Let

$$w(x) = 2\int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} v(y) ds(y), \qquad x \in \mathrm{R}^2_+.$$

Then, it can be verified that w satisfies

$$\begin{aligned} (\Delta + k_0^2)w = 0 \quad & \text{in } \mathrm{R}^2_+, \\ w = 0 \quad & \text{on } \Gamma^c, \\ w = v,\ \tfrac{\partial w}{\partial n} = Tv \quad & \text{on } \Gamma, \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial \eta} - \mathbf{i}k_0 w\right) = 0. & \end{aligned}$$

From (2.8), we know that

$$\operatorname{Im} \int_\Gamma \frac{\partial w}{\partial n} \bar{w}\, ds \le 0.$$

Following the proof of Rellich Lemma (see [9]), we conclude that $w = 0$ in R^2_+. We also notice that $\frac{\partial v}{\partial n} = Tv = 0$ on Γ. A unique continuation result in [11] concludes that $v = 0$ in D. This completes the proof. □

The general theory in Babuška and Aziz [5] implies that there exists a constant $\chi > 0$ such that the following inf-sup condition holds:

$$\sup_{0 \ne \psi \in \widetilde{H}_0^1(D)} \frac{|a(\varphi,\psi)|}{||\psi||_{H^1(D)}} \ge \chi\, ||\varphi||_{H^1(D)}, \quad \forall\, \varphi \in \widetilde{H}_0^1(D). \tag{2.9}$$

2.2. TE Polarization

For this case, the incident magnetic field and the total magnetic field are parallel to the invariant dimension, i.e., $H_I = (0,0,u^i)$ and $H = (0,0,u)$. As in the TM case, we assume that $u^i = e^{\mathbf{i}\alpha x_1 - \mathbf{i}\beta x_2}$. By the perfectly electric conducting boundary conditions and the field continuity conditions, $\frac{\partial u}{\partial \tilde{n}}$ vanishes on S and Γ^c, and u, $\frac{1}{k^2(x)}\frac{\partial u}{\partial n}$ are continuous over Γ. Here, $\tilde{n}$ is the unit outward normal to $D \cup \mathrm{R}^2_+$. Therefore, u satisfies

$$\begin{aligned} \nabla \cdot \left(\tfrac{1}{k^2(x)} \nabla u\right) + u = 0 \quad & \text{in } D \cup \mathrm{R}^2_+, \\ \tfrac{\partial u}{\partial \tilde{n}} = 0 \quad & \text{on } \Gamma^c \cup S, \end{aligned} \tag{2.10}$$

and along with the radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial \eta} - \mathbf{i}k_0 u^s\right) = 0. \tag{2.11}$$

Set $v = u - u^i - u^\rho$. Then v satisfies (2.10) in the upper half plane with the boundary condition $\frac{\partial v}{\partial \tilde{n}} = 0$ on Γ^c. By the definition of v, solving the scattering problem is equivalent to finding the function v. By the radiation condition and the boundary condition, we see that in R^2_+ the field v can be expressed as

$$v(x) = -2\int_\Gamma G(x,y) \frac{\partial v}{\partial n}(y) ds(y).$$

Denote by $\widetilde{H}^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$, and define the operator $K:\widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ by

$$K(\psi)(x) = -2\int_\Gamma G(x,y)\psi(y)ds(y).$$

From [15], we present some important properties of the boundary operator K.

Lemma 2.2. *(i) The mapping $K:\widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is continuous. (ii) There exists a constant $\gamma > 0$ and a compact mapping K_0 from $\widetilde{H}^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$, such that for all $\psi \in \widetilde{H}^{-1/2}(\Gamma)$*

$$\mathrm{Re}((-K+K_0)\psi,\psi)_{L^2(\Gamma)} \geq \gamma \parallel \psi \parallel^2_{\widetilde{H}^{-1/2}(\Gamma)} .$$

From the transparent boundary condition for u and $\frac{\partial u}{\partial n}$ over Γ, the scattering problem (2.10)-(2.11) can be stated as

$$\nabla\cdot\left(\frac{1}{k^2(x)}\nabla v\right) + v = g \qquad \text{in } D, \tag{2.12a}$$

$$\frac{\partial v}{\partial n} = \frac{\partial h}{\partial n} \qquad \text{on } S, \tag{2.12b}$$

$$v = K\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\frac{\partial v}{\partial n}\right) \qquad \text{on } \Gamma, \tag{2.12c}$$

where

$$h = -u^i - u^\rho, g = \nabla\cdot\left(\frac{1}{k^2(x)}\nabla h\right) + h, \ \ \varepsilon_\Gamma = \varepsilon(x)|_\Gamma .$$

By using the Green's formula, we find that

$$b_1(v,\psi) - \langle\lambda,\psi\rangle = F(\psi) \quad \forall\,\psi \in H^1(D),$$

where

$$\lambda = \left.\frac{\partial v}{\partial n}\right|_\Gamma, \qquad b_1(\varphi,\psi) = \int_D\left(\frac{1}{k^2(x)}\nabla\varphi\cdot\nabla\bar{\psi} - \varphi\bar{\psi}\right)dx,$$

$$\langle\lambda,\psi\rangle = \int_\Gamma \frac{1}{k^2(x)}\lambda\bar{\psi}ds, \qquad F(\psi) = -\int_D g\,\bar{\psi}\,dx + \int_S \frac{1}{k^2(x)}\frac{\partial h}{\partial n}\bar{\psi}\,ds.$$

Moreover, we have that

$$b_2(\lambda,\mu) + \langle\mu,v\rangle = 0 \quad \forall\mu \in \widetilde{H}^{-1/2}(\Gamma),$$

where

$$b_2(\lambda,\mu) = -\int_\Gamma \frac{1}{k^2(x)}\mu\overline{K\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\lambda\right)}ds.$$

Denote $v_\lambda = (v,\lambda)$ and $\psi_\mu = (\psi,\mu)$. We introduce the following equivalent variational formulation of (2.12a): Given g and h, find $v_\lambda \in W = H^1(D)\times\widetilde{H}^{-1/2}(\Gamma)$ such that

$$b(v_\lambda,\psi_\mu) = F(\psi) \quad \forall\,\psi_\mu \in W, \tag{2.13}$$

where

$$b(\varphi_\eta,\psi_\mu) = b_1(\varphi,\psi) - \langle\eta,\psi\rangle + \langle\mu,\varphi\rangle + b_2(\eta,\mu).$$

Now, we have the following existence and uniqueness result.

Theorem 2.2. *The variational problem* (2.13) *admits a unique solution* v_λ *in* W.

Proof. First, we introduce the continuous sesquilinear forms

$$B, L : W \times W \to \mathrm{R},$$

and the corresponding continuous linear mappings

$$B, L : W \to W',$$

defined by

$$\begin{aligned} B(\varphi_\eta, \psi_\mu) &\equiv [B\varphi_\eta, \psi_\mu] \equiv b(\varphi_\eta, \psi_\mu) - L(\varphi_\eta, \psi_\mu), \\ L(\varphi_\eta, \psi_\mu) &\equiv [L\varphi_\eta, \psi_\mu] \equiv -2\int_D \varphi_\eta \bar{\psi}_\mu dx - \int_\Gamma \frac{1}{k^2(x)} \overline{\mu K_0\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\eta\right)} ds, \end{aligned}$$

where $[\cdot,\cdot]$ denotes the duality between W and W'. By lemma 2.2, we see that

$$|B(\psi_\mu, \psi_\mu)| \geq C\|\psi_\mu\|_W^2 \quad \forall \psi_\mu \in W,$$

where $\|\cdot\|_W$ denotes the norm in W, i.e.,

$$\|\psi_\mu\|_W = \left(\|\psi\|^2_{H^1(D)} + \|\mu\|^2_{\tilde{H}^{-1/2}(\Gamma)}\right)^{1/2}.$$

By the theorem of Lax and Milgram, the mapping $B : W \to W'$ is an isomorphism. Further, from the compactness of K_0 and the compact embedding of $H^1(D)$ into $L^2(D)$ it is easily shown that the mapping $L : W \to W'$ is compact. Therefore, the Fredholm alternative is applicable to (2.13).

Next, we prove the uniqueness. It is sufficient to show that if $g = 0$ and $h = 0$, then the solution v_λ must vanish. Suppose $g = 0$ and $h = 0$. Let v_λ be a solution of (2.13), i.e., $b(v_\lambda, \psi_\mu) = 0$ for all $\psi_\mu \in W$. Hence, Im $b(v_\lambda, v_\lambda) = 0$. From Im$\varepsilon \geq 0$, we know

$$\mathrm{Im} \int_\Gamma \frac{1}{k^2(x)} \overline{\lambda K\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\lambda\right)} ds \leq 0. \tag{2.14}$$

Let

$$w(x) = -2\int_\Gamma G(x,y)\frac{\varepsilon_0}{\varepsilon_\Gamma}\lambda(y)ds(y), \qquad x \in \mathrm{R}^2_+.$$

Then, it is easy to see that w satisfies

$$\left(\Delta + k_0^2\right) w = 0 \quad \text{in } \mathrm{R}^2_+, \tag{2.15a}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma^c, \tag{2.15b}$$

$$w = K\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\lambda\right), \frac{\partial w}{\partial n} = \frac{\varepsilon_0}{\varepsilon_\Gamma}\lambda \quad \text{on } \Gamma, \tag{2.15c}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\tfrac{\partial w}{\partial \eta} - \mathbf{i}k_0 w\right) = 0. \tag{2.15d}$$

From (2.14), we know that

$$\mathrm{Im} \int_\Gamma \frac{\partial w}{\partial n}\bar{w}\, ds \leq 0.$$

By a similar proof to the TM case, we get that $w = 0$ in R^2_+. Since $v = K\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\frac{\partial v}{\partial n}\right)$, we obtain $v = 0$ on Γ. By the unique continuation result in [11], we conclude that $v = 0$ in D. This completes the proof. □

Similarly, there exists a constant $\chi > 0$ such that the following inf-sup condition holds:

$$\sup_{0\neq\psi_\mu\in W}\frac{|b(\varphi_\eta,\psi_\mu)|}{\|\psi_\mu\|_W} \geq \chi\ \|\varphi_\eta\|_W \qquad \forall\ \varphi_\eta \in W. \tag{2.16}$$

3. The PML Formulation

In this section we shall introduce variational formulations for the scattering problem using the PML technique. We shall study the TM polarization first, and then the TE polarization.

3.1. TM Polarization

We set a PML layer $\Omega^{\mathrm{PML}} = \{(x_1,x_2) : a_1-\delta_1 \leq x_1 \leq a_2+\delta_1, 0 \leq x_2 \leq \delta_2\}$ at the open aperture of the cavity (see Fig 3.1). Let

$$s_1(x_1) = 1 + \mathbf{i}\sigma_1(x_1)/\omega \ \ and\ \ s_2(x_2) = 1 + \mathbf{i}\sigma_2(x_2)/\omega$$

be the model medium property which satisfy $\sigma_1, \sigma_2 \in C(\mathrm{R})$, $\sigma_1, \sigma_2 \geq 0$, and

$$\sigma_1(x_1) = 0 \ \text{ for } a_1 \leq x_1 \leq a_2, \qquad \sigma_2(x_2) = 0 \ \text{ for } x_2 \leq 0.$$

We introduce the PML equation

$$\frac{\partial}{\partial x_1}\left(\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial v}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial v}{\partial x_2}\right) + s_1(x_1)s_2(x_2)k_0^2 v = 0 \quad \text{in } \Omega^{\mathrm{PML}}.$$

From [8], the Green function for this equation is given by

$$\hat{G}(x_1,x_2) = \frac{\mathbf{i}}{4}H_0^1\left(k_0\sqrt{\tilde{x}_1^2+\tilde{x}_2^2}\right),$$

where

$$\tilde{x}_1 = x_1 + \frac{\mathbf{i}}{\omega}\int_0^{x_1}\sigma_1(s)ds, \quad \tilde{x}_2 = x_2 + \frac{\mathbf{i}}{\omega}\int_0^{x_2}\sigma_2(s)ds.$$

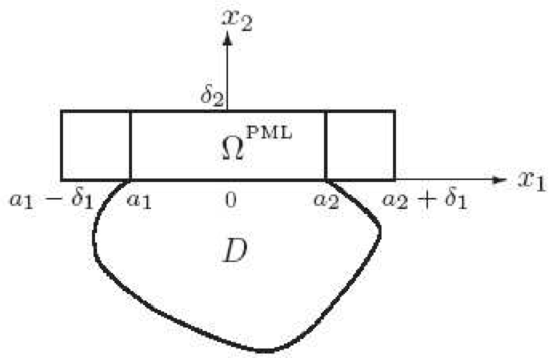

Fig. 3.1. Setting of the scattering problem with the PML layer

The PML solution $\hat{v}$ in $\Omega = D \cup \Omega^{\rm PML}$ is defined as the solution of the following system

$$\frac{\partial}{\partial x_1}\left(\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial \hat{v}}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial \hat{v}}{\partial x_2}\right) + s_1(x_1)s_2(x_2)k^2\hat{v} = \hat{g} \quad \text{in } \Omega, \tag{3.1}$$

$$\hat{v} = h \quad \text{on } S, \qquad \hat{v} = 0 \quad \text{on } \partial\Omega \backslash S. \tag{3.2}$$

Here, $\hat{g} = g$ in D, $\hat{g} = 0$ in $\Omega^{\rm PML}$, and $k = k_0$ in $\Omega^{\rm PML}$.

This problem can be reformulated in the bounded domain D by imposing the boundary condition

$$\left.\frac{\partial \hat{v}}{\partial n}\right|_\Gamma = \hat{T}\hat{v},$$

where the operator $\hat{T} : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is defined as follows: given $f \in \widetilde{H}^{1/2}(\Gamma)$,

$$\hat{T}f = \left.\frac{\partial \zeta}{\partial n}\right|_\Gamma,$$

where $\zeta \in H^1(\Omega^{\rm PML})$ satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial \zeta}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial \zeta}{\partial x_2}\right) + s_1(x_1)s_2(x_2)k_0^2\zeta = 0 \quad \text{in } \Omega^{\rm PML}, \tag{3.3}$$

$$\zeta = f \quad \text{on } \Gamma, \qquad \zeta = 0 \quad \text{on } \Gamma^{\rm PML}, \tag{3.4}$$

where $\Gamma^{\rm PML} = \partial\Omega^{\rm PML}\backslash\Gamma$. The existence and uniqueness of the solutions of the PML problem (3.3)-(3.4) will be studied in the Section 3.1.1.

Based on the operator $\hat{T}$, we introduce the sesquilinear form

$$\hat{a}(\varphi, \psi) = \int_D (\nabla\varphi \cdot \nabla\bar{\psi} - k^2\varphi\bar{\psi})dx - \int_\Gamma (\hat{T}\varphi)\bar{\psi}\, ds.$$

Then the weak formulation for (3.1)-(3.2) is: Given g and h, find $\hat{v} \in H^1(D)$ such that $\hat{v} = h$ on S, $\hat{v} \in \widetilde{H}^{1/2}(\Gamma)$, and

$$\hat{a}(\hat{v}, \psi) = -\int_D g\,\bar{\psi}\, dx, \quad \forall\, \psi \in \widetilde{H}^1_0(D). \tag{3.5}$$

The well-posedness of the PML problem (3.5) and the convergence of its solution to the solution of the original scattering problem (2.7) will be studied in the Section 3.1.2.

3.1.1. The PML equation in the layer

In this subsection we consider the Dirichlet problem

$$\frac{\partial}{\partial x_1}\left(\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial w}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial w}{\partial x_2}\right) + s_1(x_1)s_2(x_2)k_0^2 w = 0 \quad \text{in } \Omega^{\rm PML}, \tag{3.6}$$

$$w = 0 \quad \text{on } \Gamma, \qquad w = q \quad \text{on } \Gamma^{\rm PML}, \tag{3.7}$$

where $q \in \widetilde{H}^{1/2}(\Gamma^{\rm PML})$. Let $\hat{a}^{\rm PML} : H^1(\Omega^{\rm PML}) \times H^1(\Omega^{\rm PML}) \to \mathbb{C}$ be the sesquilinear form:

$$\hat{a}^{\rm PML}(\varphi, \psi) = \int_{\Omega^{\rm PML}} \left(\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial \varphi}{\partial x_1}\frac{\partial \bar{\psi}}{\partial x_1} + \frac{s_1(x_1)}{s_2(x_2)}\frac{\partial \varphi}{\partial x_2}\frac{\partial \bar{\psi}}{\partial x_2} - s_1(x_1)s_2(x_2)k_0^2\varphi\bar{\psi}\right) dx.$$

Then the weak formulation for (3.6)-(3.7) is as follows: Given $q \in \widetilde{H}^{1/2}(\Gamma^{\mathrm{PML}})$, find $w \in H^1(\Omega^{\mathrm{PML}})$ such that $w=0$ on Γ, $w=q$ on Γ^{PML}, and

$$\hat{a}^{\mathrm{PML}}(w,\psi)=0 \quad \forall\,\psi\in H_0^1(\Omega^{\mathrm{PML}}). \tag{3.8}$$

In this paper, we make some general assumptions on the medium property σ_1 and σ_2:

$$\sigma_1(x_1)=\begin{cases}\sigma_0\left(\frac{a_1-x_1}{\delta_1}\right)^m & a_1-\delta_1\le x_1\le a_1,\\ 0 & a_1\le x_1\le a_2,\\ \sigma_0\left(\frac{x_1-a_2}{\delta_1}\right)^m & a_2\le x_1\le a_2+\delta_1,\end{cases} \tag{3.9a}$$

$$\sigma_2(x_2)=\begin{cases}\sigma_0\left(\frac{x_2}{\delta_2}\right)^m & 0\le x_2\le \delta_2,\\ 0 & x_2\le 0,\end{cases} \tag{3.9b}$$

where the constant $\sigma_0>1$ and the integer $m\ge 2$. From (3.9a) and definition of s_1 and s_2, we have

$$\mathrm{Re}\left(\frac{s_2}{s_1}\right)=\frac{1+\frac{\sigma_1\sigma_2}{\omega^2}}{1+\frac{\sigma_1^2}{\omega^2}},\quad \mathrm{Re}\left(\frac{s_1}{s_2}\right)=\frac{1+\frac{\sigma_1\sigma_2}{\omega^2}}{1+\frac{\sigma_2^2}{\omega^2}},\quad \mathrm{Re}(s_1s_2)=1-\frac{\sigma_1\sigma_2}{\omega^2},$$
$$\frac{1+\frac{\sigma_1\sigma_2}{\omega^2}}{1+\frac{\sigma_1^2}{\omega^2}}\ge\frac{1}{1+\frac{\sigma_1^2}{\omega^2}}\ge|s_0|^{-2},\qquad \frac{1+\frac{\sigma_1\sigma_2}{\omega^2}}{1+\frac{\sigma_2^2}{\omega^2}}\ge\frac{1}{1+\frac{\sigma_2^2}{\omega^2}}\ge|s_0|^{-2},$$

where $s_0=1+\mathbf{i}\sigma_0/\omega$. Hence,

$$\begin{aligned}\mathrm{Re}\left[\hat{a}^{\mathrm{PML}}(\varphi,\varphi)\right]&=\int_{\Omega^{\mathrm{PML}}}\left[\frac{1+\frac{\sigma_1\sigma_2}{\omega^2}}{1+\frac{\sigma_1^2}{\omega^2}}\left|\frac{\partial\varphi}{\partial x_1}\right|^2+\frac{1+\frac{\sigma_1\sigma_2}{\omega^2}}{1+\frac{\sigma_2^2}{\omega^2}}\left|\frac{\partial\varphi}{\partial x_2}\right|^2+\left(\frac{\sigma_1\sigma_2}{\omega^2}-1\right)k_0^2|\varphi|^2\right]dx\\ &\ge|s_0|^{-2}\|\nabla\varphi\|^2_{L^2(\Omega^{\mathrm{PML}})}-k_0^2\|\varphi\|^2_{L^2(\Omega^{\mathrm{PML}})}.\end{aligned}$$

By using the analytic Fredholm alternative theorem we know that the PML problem (3.8) admits a unique solution for all but possibly a discrete set of values of k (see, e.g., the argument in [8, Theorem 2]). We will not elaborate on this issue and simply assume that there exists a unique solution to the PML problem (3.8). Then the general theory in Babuška and Aziz [5, Chapter 5] implies that there exists a positive constant $\hat{C}$ such that the following inf-sup condition holds:

$$\sup_{0\ne\psi\in H_0^1(\Omega^{\mathrm{PML}})}\frac{|\hat{a}^{\mathrm{PML}}(\varphi,\psi)|}{||\psi||_{H^1(\Omega^{\mathrm{PML}})}}\ge\hat{C}\,||\varphi||_{H^1(\Omega^{\mathrm{PML}})}\quad\forall\,\varphi\in H_0^1(\Omega^{\mathrm{PML}}). \tag{3.10}$$

Without loss of generality we assume $\hat{C}\le 1$.

Remark 3.1. Generally, the coercivity constant $\hat{C}$ depends on σ_0. We make the technical assumption that $\hat{C}^{-1}<p_l(\sigma_0)$, where p_l is some polynomial of degree l. The assumption will make sense in the convergence of the PML problem, and the numerical experiments indicate that the assumption is reasonable. We make the same assumption on $\hat{C}$ in (3.29).

We lay out the following main result of this subsection.

Theorem 3.1. *There exists a constant $C > 0$ independent of k_0 and σ_0 such that the following estimates are satisfied*

$$\|w\|_{H^1(\Omega^{\rm PML})} \leq C\hat{C}^{-1}|s_0|\big(1+k_0^2|s_0|\big)||q||_{H^{1/2}(\Gamma^{\rm PML})}, \tag{3.11}$$

$$\left\|\frac{\partial w}{\partial n}\right\|_{H^{-1/2}(\Gamma)} \leq C\hat{C}^{-1}|s_0|^2\big(1+k_0^2|s_0|\big)^2||q||_{H^{1/2}(\Gamma^{\rm PML})}. \tag{3.12}$$

Proof. From (3.9a), we know that $1 \leq |s_1|, |s_2| \leq |s_0|$. Therefore, we have

$$\begin{aligned}|\hat{a}^{\rm PML}(\varphi,\psi)| \leq & \left(\int_{\Omega^{\rm PML}} |s_2|\left|\frac{\partial\varphi}{\partial x_1}\right|^2 dx\right)^{1/2}\left(\int_{\Omega^{\rm PML}} |s_2|\left|\frac{\partial\psi}{\partial x_1}\right|^2 dx\right)^{1/2}\\ & + \left(\int_{\Omega^{\rm PML}} |s_1|\left|\frac{\partial\varphi}{\partial x_2}\right|^2 dx\right)^{1/2}\left(\int_{\Omega^{\rm PML}} |s_1|\left|\frac{\partial\varphi}{\partial x_2}\right|^2 dx\right)^{1/2}\\ & + \left(\int_{\Omega^{\rm PML}} k_0^2|s_1|^2|\varphi|^2 dx\right)^{1/2}\left(\int_{\Omega^{\rm PML}} k_0^2|s_2|^2|\psi|^2 dx\right)^{1/2}\\ \leq & |s_0|\big(1+k_0^2|s_0|\big)||\varphi||_{H^1(\Omega^{\rm PML})}||\psi||_{H^1(\Omega^{\rm PML})}.\end{aligned}$$

Now we turn to the proof the estimate (3.11). Let $\mathcal{R} : H^{1/2}(\partial\Omega^{\rm PML}) \to H^1(\Omega^{\rm PML})$ denote a right inverse of the trace mapping $\upsilon \mapsto \upsilon|_{\partial\Omega^{\rm PML}}$ and $\varpi = \mathcal{R}(\tilde{q})$, where $\tilde{q} = 0$ on Γ and $\tilde{q} = q$ on $\Gamma^{\rm PML}$. It is obvious that the function $w - \varpi \in H_0^1(\Omega^{\rm PML})$, and for any $\psi \in H_0^1(\Omega^{\rm PML})$

$$\begin{aligned}&|\hat{a}^{\rm PML}(w-\varpi,\psi)| = |\hat{a}^{\rm PML}(\varpi,\psi)|\\ \leq & |s_0|\big(1+k_0^2|s_0|\big)||\varpi||_{H^1(\Omega^{\rm PML})}||\psi||_{H^1(\Omega^{\rm PML})}.\end{aligned}$$

From the inf-sup condition (3.10), we have

$$\|w\|_{H^1(\Omega^{\rm PML})} \leq \Big(1+\hat{C}^{-1}|s_0|\big(1+k_0^2|s_0|\big)\Big)||\varpi||_{H^1(\Omega^{\rm PML})},$$

which implies (3.11).

Take any $p \in \widetilde{H}^{1/2}(\Gamma)$ and denote $\phi_p = \mathcal{R}(\tilde{p})$, where $\tilde{p} = p$ on Γ and $\tilde{p} = 0$ on $\Gamma^{\rm PML}$. To show (3.12), we multiply the equation (3.6) by ϕ_p and integrate over $\Omega^{\rm PML}$ to obtain

$$\begin{aligned}&\left|\int_\Gamma \frac{\partial w}{\partial n} p\, ds\right| = |\hat{a}^{\rm PML}(w,\bar{\phi}_p)|\\ \leq & |s_0|\big(1+k_0^2|s_0|\big)||w||_{H^1(\Omega^{\rm PML})}||\phi_p||_{H^1(\Omega^{\rm PML})}.\end{aligned}$$

Further, from (3.11) we have

$$\left|\int_\Gamma \frac{\partial w}{\partial n} p\, ds\right| \leq C\hat{C}^{-1}|s_0|^2\big(1+k_0^2|s_0|\big)^2||q||_{H^{1/2}(\Gamma^{\rm PML})}||p||_{H^{1/2}(\Gamma)}.$$

This completes the proof of the theorem. □

3.1.2. Convergence of the PML problem

In this subsection we consider the convergence of the PML problem (3.5) to the original scattering problem (2.7). For any function $f \in \widetilde{H}^{1/2}(\Gamma)$, introduce the following propagation operator $P : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{H}^{1/2}(\Gamma^{\mathrm{PML}})$:

$$P(f)(x) = 2\int_{\Gamma} \frac{\partial G(\tilde{x}, y)}{\partial n(y)} f(y) ds(y) \qquad x \in \Gamma^{\mathrm{PML}},$$

where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$. To investigate the operator, we study the H^1-norm of the following function in some domain Ω_0:

$$V(f)(x) = 2\int_{\Gamma} \frac{\partial G(\tilde{x}, y)}{\partial n(y)} f(y) ds(y).$$

Here $\Omega_0 = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\begin{aligned}
\Omega_1 &= \big\{(x_1, x_2) : a_1 - \delta_1 \le x_1 \le a_1 - \epsilon_1,\ 0 \le x_2 \le \delta_2\big\},\\
\Omega_2 &= \big\{(x_1, x_2) : a_2 + \epsilon_1 \le x_1 \le a_2 + \delta_1,\ 0 \le x_2 \le \delta_2\big\},\\
\Omega_3 &= \big\{(x_1, x_2) : a_1 - \epsilon_1 \le x_1 \le a_2 + \epsilon_1,\ \epsilon_2 \le x_2 \le \delta_2\big\}
\end{aligned}$$

with

$$\frac{\epsilon_1}{\delta_1} = \frac{\epsilon_2}{\delta_2} = c_0 \ \text{ and } \ 0 < c_0 < 1.$$

It is clear that $\Omega_0 \subset \Omega^{\mathrm{PML}}$ and $\Gamma^{\mathrm{PML}} \backslash \Gamma^c \subset \partial\Omega_0$. We need the following properties of Hankel functions ([14]):

- For $z \in \mathbb{C}$, $m \in \mathbb{Z}$ and $m \ge 1$,

$$\frac{dH_m^{(1)}(z)}{dz} = H_{m-1}^{(1)}(z) - \frac{m}{z} H_m^{(1)}(z). \tag{3.13}$$

- For $z \in \mathbb{C}$,

$$\left| H_\nu^{(1)}(z) \right| \le \gamma_0 \left| \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{\mathbf{i}\left(z - \frac{\nu}{2}\pi - \frac{1}{4}\pi\right)} \right|, \tag{3.14}$$

  where

$$\gamma_0 = \begin{cases} \left(1 - \frac{\nu - \frac{1}{2}}{2|z|}\right)^{-\nu - \frac{1}{2}} & \nu > \frac{1}{2}, \quad 2|z| > \nu - \frac{1}{2},\\ \left(1 - \frac{\nu + \frac{3}{2}}{2|z|}\right)^{-\nu - \frac{5}{2}} \left(1 + \frac{2\nu + 2}{|z|}\right) & 0 \le \nu < \frac{1}{2}, \quad 2|z| > \nu + \frac{3}{2}. \end{cases} \tag{3.15}$$

We also need the lemmas below.

Lemma 3.1. *For $x \in \Omega_0$, $a_1 \le y_1 \le a_2$ and $\sigma_0 \ge \Lambda$,*

$$\left|(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right|^{\frac{1}{2}} \ge \sigma_0 \delta \frac{\theta_0 c_0^{m+1}}{\omega(m+1)}, \tag{3.16}$$

where $0 < \theta_0 < 1$ and

$$\Lambda = (1 - \theta_0^2)^{-\frac{1}{2}} \omega(m+1) c_0^{-m-1} \left(\frac{a_2 - a_1 + \delta_1 + \delta_2}{\delta}\right).$$

Proof. In fact, from (3.9a) we know that

$$\tilde{x}_1 = \begin{cases} x_1 - \mathbf{i}\frac{\sigma_0}{\omega}\frac{\delta_1}{m+1}\left(\frac{a_1-x_1}{\delta_1}\right)^{m+1} & a_1-\delta_1 \le x_1 \le a_1, \\ x_1 & a_1 \le x_1 \le a_2, \\ x_1 + \mathbf{i}\frac{\sigma_0}{\omega}\frac{\delta_1}{m+1}\left(\frac{x_1-a_2}{\delta_1}\right)^{m+1} & a_2 \le x_1 \le a_2+\delta_1, \end{cases}$$

$$\tilde{x}_2 = \begin{cases} x_2 + \mathbf{i}\frac{\sigma_0}{\omega}\frac{\delta_2}{m+1}\left(\frac{x_2}{\delta_2}\right)^{m+1} & 0 \le x_2 \le \delta_2, \\ x_2 & x_2 \le 0. \end{cases}$$

(i) $x \in \Omega_1$. In this case, we have

$$\begin{aligned} &(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2 \\ =& \left[(x_1-y_1)^2 - \left(\frac{\sigma_0}{\omega}\frac{\delta_1}{m+1}\right)^2\left(\frac{a_1-x_1}{\delta_1}\right)^{2m+2} + x_2^2 - \left(\frac{\sigma_0}{\omega}\frac{\delta_2}{m+1}\right)^2\left(\frac{x_2}{\delta_2}\right)^{2m+2}\right] \\ &+ \mathbf{i}\left[2(y_1-x_1)\frac{\sigma_0}{\omega}\frac{\delta_1}{m+1}\left(\frac{a_1-x_1}{\delta_1}\right)^{m+1} + 2x_2\frac{\sigma_0}{\omega}\frac{\delta_2}{m+1}\left(\frac{x_2}{\delta_2}\right)^{m+1}\right]. \end{aligned}$$

Since $\epsilon_1 \le y_1 - x_1 \le a_2 - a_1 + \delta_1$ and $\epsilon_1 \le a_1 - x_1 \le \delta_1$, we deduce that for

$$\sigma_0 \ge (1-\theta_0^2)^{-\frac{1}{2}}\omega(m+1)c_0^{-m-1}\left(\frac{a_2-a_1+\delta_1+\delta_2}{\delta_1}\right)$$

and $0 < \theta_0 < 1$,

$$\begin{aligned} \mathrm{Re}\left[(\tilde{x}_1-y_1)^2 + (\tilde{x}_2)^2\right] &\le (x_1-y_1)^2 + x_2^2 - \left(\frac{\sigma_0}{\omega}\frac{\delta_1}{m+1}\right)^2\left(\frac{a_1-x_1}{\delta_1}\right)^{2m+2} \\ &< (a_2-a_1+\delta_1+\delta_2)^2 - \left(\frac{\sigma_0}{\omega}\frac{\delta_1}{m+1}\right)^2\left(\frac{\epsilon_1}{\delta_1}\right)^{2m+2} \\ &\le -\theta_0^2\left(\frac{\sigma_0}{\omega}\frac{\delta_1}{m+1}\right)^2 c_0^{2m+2}. \end{aligned}$$

Hence,

$$\left|(\tilde{x}_1-y_1)^2 + (\tilde{x}_2)^2\right| \ge \left|\mathrm{Re}\left[(\tilde{x}_1-y_1)^2 + (\tilde{x}_2)^2\right]\right| \ge \theta_0^2\left(\frac{\sigma_0}{\omega}\frac{\delta_1}{m+1}\right)^2 c_0^{2m+2}.$$

(ii) $x \in \Omega_2$. The proof is similar to (i).
(iii) $x \in \Omega_3$. Similarly, for

$$\sigma_0 \ge (1-\theta_0^2)^{-\frac{1}{2}}\omega(m+1)c_0^{-m-1}\left(\frac{a_2-a_1+\delta_1+\delta_2}{\delta_2}\right),$$

we have

$$\begin{aligned} \mathrm{Re}\left[(\tilde{x}_1-y_1)^2 + (\tilde{x}_2)^2\right] &\le (x_1-y_1)^2 + x_2^2 - \left(\frac{\sigma_0}{\omega}\frac{\delta_2}{m+1}\right)^2\left(\frac{x_2}{\delta_2}\right)^{2m+2} \\ &< (a_2-a_1+\delta_1+\delta_2)^2 - \left(\frac{\sigma_0}{\omega}\frac{\delta_2}{m+1}\right)^2\left(\frac{\epsilon_2}{\delta_2}\right)^{2m+2} \\ &\le -\theta_0^2\left(\frac{\sigma_0}{\omega}\frac{\delta_2}{m+1}\right)^2 c_0^{2m+2}. \end{aligned}$$

Hence,

$$\left|(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right| \geq \left|\mathrm{Re}\left[(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right]\right| \geq \theta_0^2 \left(\frac{\sigma_0}{\omega}\frac{\delta_2}{m+1}\right)^2 c_0^{2m+2}.$$

This completes the proof. □

A direct inspection shows that

$$|\tilde{x}_2| \leq \sigma_0 \delta_2 M, \qquad |\tilde{x}_1 - y_1| \leq \sigma_0(\delta_1 + a_2 - a_1)M, \tag{3.17}$$

where $M = \left(1 + \omega^{-2}\right)^{\frac{1}{2}}$.

Lemma 3.2. *For $x \in \Omega_0$, $a_1 \leq y_1 \leq a_2$ and $\sigma_0 \geq \Lambda$,*

$$\frac{\pi}{2} < \arg\left((\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right) < \pi. \tag{3.18}$$

Proof. (i) $x \in \Omega_1$ or $x \in \Omega_2$. From Lemma 3.1, we see that

$$\mathrm{Im}\left[(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right] \geq \sigma_0 \frac{2\epsilon_1 \delta_1}{\omega(m+1)}\left(\frac{\epsilon_1}{\delta_1}\right)^{m+1} \geq \sigma_0 \frac{2c_0^{m+2}}{\omega(m+1)}\delta^2.$$

(ii) $x \in \Omega_3$. Similarly, we have that

$$\mathrm{Im}\left[(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right] \geq \sigma_0 \frac{2\epsilon_2 \delta_2}{\omega(m+1)}\left(\frac{\epsilon_2}{\delta_2}\right)^{m+1} \geq \sigma_0 \frac{2c_0^{m+2}}{\omega(m+1)}\delta^2.$$

It follows from Lemma 3.1, that for $\sigma_0 \geq \Lambda$,

$$\mathrm{Re}\left[(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right] < 0.$$

Hence,

$$\pi/2 < \arg\left((\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right) < \pi.$$

This completes the proof. □

By Lemmas 3.1 and 3.2, we have the following result.

Lemma 3.3. *The operator $P : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{H}^{1/2}(\Gamma^{\mathrm{PML}})$ is well-defined, and for any $f \in \widetilde{H}^{1/2}(\Gamma)$ and sufficiently large σ_0,*

$$\|P(f)\|_{H^{1/2}(\Gamma^{\mathrm{PML}})} \leq C_0 \sigma_0^{\frac{1}{2}} e^{-\sigma_0 \delta \frac{k_0}{2\omega(m+1)}} \|f\|_{H^{1/2}(\Gamma)}, \tag{3.19}$$

where the positive constant C_0 is independent with σ_0.

Proof. We give an estimate of the function $V(f)$ in the H^1-norm in the domain Ω_0. For convenience, we denote $\tilde{z} = \sqrt{(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2}$. Since $z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{\mathbf{i}\frac{1}{2}\arg(z)}$ for $z \in \mathbb{C}$, from Lemmas 3.1 and 3.2 we know that for $\sqrt{2}\theta_0 c_0^{m+1} \geq 1$ and $\sigma_0 \geq \Lambda_0$,

$$\begin{aligned}\mathrm{Im}(\tilde{z}) &= \left|(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right|^{\frac{1}{2}} \sin\left(\frac{1}{2}\arg\left((\tilde{x}_1 - y_1)^2 + (\tilde{x}_2)^2\right)\right) \\ &\geq \sigma_0 \delta \frac{1}{2\omega(m+1)},\end{aligned}$$

where

$$\frac{1}{\sqrt{2}} < c_0^{m+1},\ \theta_0 < 1 \ \ and \ \ \Lambda_0 = \sqrt{\frac{2\theta_0^2}{1-\theta_0^2}}\omega(m+1)\left(\frac{a_2 - a_1 + \delta_1 + \delta_2}{\delta}\right).$$

Therefore, by (3.14)-(3.18) and

$$V(f)(x) = \frac{\mathbf{i}}{2}k_0 \int_{a_1}^{a_2} H_1^{(1)}(k_0 \tilde{z}) \frac{\tilde{x}_2}{\tilde{z}} f(y_1) dy_1,$$

we know that for $\sigma_0 \geq \max\{\Lambda_0, \frac{3\omega(m+1)}{\sqrt{2}k_0\delta}\}$,

$$\begin{aligned}
|V(f)(x)| &\leq C_1 \sigma_0^{-\frac{1}{2}} e^{-\sigma_0 \delta \frac{k_0}{2\omega(m+1)}} ||f||_{L^2(\Gamma)},\\
\left|\frac{\partial V(f)}{\partial x_1}(x)\right| &\leq \left(C_2 \sigma_0^{\frac{1}{2}} + C_3 \sigma_0^{-\frac{1}{2}}\right) e^{-\sigma_0 \delta \frac{k_0}{2\omega(m+1)}} ||f||_{L^2(\Gamma)},\\
\left|\frac{\partial V(f)}{\partial x_2}(x)\right| &\leq \left(C_4 \sigma_0^{\frac{1}{2}} + C_5 \sigma_0^{-\frac{1}{2}}\right) e^{-\sigma_0 \delta \frac{k_0}{2\omega(m+1)}} ||f||_{L^2(\Gamma)}.
\end{aligned}$$

Hence, we deduce that for $\sigma_0 \geq \max\{\Lambda_0, \frac{3\omega(m+1)}{\sqrt{2}k_0\delta}\}$,

$$\|V(f)\|_{H^1(\Omega_0)} \leq C_6 \sigma_0^{\frac{1}{2}} e^{-\sigma_0 \delta \frac{k_0}{2\omega(m+1)}} ||f||_{L^2(\Gamma)}.$$

Further, by the trace theorem and $P(f) = 0$ on Γ^c we know that

$$\|P(f)\|_{H^{1/2}(\Gamma^{\mathrm{PML}})} \leq C_0 \sigma_0^{\frac{1}{2}} e^{-\sigma_0 \delta \frac{k_0}{2\omega(m+1)}} \|f\|_{H^{1/2}(\Gamma)}.$$

This completes the proof. □

Furthermore, we have the following estimate.

Lemma 3.4. *For any $f \in \widetilde{H}^{1/2}(\Gamma)$ and sufficiently large σ_0,*

$$\|Tf - \hat{T}f\|_{H^{-1/2}(\Gamma)} \leq C\hat{C}^{-1}|s_0|^{\frac{5}{2}}\left(1 + k_0^2|s_0|\right)^2 e^{-\sigma_0 \delta \frac{k_0}{2\omega(m+1)}} \|f\|_{H^{1/2}(\Gamma)},$$

where the positive constant C is independent with σ_0.

Proof. For any $f \in \widetilde{H}^{1/2}(\Gamma)$, we know that

$$Tf - \hat{T}f = \left.\frac{\partial \varpi}{\partial n}\right|_{\Gamma},$$

where $\varpi \in H^1(\Omega^{\mathrm{PML}})$ satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial \varpi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial \varpi}{\partial x_2}\right) + s_1(x_1)s_2(x_2)k_0^2 \varpi = 0 \quad \text{in } \Omega^{\mathrm{PML}},$$

$$\varpi = 0 \quad \text{on } \Gamma, \qquad \varpi = P(f) \quad \text{on } \Gamma^{\mathrm{PML}}.$$

By (3.12) and (3.19) we deduce that

$$\begin{aligned}
\left\|\frac{\partial \varpi}{\partial n}\right\|_{H^{-1/2}(\Gamma)} &\leq C\hat{C}^{-1}|s_0|^2\left(1 + k_0^2|s_0|\right)^2 \|P(f)\|_{H^{1/2}(\Gamma^{\mathrm{PML}})}\\
&\leq C\hat{C}^{-1}|s_0|^{\frac{5}{2}}\left(1 + k_0^2|s_0|\right)^2 e^{-\sigma_0 \delta \frac{k_0}{2\omega(m+1)}} \|f\|_{H^{1/2}(\Gamma)}.
\end{aligned}$$

This completes the proof. □

The following theorem is the main result of this section.

Theorem 3.2. *For sufficiently large σ_0, the PML problem* (3.5) *has a unique solution $\hat{v} \in H^1(D)$. Moreover, we have the following estimate*

$$\|v - \hat{v}\|_{H^1(D)} \leq C\hat{C}^{-1}|s_0|^{\frac{5}{2}}\left(1 + k_0^2|s_0|\right)^2 e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}}\|\hat{v}\|_{H^{1/2}(\Gamma)}. \tag{3.20}$$

Proof. The existence of a unique solution for (3.5) follows from Lemma 3.4 by using the same argument as in [17, Theorem 5.1]. Next, by (2.7) and (3.5), we have

$$a(v - \hat{v}, \psi) = \hat{a}(\hat{v}, \psi) - a(\hat{v}, \psi) = \int_\Gamma (T\hat{v} - \hat{T}\hat{v})\bar{\psi} ds \quad \forall \psi \in \widetilde{H}_0^1(D).$$

This completes the proof of the theorem upon using Lemma 3.4 and (2.9). □

3.2. TE Polarization

In this subsection we state the corresponding results for problem (2.12a). The PML solution $\hat{v}$ in $\Omega = D \cup \Omega^{\text{PML}}$ is defined as the solution of the following system

$$\frac{\partial}{\partial x_1}\left(\frac{1}{k^2(x)}\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial v}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{k^2(x)}\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial v}{\partial x_2}\right) + s_1(x_1)s_2(x_2)v = \hat{g} \quad \text{in } \Omega, \tag{3.21}$$

$$\frac{\partial \hat{v}}{\partial n} = \frac{\partial h}{\partial n} \quad \text{on } S, \qquad \frac{\partial \hat{v}}{\partial n} = 0 \quad \text{on } \partial\Omega \setminus S, \tag{3.22}$$

where, $\hat{g} = \nabla \cdot \left(\frac{1}{k^2(x)}\nabla h\right) + h$ in D, $\hat{g} = 0$ in Ω^{PML}, $h = -u^i - u^\rho$ and $k = k_0$ in Ω^{PML}.

This problem can be reformulated in the bounded domain D by imposing the boundary condition

$$\hat{v} = \hat{K}\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\frac{\partial \hat{v}}{\partial n}\bigg|_\Gamma\right),$$

where the operator $\hat{K} : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is defined as follows: given $f \in \widetilde{H}^{-1/2}(\Gamma)$,

$$\hat{K}f = \xi|_\Gamma .$$

Here $\xi \in H^1(\Omega^{\text{PML}})$ satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{1}{k_0^2}\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial \xi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{k_0^2}\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial \xi}{\partial x_2}\right) + s_1(x_1)s_2(x_2)\xi = 0 \quad \text{in } \Omega^{\text{PML}}, \tag{3.23}$$

$$\frac{\partial \xi}{\partial n} = f \quad \text{on } \Gamma, \qquad \frac{\partial \xi}{\partial n} = 0 \quad \text{on } \Gamma^{\text{PML}}. \tag{3.24}$$

The existence and uniqueness of the solutions of the PML problem (3.23)-(3.24) will be studied in the subsection 3.2.1.

Based on the operator $\hat{K}$, we introduce the sesquilinear form

$$\hat{b}(\varphi_\eta, \psi_\mu) = b_1(\varphi, \psi) - \langle \eta, \psi \rangle + \langle \mu, \varphi \rangle + \hat{b}_2(\eta, \mu),$$

where

$$\hat{b}_2(\eta, \mu) = -\int_\Gamma \frac{1}{k^2(x)}\mu\overline{\hat{K}\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\eta\right)} ds.$$

Then the weak formulation for (3.21)-(3.22) is: Given g and h, find $\hat{v}_{\hat{\lambda}} \in W$ such that

$$\hat{b}(\hat{v}_{\hat{\lambda}}, \psi_\mu) = F(\psi) \quad \forall\, \psi_\mu \in W. \tag{3.25}$$

The well-posedness of the PML problem (3.25) and the convergence of its solution to the solution of the original scattering problem (2.13) will be studied in the section 3.2.2.

3.2.1. The PML equation in the layer

In this subsection we consider the boundary value problem

$$\frac{\partial}{\partial x_1}\left(\frac{1}{k_0^2}\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial w}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{1}{k_0^2}\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial w}{\partial x_2}\right)+s_1(x_1)s_2(x_2)w=0 \quad \text{in } \Omega^{\mathrm{PML}}, \tag{3.26}$$

$$\frac{\partial w}{\partial n}=0 \quad \text{on } \Gamma, \qquad \frac{\partial w}{\partial n}=q \quad \text{on } \Gamma^{\mathrm{PML}}, \tag{3.27}$$

where $q\in \widetilde{H}^{-1/2}(\Gamma^{\mathrm{PML}})$. Let $\hat{b}^{\mathrm{PML}}: H^1(\Omega^{\mathrm{PML}})\times H^1(\Omega^{\mathrm{PML}})\to\mathbb{C}$ be the sesquilinear form

$$\hat{b}^{\mathrm{PML}}(\varphi,\psi)=\int_{\Omega^{\mathrm{PML}}}\left(\frac{1}{k_0^2}\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial\varphi}{\partial x_1}\frac{\partial\bar{\psi}}{\partial x_1}+\frac{1}{k_0^2}\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial\varphi}{\partial x_2}\frac{\partial\bar{\psi}}{\partial x_2}-s_1(x_1)s_2(x_2)\varphi\bar{\psi}\right)dx.$$

Then the weak formulation for (3.26)-(3.27) is as follows: Given $q\in\widetilde{H}^{-1/2}(\Gamma^{\mathrm{PML}})$, find $w\in H^1(\Omega^{\mathrm{PML}})$ such that

$$\hat{b}^{\mathrm{PML}}(w,\psi)=\frac{1}{k_0^2}\int_{\Gamma^{\mathrm{PML}}}q\,\bar{\psi}ds \qquad \forall\,\psi\in H^1(\Omega^{\mathrm{PML}}). \tag{3.28}$$

By using the same argument as in Section 3.1.1 we know that the PML problem (3.28) admits a unique solution for all but possibly a discrete set of values of k. We also assume that there exists a unique solution to the PML problem (3.28). Then the general theory in Babuška and Aziz [5, Chapter 5] implies that there exists a constant $\hat{C}$ such that the following inf-sup condition holds:

$$\sup_{0\neq\psi\in H^1(\Omega^{\mathrm{PML}})}\frac{|\hat{b}^{\mathrm{PML}}(\varphi,\psi)|}{||\psi||_{H^1(\Omega^{\mathrm{PML}})}}\geq\hat{C}\ ||\varphi||_{H^1(\Omega^{\mathrm{PML}})} \quad \forall\,\varphi\in H^1(\Omega^{\mathrm{PML}}). \tag{3.29}$$

Without loss of generality we assume $\hat{C}\leq 1$.

We have the following main result of this subsection.

Theorem 3.3. *There exists a constant $C>0$ independent of σ_0 such that the following estimates are satisfied:*

$$\|w\|_{H^{1/2}(\Gamma)}\leq C\hat{C}^{-1}||q||_{\widetilde{H}^{-1/2}(\Gamma^{\mathrm{PML}})}. \tag{3.30}$$

Proof. From (3.28), we have

$$|\hat{b}^{\mathrm{PML}}(w,\psi)|\leq\frac{1}{k_0^2}\|q\|_{\widetilde{H}^{-1/2}(\Gamma^{\mathrm{PML}})}\|\psi\|_{H^{1/2}(\Gamma^{\mathrm{PML}})}.$$

This completes the proof of the theorem upon using the trace theorem and (3.29). □

3.2.2. Convergence of the PML problem

In this subsection we consider the convergence of the PML problem (3.25) to the original scattering problem (2.13). For any function $f\in\widetilde{H}^{-1/2}(\Gamma)$, introduce the following propagation operator $Q:\widetilde{H}^{-1/2}(\Gamma)\to\widetilde{H}^{-1/2}(\Gamma^{\mathrm{PML}})$:

$$Q(f)(x)=-2\int_{\Gamma}\frac{\partial G(\tilde{x},y)}{\partial n(x)}f(y)ds(y) \qquad x\in\Gamma^{\mathrm{PML}},$$

where $\tilde{x}=(\tilde{x}_1,\tilde{x}_2)$. We have the following result.

Lemma 3.5. *The operator* $Q : \widetilde{H}^{-1/2}(\Gamma) \to \widetilde{H}^{-1/2}(\Gamma^{\mathrm{PML}})$ *is well-defined, and for any* $f \in \widetilde{H}^{-1/2}(\Gamma)$ *and sufficiently large* σ_0,

$$\|Q(f)\|_{\widetilde{H}^{-1/2}(\Gamma^{\mathrm{PML}})} \leq C_0|s_0|^{\frac{1}{2}}e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}}\|f\|_{\widetilde{H}^{-1/2}(\Gamma)}, \tag{3.31}$$

where the positive constant C_0 *is independent with* σ_0.

Proof. Let $\widetilde{\Gamma} = \overline{\Gamma^{\mathrm{PML}}\backslash\Gamma^c}$. It is readily to see that

$$\|Q(f)\|_{H^{-1/2}(\Gamma^{\mathrm{PML}})} \leq \|Q(f)\|_{L^2(\Gamma^{\mathrm{PML}})} \leq (a_2 - a_1 + 2\delta_1 + 2\delta_2)^{\frac{1}{2}}\|Q(f)\|_{L^\infty(\widetilde{\Gamma})},$$

and

$$|Q(f)(x)| \leq 2\left\|\frac{\partial G(\tilde{x},\cdot)}{\partial n(x)}\right\|_{H^{1/2}(\Gamma)}\|f\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq 2\left\|\frac{\partial G(\tilde{x},\cdot)}{\partial n(x)}\right\|_{H^1(\Gamma)}\|f\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

From Lemma 3.1 and 3.2 we know that for $x \in \widetilde{\Gamma}$ and $\sigma_0 \geq \Lambda$,

$$\mathrm{Im}(\tilde{z}) \geq \sigma_0\delta\frac{1}{2\omega(m+1)}.$$

Here $\Lambda = \sqrt{2}\omega(m+1)\left(\frac{a_2-a_1+\delta_1+\delta_2}{\delta}\right)$. Therefore, by (3.14)-(3.18) and

$$\frac{\partial G(\tilde{x},y)}{\partial n(x)} = -\frac{\mathbf{i}}{4}k_0H_1^{(1)}(k_0\tilde{z})\frac{(s_1(x_1)(\tilde{x}_1 - y_1), s_2(x_2)\tilde{x}_2)}{\tilde{z}}\cdot n(x),$$

for $\sigma_0 \geq \max\{\Lambda, \frac{\omega(m+1)}{\sqrt{2}k_0\delta}\}$, we have,

$$\left|\frac{\partial G(\tilde{x},y)}{\partial n(x)}\right| \leq C_1|s_0|^{\frac{1}{2}}e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}},$$
$$\left|\frac{\partial}{\partial y_1}\left(\frac{\partial G(\tilde{x},y)}{\partial n(x)}\right)\right| \leq C_2|s_0|^{\frac{1}{2}}e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}},$$

where C_1 and C_2 are independent of σ_0. Hence, we deduce that

$$\left\|\frac{\partial G(\tilde{x},\cdot)}{\partial n(x)}\right\|_{H^1(\Gamma)} \leq C_3|s_0|^{\frac{1}{2}}e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}}.$$

This completes the proof of the lemma. $\square$

Furthermore, we have the following estimate.

Lemma 3.6. *For any* $f \in \widetilde{H}^{-1/2}(\Gamma)$ *and sufficiently large* σ_0,

$$\|Kf - \hat{K}f\|_{H^{1/2}(\Gamma)} \leq C\hat{C}^{-1}|s_0|^{\frac{1}{2}}e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}}\|f\|_{\widetilde{H}^{-1/2}(\Gamma)},$$

where the positive constant C *is independent with* σ_0.

Proof. For any $f \in \widetilde{H}^{-1/2}(\Gamma)$, we know that

$$Kf - \hat{K}f = \varpi|_{\Gamma},$$

where $\varpi \in H^1(\Omega^{\mathrm{PML}})$ satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{1}{k_0^2}\frac{s_2(x_2)}{s_1(x_1)}\frac{\partial \varpi}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{1}{k_0^2}\frac{s_1(x_1)}{s_2(x_2)}\frac{\partial \varpi}{\partial x_2}\right)+s_1(x_1)s_2(x_2)\varpi=0 \quad \text{in } \Omega^{\mathrm{PML}},$$

$$\frac{\partial \varpi}{\partial n}=0 \quad \text{on } \Gamma, \qquad \frac{\partial \varpi}{\partial n}=Q(f) \quad \text{on } \Gamma^{\mathrm{PML}}.$$

By (3.30) and (3.31) we deduce that

$$\begin{aligned}\|\varpi\|_{H^{1/2}(\Gamma)} &\le C\hat{C}^{-1}\|Q(f)\|_{\widetilde{H}^{-1/2}(\Gamma^{\mathrm{PML}})}\\ &\le C\hat{C}^{-1}|s_0|^{\frac{1}{2}}e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}}\|f\|_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

This completes the proof. □

The following theorem is the main result of this subsection.

Theorem 3.4. *For sufficiently large σ_0, the PML problem* (3.25) *has a unique solution $\hat{v}_{\hat{\lambda}} \in W$. Moreover, we have the estimate*

$$\|v-\hat{v}\|_W \le C\hat{C}^{-1}|s_0|^{\frac{1}{2}}e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}}\|\hat{\lambda}\|_{\widetilde{H}^{-1/2}(\Gamma)}, \tag{3.32}$$

where the positive constant C is independent with σ_0.

Proof. The existence of a unique solution for (3.25) follows from Lemma 3.6 by using the same argument as in [17, Theorem 5.1]. Next, by (2.13) and (3.25), we have

$$\begin{aligned}b(v_\lambda-\hat{v}_{\hat{\lambda}},\psi_\mu) &= \hat{b}(\hat{v}_{\hat{\lambda}},\psi_\mu)-b(\hat{v}_{\hat{\lambda}},\psi_\mu)\\ &=\int_\Gamma \frac{1}{k^2(x)}\mu\overline{(K-\hat{K})\left(\frac{\varepsilon_0}{\varepsilon_\Gamma}\hat{\lambda}\right)}\,ds \quad \forall \psi_\mu \in W.\end{aligned}$$

This completes the proof of the theorem upon using Lemma 3.6 and (2.16). □

From the classical FEM theory, it is readily to achieve the convergence for the finite element approximation of the PML problems. We omit the details here.

4. Numerical Examples

In this section, we present computational results for a set of test problems. In general, we assume that $\mu_0 = 1$. We use the error estimate in Theorems 3.2 and 3.4 to determine the PML parameters. In our implementation we choose δ_1, δ_2 and σ_0 such that $\sigma_0 \ge \Lambda_0$, and

$$e^{-\sigma_0\delta\frac{k_0}{2\omega(m+1)}} \le 10^{-8}, \tag{4.1}$$

Table 4.1: The PML parameters for Examples 4.1 and 4.2.

Example 4.1		Example 4.2	
δ	σ_0	δ	σ_0
1	112	1	112
2	56	2	72
3	38	4	61
4	33		

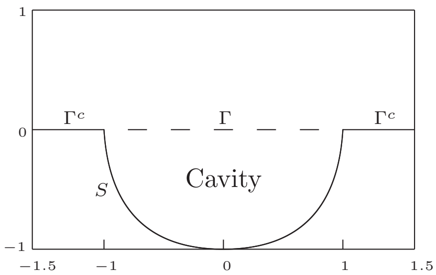

Fig. 4.1. Geometry of the cavity in Example 4.1

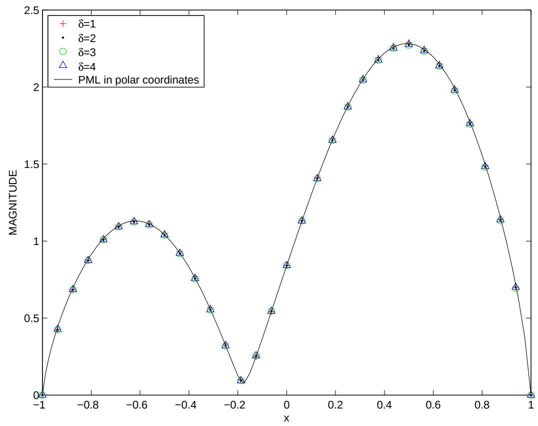

Fig. 4.2. Magnitude of the electric field at open aperture for Example 4.1 (TM).

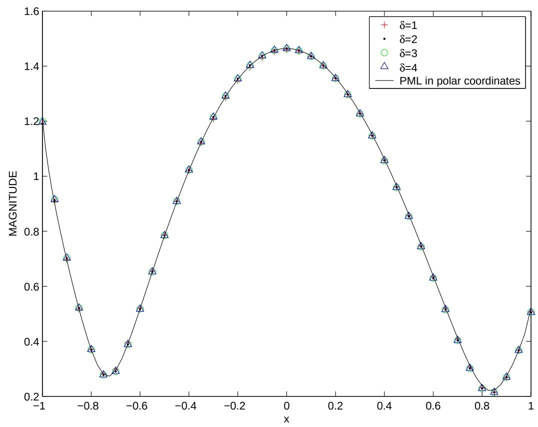

Fig. 4.3. Magnitude of the magnetic field at open aperture for Example 4.1 (TE).

which makes the PML error negligible compared with the finite element discretization errors.

In the following we report several numerical examples to demonstrate the competitive behavior of the proposed algorithm. In the computations we first prescribe δ_1, δ_2 and then determine σ_0 according to (4.1).

We compute the magnitude of the electric field and magnetic field at the aperture of the cavity for the TM case and the TE case, respectively. In the first two examples, we compare the numerical results with those obtained by PML method in cylindrical coordinates (see [18]).

Example 4.1. Consider the plane wave $u^i = e^{\mathbf{i}k_0(x_1 \sin\theta - x_2 \cos\theta)}$ incident at $\theta = \pi/4$ on the cavity as shown in Figure 4.1. Assume that the cavity is unfilled, that is, $\varepsilon(x) = \varepsilon_0 = 1$. Here we take $\omega = \pi$. Table 4.1 shows the different choices of the PML parameters δ and σ_0 determined by the relation (4.1). The results agree very well in both polarizations (see Figs. 4.2 and 4.3).

Example 4.2. A cavity with multi-layers is shown in Figure 4.4. We choose the parameters

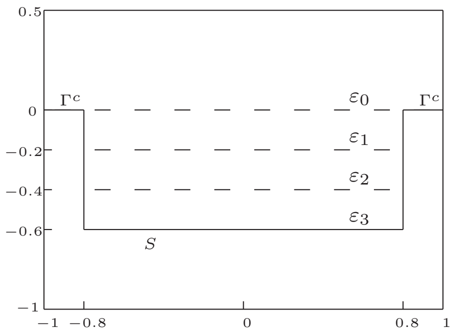

Fig. 4.4. Geometry of the cavity in Example 4.2

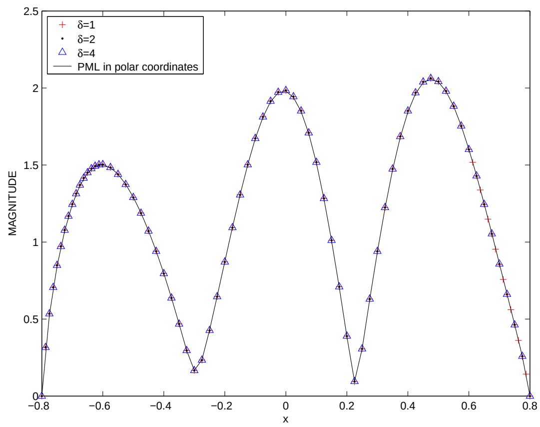

Fig. 4.5. Magnitude of the electric field at open aperture for Example 4.2 (TM).

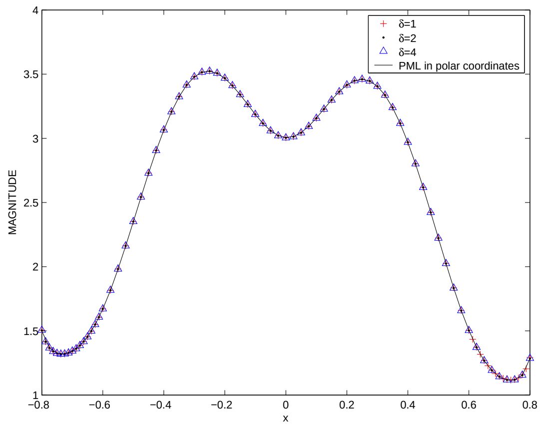

Fig. 4.6. Magnitude of the magnetic field at open aperture for Example 4.2 (TE).

as follows: $\varepsilon_0 = 1$, $\varepsilon_1 = 4.84$, $\varepsilon_2 = 1.96$, $\varepsilon_3 = 2.56$ and $\omega = 6$. The incident plane wave is $u^i = e^{\mathbf{i}k_0(x_1 \sin\theta - x_2 \cos\theta)}$ with $\theta = \pi/6$. The different choices of PML parameters δ and σ_0 determined by the relation (4.1) are shown in Table 4.1. The magnitude of the electric and magnetic field at open aperture for both fundamental polarizations are compared with those obtained by PML method in cylindrical coordinates (see Figs. 4.5 and 4.6). Again, the numerical results agree very well with the theoretical predictions.

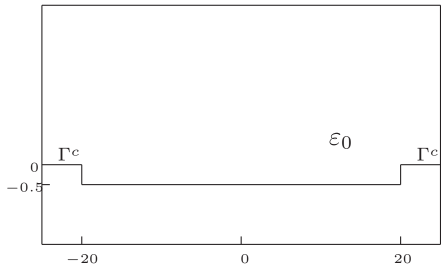

Fig. 4.7. Geometry of the cavity in Example 4.3

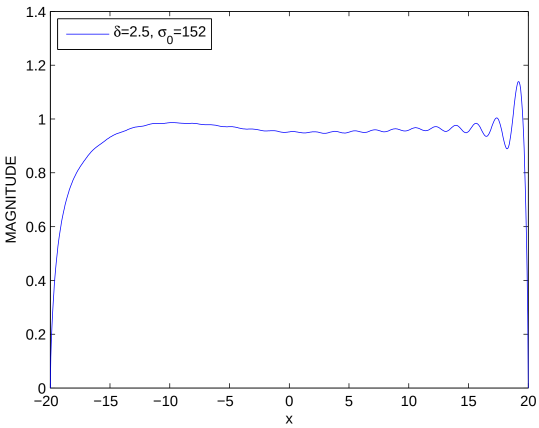

Fig. 4.8. Magnitude of the electric field at open aperture for Example 4.3 (TM).

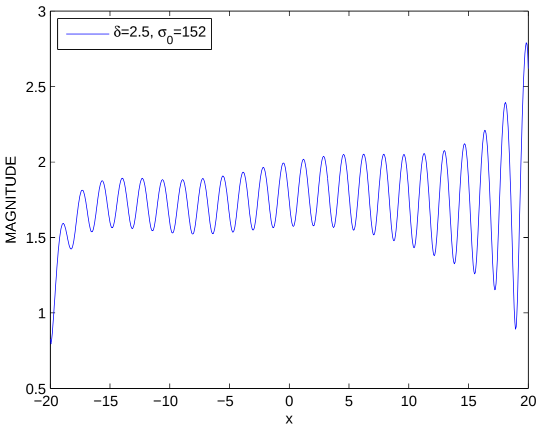

Fig. 4.9. Magnitude of the magnetic field at open aperture for Example 4.3 (TE).

Example 4.3. Finally, we consider a large open cavity as shown in Figure 4.7. The parameters are chosen as: $\omega = 2$ and $\varepsilon_0 = 1$. The incident plane wave is $u^i = e^{\mathbf{i}k_0(x_1 \sin\theta - x_2 \cos\theta)}$ with $\theta = \pi/3$. The magnitude of the electric field and magnetic field at open aperture are illustrated in Figs. 4.8 and 4.9. It is observe that the numerical results seems reasonable.

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