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LOCAL A PRIORI AND A POSTERIORI ERROR ESTIMATE OF TQC9 ELEMENT FOR THE BIHARMONIC EQUATION*

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Abstract

In this paper, local a priori, local a posteriori and global a posteriori error estimates are obtained for TQC9 element for the biharmonic equation. An adaptive algorithm is given based on the a posteriori error estimates.

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Key words: Finite element, Biharmonic equation, A priori error estimate, A posteriori error estimate, TQC9 element.

1. Introduction

For a posteriori error estimates of finite elements, there has been a great deal of work (see, e.g., [1–5, 9, 14] and references therein). Most of the finite elements considered are mainly for the second-order partial differential equations. In the recent paper [12], local a priori and a posteriori error estimates of conforming and nonconforming elements for the biharmonic equation were discussed. In this paper, we consider the TQC9 element for the biharmonic equation.

The TQC9 (9-parameter quasi-conforming triangle) element was proposed by Tang et al. [6, 8] for the biharmonic equation. The TQC9 element also uses the degrees of freedom of the Zienkiewicz element, but unlike the Zienkiewicz element, it is convergent. The convergence property and a global a priori error estimate of the TQC9 element were proved in [10, 15, 16]. Here we will show local a priori, local a posteriori and global a posteriori error estimates of the TQC9 element.

Let $\Omega \subset R^2$ be a bounded polygonal domain with boundary $\partial\Omega$. For $f \in L^2(\Omega)$, we consider the homogeneous Dirichlet boundary value problem of the biharmonic equation:

$$\begin{cases} \Delta^2 u = f, \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\nu = (\nu_1, \nu_2)^\top$ is the unit outer normal of $\partial\Omega$ and Δ is the standard Laplace operator.

Given a bounded domain $B \subset R^2$ and an integer m, let $H^m(B)$, $H_0^m(B)$, $\|\cdot\|_{m,B}$ and $|\cdot|_{m,B}$ denote the Sobolev space, the closure of $C_0^\infty(B)$ in $H^m(B)$, the corresponding Sobolev norm and semi-norm respectively. Let $H^{-m}(\Omega)$ denote the dual space of $H_0^m(\Omega)$ with norm $\|\cdot\|_{-m,\Omega}$.

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Let $i,j \in \{1,2\}$ and $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \partial_i\partial_j$. For a function $v \in H^2(\Omega)$, we define

$$E(v) = \big(\partial_{11}v, \partial_{22}v, \partial_{12}v\big)^\top. \tag{1.2}$$

Let $\sigma \in [0, \frac{1}{2}]$ be the Poisson ratio and

$$K = \begin{pmatrix} 1 & \sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & 2(1-\sigma) \end{pmatrix}. \tag{1.3}$$

Define

$$a(v,w) = \int_\Omega E(w)^\top K E(v), \quad \forall v, w \in H^2(\Omega). \tag{1.4}$$

The weak form of problem (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega), \tag{1.5}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

The TQC9 element for problem (1.5) and some known results will be given in Section 2. Section 3 will discuss local a priori error estimate of the TQC9 element. Section 4 will consider a posteriori error estimate. The last section gives some numerical results of an adaptive algorithm based on the a posteriori error estimate obtained.

2. TQC9 Element

Let (T, P_T, Φ_T) be the Zienkiewicz element with T a triangle, P_T the shape function space and Φ_T the set of nodal parameters consisting of the function values and two first order derivatives at three vertices of T (cf. [7]).

Let $\{\mathcal{T}_h(\Omega)\}$ be a family of shape regular triangulations by triangles with mesh size $h \to 0$. Let $h(x)$ be the function with its value the diameter h_T of the element T containing x.

Corresponding to $\mathcal{T}_h(\Omega)$, denote by $V_h(\Omega)$ and $V_{h0}(\Omega)$ the Zienkiewicz element spaces with respect to $H^2(\Omega)$ and $H_0^2(\Omega)$ respectively. It is known that $V_h(\Omega) \not\subset H^2(\Omega)$, $V_{h0}(\Omega) \not\subset H_0^2(\Omega)$, and $V_h(\Omega) \subset H^1(\Omega)$, $V_{h0}(\Omega) \subset H_0^1(\Omega)$. Given $G \subset \Omega$, $V_h(G)$ and $\mathcal{T}_h(G)$ are the restrictions of $V_h(\Omega)$ and $\mathcal{T}_h(\Omega)$ to G, respectively. Set

$$V_{h0}(G) = \{\, v \in V_{h0}(\Omega) \,:\, \operatorname{supp} v \subset \bar{G} \,\}. \tag{2.1}$$

For any $G \subset \Omega$ mentioned in this paper, we assume that it aligns with $\mathcal{T}_h(\Omega)$ when it is necessary.

For nonnegative integer k and $T \in \mathcal{T}_h(\Omega)$, let $P_k(T)$ denote the set of all polynomials with degree not greater than k. Let Π_T^1 be the linear interpolation operator with the function values at three vertices of T.

For $p \in P_T$, define $\partial_{ij,T}p \in P_1(T)$, $i,j \in \{1,2\}$, such that $\partial_{12,T}p = \partial_{21,T}p$ and for any $q \in P_1(T)$,

$$\begin{cases} \displaystyle\int_T q\partial_{11,T}p = \int_{\partial T} q\Pi_T^1\partial_1 p\nu_1 - \int_T \partial_1 q\,\partial_1 p, \\ \displaystyle\int_T q\partial_{22,T}p = \int_{\partial T} q\Pi_T^1\partial_2 p\nu_2 - \int_T \partial_2 q\,\partial_2 p, \\ \displaystyle 2\int_T q\partial_{12,T}p = \int_{\partial T} q\big(\Pi_T^1\partial_2 p\nu_1 + \Pi_T^1\partial_1 p\nu_2\big) - \int_T \big(\partial_2 q\,\partial_1 p + \partial_1 q\,\partial_2 p\big). \end{cases} \tag{2.2}$$

Set

$$E_T(p) = (\partial_{11,T}p, \partial_{22,T}p, \partial_{12,T}p)^\top, \quad \forall p \in P_T.$$

For $v_h \in V_h(\Omega)$ and $i, j \in \{1, 2\}$, define $\partial_{ij,h}v_h$ and $E_h(v_h)$ as follows: $\partial_{ij,h}v_h|_T = \partial_{ij,T}v_h$, $\forall T \in T_h(\Omega)$, and

$$E_h(v_h) = (\partial_{11,h}v_h, \partial_{22,h}v_h,\ \partial_{12,h}v_h)^\top.$$

Define

$$\bar{a}_h(v, w) = \int_\Omega E_h(v)^\top K E_h(w), \quad \forall v, w \in V_h. \tag{2.3}$$

The finite element method for problem (1.1) with the TQC9 element is: find $u_h \in V_{h0}$ such that

$$\bar{a}_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}. \tag{2.4}$$

Define, for the function $v \in L^2(G)$ and $v|_T \in H^m(T), \forall T \in \mathcal{T}_h(G)$,

$$\|v\|_{m,h,G} = \Big(\sum_{T\in\mathcal{T}_h(G)} \|v\|^2_{m,T}\Big)^{1/2}, \quad |v|_{m,h,G} = \Big(\sum_{T\in\mathcal{T}_h(G)} |v|^2_{m,T}\Big)^{1/2}. \tag{2.5}$$

For convenience, following [13], the symbols $\lesssim$, $\gtrsim$ and $\eqsim$ will be used in this paper. That is, $X_1 \lesssim Y_1$, $X_2 \gtrsim Y_2$ and $X_3 \eqsim Y_3$, mean that $X_1 \le c_1Y_1$, $c_2X_2 \ge Y_2$ and $c_3X_3 \le Y_3 \le c_4X_3$ for some constants c_1, c_2, c_3 and c_4 that are independent of the mesh size h.

For a set $B \subset \bar{\Omega}$, let

$$S_h(B) = \{\, T \,|\, T \cap \bar{B} \ne \emptyset,\ \forall T \in \mathcal{T}_h(\Omega)\},$$

and let $N_h(B)$ be the number of elements in $S_h(B)$.

From [16], the following inequalities hold,

$$|p|^2_{2,T} \eqsim \sum_{1\le i,j\le 2} |\partial_{ij,T}p|^2_{0,T}, \quad \forall p \in P_T,\ \forall T \in \mathcal{T}_h(\Omega), \tag{2.6}$$

$$\sum_{1\le i,j\le 2} |\partial_{ij}p - \partial_{ij,T}p|^2_{0,T} \lesssim |hp|^2_{3,T}, \quad \forall p \in P_T,\ \forall T \in \mathcal{T}_h(\Omega). \tag{2.7}$$

For $D \subset \bar{G} \subset \bar{\Omega}$, we use the notation $D \subset\subset G$ to mean that $\operatorname{dist}(\partial D\backslash\partial\Omega, \partial G\backslash\partial\Omega) > 0$.

An assumption about the mesh is that it is not exceedingly overrefined locally, namely

A0. There exists $\gamma \ge 1$ such that

$$h^\gamma_\Omega \lesssim h(x), \quad x \in \Omega, \tag{2.8}$$

where $h_\Omega = \max_{x\in\Omega} h(x)$.

Now we list some properties of the finite element spaces $V_{h0}(\Omega)$ and $V_h(\Omega)$.

P1. *Approximation.* For all $s \in \{0,1\}$,

$$\inf_{v_h \in V_{h0}(\Omega)} \sum_{j=0}^{2} |h^{j-2}(v - v_h)|_{j,h,\Omega} \lesssim |h^s v|_{2+s,\Omega}, \quad \forall v \in H_0^2(\Omega) \cap H^{2+s}(\Omega). \tag{2.9}$$

P2. *Weak Continuity.* For all $v_h \in V_{h0}(\Omega)$, v_h and its two first order derivatives are continuous at all vertices of all $T \in \mathcal{T}_h(\Omega)$ and vanish at the vertices located on $\partial\Omega$.
P3. *Superapproximation.* Let Π_h be the interpolation operator of the Zienkiewicz element corresponding to $\mathcal{T}_h(\Omega)$. For $G \subset \Omega$, let $\varphi \in C_0^\infty(\Omega)$ with supp $\varphi \subset\subset G$. Then for any $v_h \in V_h(G)$,

$$\begin{aligned} &\sum_{j=0}^{2} \big|h^{j-3}\big(\varphi v_h - \Pi_h(\varphi v_h)\big)\big|_{j,h,G} \\ &\qquad + \sum_{1\le i,j\le 2} \big\|h^{-1}\big(\varphi \partial_{ij,h} v_h - \partial_{ij,h}\Pi_h(\varphi v_h)\big)\big\|_{0,G} \lesssim \|v_h\|_{2,h,G}. \end{aligned} \tag{2.10}$$

Since the triangulations are shape regular, the following inverse and trace inequalities hold:

$$\|v_h\|_{j,h,\Omega_0} \lesssim \|h^{-1}v_h\|_{j-1,h,\Omega_0}, \quad \forall v_h \in V_h(\Omega_0),\ j \in \{1,2,\cdots\}, \tag{2.11}$$
$$\|v\|_{0,\partial T} \lesssim \|h^{-1/2}v\|_{0,T} + |h^{1/2}v|_{1,T}, \quad \forall v \in H^1(T),\ \forall T \in \mathcal{T}_h. \tag{2.12}$$

For the regularity of solution u of problem (1.1), if Ω is convex then for $f \in H^{-1}(\Omega)$ and $u \in H^3(\Omega)$:

$$\|u\|_{3,\Omega} \lesssim \|f\|_{-1,\Omega}. \tag{2.13}$$

For the solution u_h of problem (2.5), the following global a priori error estimate is true (cf. [15] or [16]).

Theorem 2.1. *Let u and u_h be the solutions of problems* (1.5) *and* (2.4) *respectively. Then*

$$\|u - u_h\|_{2,h,\Omega} \lesssim \|hu\|_{3,h,\Omega} \tag{2.14}$$

when $u \in H^3(\Omega)$. In addition, if Ω is convex then

$$\|u - u_h\|_{1,h,\Omega} \lesssim \|h(u - u_h)\|_{2,h,\Omega}. \tag{2.15}$$

3. Local a Priori Error Estimate

Given $\rho \in C^\infty(\bar{\Omega})$ and $T \in \mathcal{T}_h(\Omega)$, define

$$a_{h,\rho}(v,w) = \sum_{T\in\mathcal{T}_h(\Omega)} \int_T \rho^2 E(v)^\top K E(w), \quad \forall v,w \in H^2(\Omega) + V_h(\Omega), \tag{3.1}$$
$$\bar{a}_{h,\rho}(v,w) = \sum_{T\in\mathcal{T}_h(\Omega)} \int_\Omega \rho^2 E_h(v)^\top K E_h(w), \quad \forall v,w \in V_h(\Omega). \tag{3.2}$$

Lemma 3.1. *Assume that A0 holds. Let $g \in L^2(\Omega)$ and $D \subset\subset \Omega_0 \subset\subset \Omega$. If $w_h \in V_{h0}(\Omega_0)$ satisfies*

$$\bar{a}_h(w_h, v_h) = (g, v_h), \quad \forall v_h \in V_{h0}(\Omega_0), \tag{3.3}$$

then

$$\|w_h\|_{2,h,D} \lesssim \|w_h\|_{1,h,\Omega_0} + \|g\|_{0,\Omega_0}. \tag{3.4}$$

Proof. Let θ be the integer such that $\theta \geq 2\gamma - 1$ and Ω_j $(j = 1, 2, \cdots, \theta)$ satisfy

$$D \subset\subset \Omega_\theta \subset\subset \Omega_{\theta-1} \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega_0.$$

Take $D_1 \subset \Omega$ with $D \subset\subset D_1 \subset\subset \Omega_\theta$ and $\rho \in C^\infty(\bar{\Omega})$ such that $\rho \equiv 1$ on $\bar{D}_1$ and supp $\rho \subset\subset \Omega_\theta$. From P2 and P3, we know that $\Pi_h(\rho^2 w_h) \in V_{h0}(\Omega_\theta)$,

$$\Big| \int_\Omega E_h(w_h)^\top K\big(\rho^2 E_h(w_h) - E_h(\Pi_h(\rho^2 w_h))\big) \Big| \lesssim h_{\Omega_0} \|w_h\|^2_{2,h,\Omega_\theta},$$

and

$$\begin{aligned} |(g, \Pi_h(\rho^2 w_h))| &\lesssim \|g\|_{0,\Omega_0} \|\Pi_h(\rho^2 w_h)\|_{0,\Omega_\theta} \\ &\lesssim \|g\|_{0,\Omega_0} (h_{\Omega_0} \|w_h\|_{2,h,\Omega_\theta} + \|\rho^2 w_h\|_{0,h,\Omega}). \end{aligned}$$

Then

$$\begin{aligned} \bar{a}_{h,\rho}(w_h, w_h) &= \int_\Omega E_h(w_h)^\top K\big(\rho^2 E_h(w_h) - E_h(\Pi_h(\rho^2 w_h))\big) + (g, \Pi_h(\rho^2 w_h)) \\ &\lesssim h_{\Omega_0} \|w_h\|^2_{2,h,\Omega_\theta} + \|g\|_{0,\Omega_0} (h_{\Omega_0} \|w_h\|_{2,h,\Omega_\theta} + \|w_h\|_{0,h,\Omega_0}), \end{aligned}$$

which gives that

$$\bar{a}_{h,\rho}(w_h, w_h) \lesssim h_{\Omega_0} \|w_h\|^2_{2,h,\Omega_\theta} + \|w_h\|^2_{1,h,\Omega_0} + \|g\|^2_{0,\Omega_0}. \tag{3.5}$$

By (2.6) we obtain

$$\|w_h\|_{2,h,D} \lesssim h^{1/2}_{\Omega_0} \|w_h\|_{2,h,\Omega_\theta} + \|w_h\|_{1,h,\Omega_0} + \|g\|_{0,\Omega_0}. \tag{3.6}$$

The above argument may be repeated for $\|w_h\|_{2,h,\Omega_\theta}$ on the right-hand side to yield

$$\|w_h\|_{2,h,\Omega_j} \lesssim h^{1/2}_{\Omega_0} \|w_h\|_{2,h,\Omega_{j-1}} + \|w_h\|_{1,h,\Omega_0} + \|g\|_{0,\Omega_0}, \quad j = 1, \cdots, \theta. \tag{3.7}$$

By A0 and (2.11) we get

$$\begin{aligned} \|w_h\|_{2,h,D} &\lesssim h^{(\theta+1)/2}_{\Omega_0} \|w_h\|_{2,h,\Omega_0} + \|w_h\|_{1,h,\Omega_0} + \|g\|_{0,\Omega_0} \\ &\lesssim h^{(\theta+1)/2}_{\Omega} \|h^{-1} w_h\|_{1,h,\Omega_0} + \|w_h\|_{1,h,\Omega_0} + \|g\|_{0,\Omega_0} \\ &\lesssim \|w_h\|_{1,h,\Omega_0} + \|g\|_{0,\Omega_0} \end{aligned}$$

This completes the proof of this lemma. $\square$

Theorem 3.1. *Assume that A0 holds. Let u and u_h be the solutions of problems* (1.5) *and* (2.4) *respectively. For $\Omega_0 \subset\subset \Omega$, if $D \subset\subset \Omega_0$, we have*

$$\|u - u_h\|_{2,h,D} \lesssim h_{\Omega_0} \|u\|_{3,\Omega_0} + h^2_\Omega \|u\|_{3,\Omega}, \tag{3.8}$$

when $u \in H^3(\Omega)$.

Proof. Let $D \subset\subset D_1 \subset\subset \Omega_0$ and $\rho \in C^\infty(\bar{\Omega})$ such that $\rho \equiv 1$ on $\bar{D}_1$ and supp $\rho \subset\subset \Omega_0$. Set $\tilde{u} = \rho u$. Let $\tilde{u}_h \in V_{h0}(\Omega_0)$ be the solution of the problem

$$\bar{a}_h(\tilde{u}_h, v_h) = (\triangle^2 \tilde{u}, v_h), \quad \forall v_h \in V_{h0}(\Omega_0). \tag{3.9}$$

Applying Theorem 2.1 to Ω_0, we have

$$\|\tilde{u} - \tilde{u}_h\|_{2,h,\Omega_0} \lesssim h_{\Omega_0} \|\tilde{u}\|_{3,\Omega_0}. \tag{3.10}$$

Since $\triangle^2 \tilde{u} = \triangle^2 u = f$ in D_1,

$$\bar{a}_h(u_h - \tilde{u}_h, v_h) = 0, \quad \forall v_h \in V_{h0}(D_1). \tag{3.11}$$

From Lemma 3.1, we can derive that

$$\begin{aligned}
&\|u - u_h\|_{2,h,D} \le \|u - \tilde{u}_h\|_{2,h,D} + \|u_h - \tilde{u}_h\|_{2,h,D} \\
\lesssim\ &\|u - \tilde{u}_h\|_{2,h,D} + \|u_h - \tilde{u}_h\|_{1,h,D_1} \lesssim \|u - \tilde{u}_h\|_{2,h,D_1} + \|u - u_h\|_{1,h,D_1} \\
\lesssim\ &\|\tilde{u} - \tilde{u}_h\|_{2,h,D_1} + \|u - u_h\|_{1,h,D_1} \lesssim \|\tilde{u} - \tilde{u}_h\|_{2,h,\Omega_0} + \|u - u_h\|_{1,h,\Omega}.
\end{aligned}$$

Hence

$$\|u - u_h\|_{2,h,D} \lesssim h_{\Omega_0} \|\tilde{u}\|_{3,\Omega_0} + h_\Omega^2 \|u\|_{3,\Omega}. \tag{3.12}$$

This leads to the theorem. □

4. A Posteriori Error Estimate

In this section we consider the a posteriori error estimate of the TQC9 element.

Lemma 4.1. *If $G \subset \Omega$ and $\varphi \in C^\infty(\bar{\Omega})$ with supp $\varphi \subset\subset G$, then for $i, j \in \{1, 2\}$,*

$$\begin{aligned}
&\Big| \int_\Omega (\partial_{ij} v - \partial_{ij,h} v_h) \varphi w \Big| \\
\lesssim\ &\Big(h_G \|v - v_h\|_{2,h,G} + \|v - v_h\|_{1,h,G} \Big) \Big(h_G \|w\|_{2,h,G} + \|w\|_{1,h,G} \Big)
\end{aligned} \tag{4.1}$$

is true for all $v \in H_0^2(\Omega)$, $v_h \in V_{h0}(\Omega)$, $w \in H_0^2(\Omega) + V_{h0}(\Omega)$.

Proof. Let $v \in H_0^2(\Omega)$, $v_h \in V_{h0}(\Omega)$, $w \in H_0^2(\Omega) + V_{h0}(\Omega)$. Then by P2, $\Pi_h^1(\varphi w) \in H_0^1(G)$ and

$$\begin{aligned}
&\int_\Omega (\partial_{ij} v - \partial_{ij,h} v_h) \varphi w \\
=&\int_\Omega \Big((\partial_{ij} v - \partial_{ij,h} v_h) \Pi_h^1(\varphi w) + \partial_i (v - v_h) \partial_j \Pi_h^1(\varphi w) \Big) \\
&- \int_\Omega \partial_i (v - v_h) \partial_j \Pi_h^1(\varphi w) + \int_T (\partial_{ij} v - \partial_{ij,h} v_h) \Big(\varphi w - \Pi_h^1(\varphi w) \Big).
\end{aligned}$$

By the definition of $\partial_{ij,h}$ and the interpolation theory, we obtain

$$\begin{aligned}
&\Big| \int_\Omega (\partial_{ij} v - \partial_{ij,h} v_h) \varphi w \Big| \\
\lesssim\ &\|v - v_h\|_{1,h,G} \|\Pi_h^1(\varphi w)\|_{1,h,G} + h_G^2 \|v - v_h\|_{2,h,G} \|w\|_{2,h,G} \\
\lesssim\ &\|v - v_h\|_{1,h,G} (\|\varphi w\|_{1,h,G} + h_G \|\varphi w\|_{2,h,G}) + h_G^2 \|v - v_h\|_{2,h,G} \|w\|_{2,h,G},
\end{aligned}$$

which proves the desired inequality (4.1). □

Lemma 4.2. *If $\Omega_0 \subset\subset \Omega$ and $\rho \in C^\infty(\bar{\Omega})$ with supp $\rho \subset\subset \Omega_0$, then*

$$\begin{aligned}
&\Big| \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \rho^2 \big(E(v)-E_h(v_h)\big)^\top K E(v-v_h)\Big| \\
\lesssim &\Big| \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \big(E(v)-E_h(v_h)\big)^\top K E(\rho^2(v-v_h))\Big| \\
&+\Big(\int_{\Omega_0} \rho^2\big(E(v)-E_h(v_h)\big)^\top K\big(E(v)-E_h(v_h)\big)\Big)^{1/2}\|v-v_h\|_{1,h,\Omega_0} \\
&+h_{\Omega_0}^2\|v-v_h\|_{2,h,\Omega_0}^2+\|v-v_h\|_{1,h,\Omega_0}^2, \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}
&\Big| \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \big(E(v)-E_h(v_h)\big)^\top K E(\rho^2 w)\Big| \\
\lesssim &\Big| \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \rho^2\big(E(v)-E_h(v_h)\big)^\top K E(w)\Big| \\
&+\Big(\int_{\Omega_0} \rho^2\big(E(v)-E_h(v_h)\big)^\top K\big(E(v)-E_h(v_h)\big)\Big)^{1/2}\|w\|_{1,h,\Omega_0} \\
&+\Big(h_{\Omega_0}\|v-v_h\|_{2,h,\Omega_0}+\|v-v_h\|_{1,h,\Omega_0}\Big)\Big(h_{\Omega_0}\|w\|_{2,h,\Omega_0}+\|w\|_{1,h,\Omega_0}\Big) \qquad (4.3)
\end{aligned}$$

are true for all $v \in H_0^2(\Omega)$, $v_h \in V_{h0}(\Omega)$, $w \in H_0^2(\Omega)+V_{h0}(\Omega)$.

Proof. Let $k, l \in \{1,2\}$ and $v \in H_0^2(\Omega)$, $v_h \in V_{h0}(\Omega)$, $w \in H_0^2(\Omega)+V_{h0}(\Omega)$. It is obvious that

$$\partial_{kl}(\rho^2\psi) = \rho^2\partial_{kl}\psi + 2\rho\partial_k\rho\partial_l\psi + 2\rho\partial_l\rho\partial_k\psi + \partial_{kl}\rho^2\psi$$

holds for any function ψ. Then we have

$$\begin{aligned}
&\Big| \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \rho^2 \big(E(v)-E_h(v_h)\big)^\top K E(v-v_h)\Big| \\
\lesssim &\Big| \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \big(E(v)-E_h(v_h)\big)^\top K E(\rho^2(v-v_h))\Big| \\
&+\Big(\int_{\Omega_0} \rho^2\big(E(v)-E_h(v_h)\big)^\top K\big(E(v)-E_h(v_h)\big)\Big)^{1/2}\|v-v_h\|_{1,h,\Omega_0} \\
&+\sum_{i,j,k,l=1}^{2}\Big|\int_{\Omega_0}(\partial_{ij}v-\partial_{ij,h}v_h)\partial_{kl}\rho^2(v-v_h)\Big|,
\end{aligned}$$

and

$$\begin{aligned}
&\Big| \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \big(E(v)-E_h(v_h)\big)^\top K E(\rho^2 w)\Big| \\
\lesssim &\Big| \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \rho^2\big(E(v)-E_h(v_h)\big)^\top K E(w)\Big| \\
&+\Big(\int_{\Omega_0} \rho^2\big(E(v)-E_h(v_h)\big)^\top K\big(E(v)-E_h(v_h)\big)\Big)^{1/2}\|w\|_{1,h,\Omega_0} \\
&+\sum_{i,j,k,l=1}^{2}\Big|\int_{\Omega_0}(\partial_{ij}v-\partial_{ij,h}v_h)\partial_{kl}\rho^2 w\Big|.
\end{aligned}$$

By (4.1), we get inequalities (4.2) and (4.3). This completes the proof of this lemma. $\square$

Denote by $\partial\mathcal{T}_h$ the set consisting of all sides of all elements in $\mathcal{T}_h(\Omega)$. Define

$$\partial\mathcal{T}_h^b = \{\, F \mid F \in \partial\mathcal{T}_h, F \subset \partial\Omega\}, \quad \partial\mathcal{T}_h^i = \partial\mathcal{T}_h - \partial\mathcal{T}_h^b.$$

For $T \in \mathcal{T}_h(\Omega)$ and $F \in \partial\mathcal{T}_h^i$ with $F \subset \partial T$, let $[\,\cdot\,]_{J,F}^T$ be the jump of a function through F from the interior of T to the exterior. Define, for $v_h \in V_h(\Omega)$ and $T \in \mathcal{T}_h(\Omega)$,

$$\begin{aligned} R_T(v_h) = |T|^2\, \|f\|_{0.T}^2 + |T|\, |v_h|_{3,T}^2 + \sum_{F\subset\partial T, F\in\partial\mathcal{T}_h^b} |F| \Big|\frac{\partial^2 v_h}{\partial\nu\partial s}\Big|_{0.F}^2 \\ + \sum_{F\subset\partial T, F\in\partial\mathcal{T}_h^i} |F| \Big(\Big|\Big[\frac{\partial^2 v_h}{\partial\nu^2}\Big]_{J,F}^T\Big|_{0.F}^2 + \Big|\Big[\frac{\partial^2 v_h}{\partial\nu\partial s}\Big]_{J,F}^T\Big|_{0.F}^2\Big). \end{aligned} \tag{4.4}$$

Lemma 4.3. *Let u and u_h be the solutions of problems* (1.5) *and* (2.5) *respectively. For $\Omega_0 \subset\subset \Omega$, if $D \subset\subset \Omega_0$ then*

$$\begin{aligned} \|u-u_h\|_{2,D}^2 \lesssim \sum_{T\in S_h(\Omega_0)} R_T(u_h) + \inf_{w\in H_0^2(\Omega)} \|u_h - w\|_{2,h,\Omega_0}^2 \\ + h_{\Omega_0}^2 \|u-u_h\|_{2,\Omega_0}^2 + \|u-u_h\|_{1,\Omega}^2. \end{aligned} \tag{4.5}$$

Proof. Choose $D_1 \subset \Omega$ satisfying $D \subset\subset D_1 \subset\subset \Omega_0$ and $\rho \in C_0^\infty(\Omega)$ such that $\rho \equiv 1$ on $\bar D_1$ and supp $\rho \subset\subset \Omega_0$. Set $e_h = u - u_h$ and

$$A_h = \int_{\Omega_0} \rho^2 \big(E(u) - E_h(u_h)\big)^\top K \big(E(u) - E_h(u_h)\big).$$

Let $v_h \in V_{h0}(\Omega_0)$ and $w \in H_0^2(\Omega)$. Then

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \big(E(u) - E_h(u_h)\big)^\top K E(\rho^2 e_h) \\ &= (f, \rho^2(u-w) - v_h) - \int_{\Omega_0} E_h(u_h)^\top K\big(E(\rho^2(u-w)) - E_h(v_h)\big) \\ &\quad - \sum_{T\in\mathcal{T}_h(\Omega_0)} \int_T \big(E(u) - E_h(u_h)\big)^\top K E(\rho^2(u_h - w)). \end{aligned} \tag{4.6}$$

Set $e = \rho^2(u-w)$. By the assumption P1 we can choose v_h such that

$$\sum_{j=0}^{2} |h^{j-2}(e - v_h)|_{j,h,\Omega_0} + |v_h|_{2,h,\Omega_0} \lesssim \|e\|_{2,\Omega_0}. \tag{4.7}$$

Let $i,j,k,l \in \{1,2\}$. From Lemma 5.1 in [12] and its proof, there exists $\tilde v_{ij} \in H^1(\Omega)$ such that

$$\|\partial_{ij} u_h - \tilde v_{ij}\|_{0,T}^2 + |T|\, |\tilde v_{ij}|_{1,T}^2 \lesssim |T|\, |u_h|_{3,T}^2 + \sum_{T'\in S_h(T)} \sum_{\substack{F\in\partial\mathcal{T}_h^i \\ F\subset\partial T}} |F|\, \Big|[\partial_{ij} u_h]_{J,F}^{T'}\Big|_{0,F}^2$$

holds for all $T \in \mathcal{T}_h(\Omega)$, and $\tilde v_{ij}$ is piecewise linear. Then

$$\sum_{T\in\mathcal{T}_h(\Omega_0)} \Big(\|\partial_{ij} u_h - \tilde v_{ij}\|_{0,T}^2 + |T|\, |\tilde v_{ij}|_{1,T}^2\Big) \lesssim \sum_{T\in S_h(\Omega_0)} R_T(u_h). \tag{4.8}$$

Note that

$$\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T\Big(\tilde v_{ij}(\partial_{kl}e-\partial_{kl,h}v_h)+\partial_k\tilde v_{ij}\partial_l(e-v_h)\Big)=0.$$

It is can be derived that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T\partial_{ij,h}u_h(\partial_{kl}e-\partial_{kl,h}v_h)\\
=&-\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T\partial_{ijk}u_h\partial_l(e-v_h)+\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T(\partial_{ij,h}u_h-\partial_{ij}u_h)(\partial_{kl}e-\partial_{kl,h}v_h)\\
&+\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T(\partial_{ij}u_h-\tilde v_{ij})(\partial_{kl}e-\partial_{kl,h}v_h)+\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T\partial_k(\partial_{ij}u_h-\tilde v_{ij})\partial_l(e-v_h).\end{aligned}$$

By (2.7), (4.7), (4.8) and the Schwarz inequality, we obtain

$$\Big|\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T\partial_{ij,h}u_h(\partial_{kl}e-\partial_{kl,h}v_h)\Big|\lesssim\Big(\sum_{T\in S_h(\Omega_0)}R_T(u_h)\Big)^{1/2}|e|_{2,\Omega_0}.$$

Therefore,

$$\Big|\int_{\Omega_0}E_h(u_h)^\top K\big(E(\rho^2(u-w))-E_h(v_h)\big)\Big|\lesssim\Big(\sum_{T\in S_h(\Omega_0)}R_T(u_h)\Big)^{1/2}|e|_{2,\Omega_0}. \tag{4.9}$$

By (4.7) and the Schwarz inequality, we have

$$|(f,e-v_h)|\lesssim\Big(\sum_{T\in\mathcal{T}_h(\Omega_0)}|T|^2\|f\|_{0,T}^2\Big)^{1/2}|e|_{2,\Omega_0}. \tag{4.10}$$

By (4.3), we have

$$\begin{aligned}&\Big|\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T\big(E(u)-E_h(u_h)\big)^\top KE(\rho^2(u_h-w))\Big|\\
\lesssim&\Big|\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T\rho^2\big(E(u)-E_h(u_h)\big)^\top KE(u_h-w)\Big|+A_h^{1/2}\|u_h-w\|_{1,h,\Omega_0}\\
&+\Big(h_{\Omega_0}\|e_h\|_{2,h,\Omega_0}+\|e_h\|_{1,h,\Omega_0}\Big)\Big(h_{\Omega_0}\|u_h-w\|_{2,h,\Omega_0}+\|u_h-w\|_{1,h,\Omega_0}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big|\sum_{T\in\mathcal{T}_h(\Omega_0)}\int_T\big(E(u)-E_h(u_h)\big)^\top KE(\rho^2(u_h-w))\Big|\\
\lesssim&A_h^{1/2}\|u_h-w\|_{2,h,\Omega_0}+h_{\Omega_0}^2\|e_h\|_{2,h,\Omega_0}+\|e_h\|_{1,h,\Omega_0}^2+\|u_h-w\|_{2,h,\Omega_0}^2.\end{aligned} \tag{4.11}$$

Taking $v=u$ and $v_h=u_h$ in (4.2) and applying (4.6), (4.9), (4.10) and (4.11), we obtain

$$\begin{aligned}A_h\lesssim&A_h^{1/2}\Big(\|e_h\|_{1,h,\Omega_0}+\|u_h-w\|_{2,h,\Omega_0}\Big)\\
&+\Big(\sum_{T\in S_h(\Omega_0)}R_T(u_h)\Big)^{1/2}\Big(|\rho^2e_h|_{2,h,\Omega_0}+\|u_h-w\|_{2,h,\Omega_0}\Big)\\
&+h_{\Omega_0}\|e_h\|_{2,h,\Omega_0}^2+\|e_h\|_{1,h,\Omega_0}^2+\|u_h-w\|_{2,h,\Omega_0}^2.\end{aligned}$$

Then

$$A_h \lesssim \Big(\sum_{T\in S_h(\Omega_0)} R_T(u_h)\Big)^{1/2}|\rho^2 e_h|_{2,h,\Omega_0} + \sum_{T\in S_h(\Omega_0)} R_T(u_h) \\ +h_{\Omega_0}\|e_h\|^2_{2,h,\Omega_0} + \|e_h\|^2_{1,h,\Omega_0} + \|u_h - w\|^2_{2,h,\Omega_0}. \tag{4.12}$$

It is easy to verify that

$$|\rho^2 e_h|^2_{2,h,\Omega_0} \lesssim a_{h,\rho}(e_h, e_h) + \|e_h\|^2_{1,h,\Omega}. \tag{4.13}$$

On the other hand, (2.7) leads to

$$a_{h,\rho}(e_h, e_h) \lesssim A_h + \sum_{T\in\mathcal{T}_h(\Omega_0)} |T|\,|u_h|^2_{3,h,T}. \tag{4.14}$$

From (4.12)-(4.14), we obtain

$$a_{h,\rho}(e_h, e_h) \lesssim \Big(a_{h,\rho}(e_h, e_h)\Big)^{1/2}\Big(\sum_{T\in S_h(\Omega_0)} R_T(u_h)\Big)^{1/2} \\ + \sum_{T\in S_h(\Omega_0)} R_T(u_h) + \|u_h - w\|^2_{2,h,\Omega_0} + h^2_{\Omega_0}\|e_h\|^2_{2,\Omega_0} + \|e_h\|^2_{1,h,\Omega}.$$

Hence

$$a_{h,\rho}(e_h, e_h) \lesssim \sum_{T\in S_h(\Omega_0)} R_T(u_h) + \|u_h - w\|^2_{2,h,\Omega_0} + h^2_{\Omega_0}\|e_h\|^2_{2,\Omega_0} + \|e_h\|^2_{1,h,\Omega}. \tag{4.15}$$

By the arbitrariness of w and

$$\|e_h\|^2_{2,D} \lesssim a_{h,\rho}(e_h, e_h),$$

we obtain the lemma. □

From Lemma 5.3 in [12] and Lemma 4.3, we obtain the following local a posteriori error estimate.

Theorem 4.1. *Let u and u_h be the solutions of problems* (1.5) *and* (2.4), *respectively. For $\Omega_0 \subset\subset \Omega$, if $D \subset\subset \Omega_0$ then*

$$\|u - u_h\|^2_{2,h,D} \lesssim \sum_{T\in S_h(\Omega_0)} R_T(u_h) + h^2_{\Omega_0}\|u - u_h\|^2_{2,h,\Omega_0} + \|u - u_h\|^2_{1,h,\Omega}. \tag{4.16}$$

By a similar argument, we can obtain global a posteriori error estimates of the TQC9 element.

Theorem 4.2. *Let u and u_h be the solutions of problems* (1.5) *and* (2.4), *respectively. Then*

$$\|u - u_h\|^2_{2,h,\Omega} \lesssim \sum_{T\in\mathcal{T}_h(\Omega)} R_T(u_h). \tag{4.17}$$

Define

$$\begin{aligned} E_T &= \sum_{F\subset\partial T, F\in\partial\mathcal{T}_h^i} |F|\Big|\Big[\frac{\partial^2 u_h}{\partial\nu^2}\Big]^T_{J,F}\Big|^2_{0.F} + |T|\,|u_h|^2_{3,T}, \\ E_\Omega &= \sum_{T\in\mathcal{T}_h(\Omega)} E_T. \end{aligned} \tag{4.18}$$

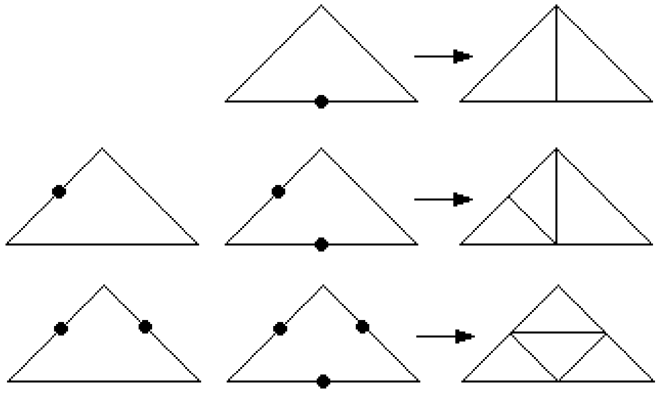

Fig. 5.1. Eliminating the hanging nodes.

By the assumption P2, we have

$$R_T(u_h) \lesssim E_T + |T|^2 \|f\|_{0,T}^2. \tag{4.19}$$

Similar to Theorem 5.3 in [12], we can prove that

$$\sum_{T\in\mathcal{T}_h(\Omega)} R_T(u_h) \lesssim |hu|_{3,h,\Omega}^2 + \|hf\|_{0,\Omega}^2, \tag{4.20}$$

$$\sum_{T\in\mathcal{T}_h(\Omega)} E_T \lesssim |hu|_{3,h,\Omega}^2. \tag{4.21}$$

Since the a priori error estimate of the TQC9 element for the biharmonic equation is of order $\mathcal{O}(h)$, one can use E_Ω as the estimator for the adaptive algorithms of the TQC9 element.

5. Adaptive Algorithm

In this section, we give an adaptive algorithm for problem (1.1) according to the a posteriori error estimator given in previous section, and show some numerical results. Our adaptive algorithm is described as follows:

1) Chose an error bound $\varepsilon > 0$ and an initial coarse triangulation $\mathcal{T}_0$ which has at least one inner node, put $k := 0$.
2) Solve the finite element problem (2.4) on $\mathcal{T}_k$.
3) For each element $T \in \mathcal{T}_k$ compute the a posteriori error estimator E_T given by (4.18).
4) If for all $T \in \mathcal{T}_k$ $E_T \le \varepsilon$, then stop. Otherwise refine the elements T so that $E_T > \varepsilon$, eliminate the hanging nodes, and then obtain the next triangulation $\mathcal{T}_{k+1}$. Replace k by $k+1$ and return to step 2.

In the adaptive algorithm, there are some strategies to refine the elements and to eliminate the hanging nodes. We refine an element by connecting each two midpoints of its edges and obtain four sub-triangles. To eliminate the hanging nodes, we use the method shown in Fig. 5.1.

Now let $\Omega = [0,1]^2$ and $\sigma = 0$ in (1.3). Define

$$u_1 = x^4 + y^4, \quad u_2 = 1 + e^{-(x+y)}, \quad u_3 = e^{-(x^2+y^2)}, \quad u_4 = \sin 2x + \sin 2y.$$

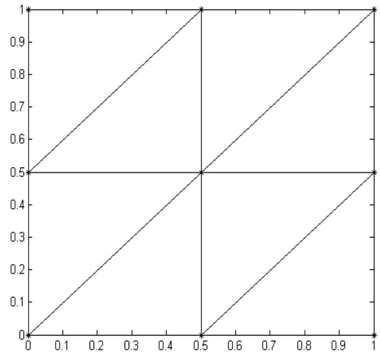

Fig. 5.2. The initial triangulation.

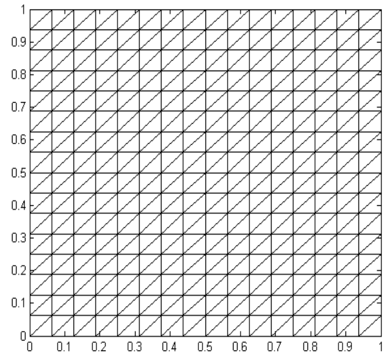

Fig. 5.3. $u = x^4 + y^4$.

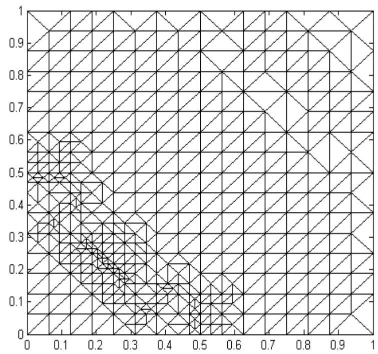

Fig. 5.4. $u = 1 + e^{-(x+y)}$.

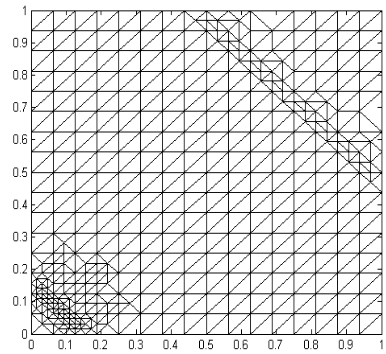

Fig. 5.5. $u = e^{-(x^2+y^2)}$.

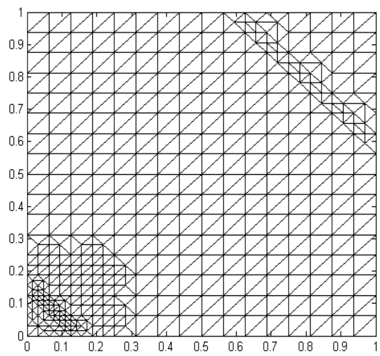

Fig. 5.6. $u = e^{-(x^2+y^2)}$, $\varepsilon = 0.0075$.

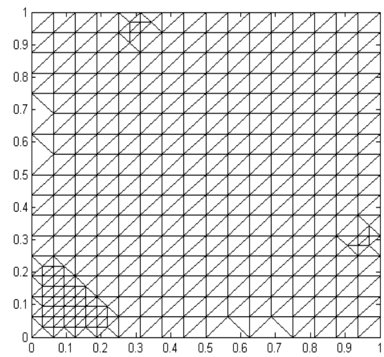

Fig. 5.7. $u = \sin 2x + \sin 2y$.

For $1 \le i \le 4$, let

$$f = \Delta^2 u_i, \quad g_0 = u_i|_{\partial\Omega}, \quad g_1 = \frac{\partial}{\partial\nu} u_i\Big|_{\partial\Omega}.$$

Then $u = u_i$ is the solution of the following problem:

$$\begin{cases} \Delta^2 u = f, \quad \text{in } \Omega, \\ u|_{\partial\Omega} = g_0, \ \dfrac{\partial u}{\partial\nu}\Big|_{\partial\Omega} = g_1. \end{cases} \tag{5.1}$$

We take $\varepsilon = 0.01$ and the initial coarse triangulation to be the one shown in Fig. 5.2.

For $u = u_i$ and $1 \le i \le 4$, the final triangulations T_k are shown in Figs. 5.3 to 5.7. It is observed that when $u = x^4 + y^4$ the triangulation is uniform while the others are not.

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