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A NEW APPROACH AND ERROR ANALYSIS FOR RECONSTRUCTING THE SCATTERED WAVE BY THE POINT SOURCE METHOD *1)

Jijun Liu
(*Department of Mathematics, Southeast University, Nanjing, 210096, China*
Email: jjliu@seu.edu.cn)

Gen Nakamura
(*Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo, 060-0810, Japan*
Email: gnaka@math.sci.hokudai.ac.jp)

Roland Potthast
(*Institute for Numerical and Applied Mathematics, University of Göttingen, Lotzestr, 16-18,37085, Göttingen, Germany*
Email: potthast@math.uni-goettingen.de)

Abstract

Consider an inverse scattering problem by an obstacle $D \subset \mathcal{R}^2$ with impedance boundary. We investigate the reconstruction of the scattered field u^s from its far field pattern u^∞ using the point source method. First, by applying the boundary integral equation method, we provide a new approach to the point-source method of Potthast by classical potential theory. This extends the range of the point source method from plane waves to scattering of arbitrary waves. Second, by analyzing the behavior of the Hankel function, we obtain an improved strategy for the choice of the regularizing parameter from which an improved convergence rate (compared to the result of [15]) is achieved for the reconstruction of the scattered wave. Third, numerical implementations are given to test the validity and stability of the inversion method for the impedance obstacle.

Mathematics subject classification: 35L, 35R.
Key words: Inverse scattering, Regularization, Error estimate, Numerics.

1. Introduction

Let $D \subset \mathcal{R}^2$ be a domain with smooth boundary $\partial D \in C^2$ such that the exterior $\mathcal{R}^2 \setminus \overline{D}$ is connected. If we consider D as a 2-D impenetrable obstacle with impedance boundary, then, for a given incident wave $u^i(x)$ such as an incident plane wave $e^{ikx\cdot d}$ with incident direction $d \in \Omega = \{\xi \in \mathcal{R}^2, |\xi| = 1\}$ and wave number $k > 0$, the total wave field

$$u(x) = u^i(x) + u^s(x) \tag{1.1}$$

with the scattered wave field $u^s(x)$ outside D is governed by ([1], Ch.3)

$$\begin{cases} \Delta u + k^2 u = 0, & x \in \mathcal{R}^2 \setminus \overline{D}, \\ \frac{\partial u(x)}{\partial \nu(x)} + ik\sigma(x)u(x) = 0, & x \in \partial D, \\ \frac{\partial u^s(x)}{\partial r} - iku^s(x) = O(\frac{1}{\sqrt{r}}), & r = |x| \to \infty. \end{cases} \tag{1.2}$$

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Here, $\nu(x)$ is the outward normal direction on ∂D, and $0 \le \sigma(x) \in C(\partial D)$ is the boundary impedance for D.

For the scattering problem (1.1), it is well known that the scattered wave field $u^s(x)$ has the asymptotic expression([5, 6])

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}}\left[u^\infty(\hat{x}) + O\left(\frac{1}{\sqrt{|x|}}\right)\right], \quad |x| \to \infty, \quad \hat{x} = \frac{x}{|x|} \in \Omega, \tag{1.3}$$

where $u^\infty(\hat{x})$ is called the far-field pattern of the scattered wave field. Both direct and inverse scattering problems have a long history. The direct problem is to determine the scattered wave as well as its far-field pattern for a known scatterer and incident wave, while the inverse problem is to recover a scatterer D from given information about $u^s(x)$. The typical inverse scattering problem is to determine ∂D from the far-field pattern u^∞ of $u^s(x)$. For the scattering described by (1.1) and (1.2), some related results may be found in ([1, 3, 6, 15]) and the references therein.

The relation between the scattered wave $u^s(x)$ and its far-field pattern $u^\infty(\hat{x})$ is also of great importance, for both direct and inverse scattering problems. On one hand, the far-field pattern $u^\infty(\hat{x})$ for $\hat{x} \in \Omega$ can determine the scattered wave uniquely as stated by the well-known Rellich lemma ([4]), which means that we can determine $u^s(\cdot)$ in $\mathcal{R}^2 \setminus \overline{D}$ from the knowledge of $u^\infty(\cdot)$ given in Ω. On the other hand, the determination of $u^s(x)$ from $u^\infty(\hat{x})$ is ill-posed, that is, the mapping from u^∞ to u^s is unbounded ([15]), which implies that a small perturbation in the far-field data can cause a large error in the scattered wave. Therefore some regularization technique should be introduced, such that we can use the noisy data of u^∞ to determine u^s approximately and stably.

The recovery of u^s from u^∞ has been studied theoretically and numerically for a long time. One method is to express $u^s(x)$ as an infinite series

$$u^s(x) = \sum_{n=0}^{\infty} \sum_{p \in \{\pm 1\}} a_n^p H_n^{(1)}(k|x|)e^{i(pn\varphi)} \tag{1.4}$$

(where $H_n^{(1)}$ denotes the Hankel function of the first kind of order n and φ is the angle between $\hat{x}$ and $(1,0)^T$) with the coefficients determined by $u^\infty(\hat{x}, d)$ ([1], Theorem 3.6, Corollary 3.8). A second method is to establish a relation between $u^s(x)$ and $u^\infty(\hat{x}, d)$ by introducing a density function ([5, 7]). The former method, which expresses the solution $u^s(x)$ explicitly by a recursive relation, is used by engineers widely. However, this method is very sensitive to the noisy far-field pattern data. Also, it has strict geometric limitations, since the recovery of u^s is restricted to the exterior of a circle enclosing the scattering object. The *potential method* of Kirsch and Kress calculates $u^s(x)$ from $u^\infty(\hat{x}, d)$ by solving the integral equation

$$\int_\Gamma \Phi^\infty(\hat{x}, y)\varphi(y)ds(y) = u^\infty(\hat{x}), \quad \hat{x} \in \Omega, \tag{1.5}$$

with some auxillary curve $\Gamma \subset D$, where $\Phi(x, y) = \frac{i}{4}H_0^{(1)}(k|x-y|)$ is the fundamental solution to 2-D Helmholtz equation and $\Phi^\infty(\cdot, y)$ denotes the far field pattern of $\Phi(\cdot, y)$. Please note that in contrast to this notation Φ^∞ is often used for the far field pattern of the scattered field $\Phi^s(\cdot, y)$ for scattering of $\Phi(\cdot, y)$ by some scatterer D. With a solution of (1.5), the scattered field is found by evaluating the potential

$$u^s(x) = \int_\Gamma \Phi(x, y)\varphi(y)ds(y), \quad x \in \mathcal{R}^2 \setminus \overline{D}. \tag{1.6}$$

In a series of papers [11]–[14], a point source method has been proposed to obtain $u^s(x)$ from $u^\infty(\hat{x}, d)$. The main idea of this method as presented in [15] is to approximate the point source $\Phi(\cdot, x)$ for $x \in \mathcal{R}^2 \setminus \overline{D}$ by a superposition of plane waves

$$\Phi(y,x) \approx \int_\Omega e^{i\kappa y\cdot d} g_x(d)ds(d), \quad x \in \partial D. \tag{1.7}$$

Then, we can pass to the far-field patterns on both sides to obtain an approximation

$$\Phi^\infty(\hat{y},x) \approx \int_\Omega u^\infty(\hat{y},d) g_x(d)ds(d), \quad \hat{y} \in \Omega. \tag{1.8}$$

Finally, the far-field reciprocity relation $u^\infty(\xi,\eta) = u^\infty(-\eta,-\xi)$ for $\xi,\eta \in \Omega$ and the mixed reciprocity relation

$$u^s(x,d) = \gamma\Phi^\infty(-d,x), \quad x \in \mathcal{R}^2 \setminus \overline{D}, d \in \Omega, \tag{1.9}$$

(here, γ denotes some constant depending on the dimension of the problem and $\Phi(x,y)$ is the fundamental solution to the 3-D or 2-D Helmholtz equation) are used as follows. We obtain

$$\begin{aligned} u^s(x,d) &= \gamma\Phi^\infty(-d,x) \\ &\approx \int_\Omega u^\infty(-d,\tilde{d}) g_x(\tilde{d})ds(\tilde{d}) \\ &= \int_\Omega u^\infty(-\tilde{d},d) g_x(\tilde{d})ds(\tilde{d}) \\ &=: (\mathbf{A}_\varepsilon u^\infty)(x,d), \end{aligned} \tag{1.10}$$

with the back projection operator $\mathbf{A}_\varepsilon$ with kernel $g_x(\tilde{d})$ and parameter ϵ which controls the calculation of g_x in (1.7).

In [14] bounds for the error $\|u^s - \mathbf{A}_\varepsilon u^\infty\|$ are given. With this error estimate, a stability result of recovering u^s from u^∞ is obtained. A basic ingredient of the point source method as presented in current publications is the so-called reciprocity relation (1.9) for the scattered wave. This approach has been used to treat other inverse scattering problems by generalizing the reciprocity principle, for example to scattering by electromagnetic waves ([8, 15]).

In this paper, we consider the recovery of a scattered wave for a 2-D obstacle with impedance boundary. As a first major point we develop a new approach to the point source method using classical potential theory. This extends the point source method from the scattering of plane waves to the scattering of arbitrary incident fields. The main idea is to use the representation formula

$$u^s(x) = \int_{\partial D} \Phi(x,y)\psi(y)ds(y), \quad x \in \mathcal{R}^2 \setminus \overline{D}, \tag{1.11}$$

with some density $\psi \in L^2(\partial D)$. Then, the far field of u^s is given by

$$u^\infty(\hat{x}) = \gamma \int_{\partial D} e^{-i\kappa\hat{x}\cdot y}\psi(y)ds(y). \tag{1.12}$$

We insert (1.7) into (1.11), exchange the order of integration and use (1.12) to obtain

$$\begin{aligned} u^s(x) &= \int_{\partial D} \Phi(x,y)\psi(y)ds(y) \approx \int_{\partial D}\int_\Omega e^{i\kappa y\cdot d} g_x(d)ds(d)\psi(y)ds(y) \\ &= \int_\Omega \Big(\int_{\partial D} e^{i\kappa y\cdot d}\psi(y)ds(y)\Big) g_x(d)ds(d) \\ &= \frac{1}{\gamma}\int_\Omega u^\infty(-d)g_x(d)ds(d), \end{aligned} \tag{1.13}$$

the representation (1.10). A precise version of these algebraic transformations and a rigorous convergence analysis will be described in section 2 to section 4.

We would like to emphasize that our new proof for the point source method extends the capability of the reconstruction scheme to general incident waves. This is of large importance for applications, where the particular form of the incident wave is sometimes not known and is a superposition of many different kinds of waves, for example for passive radar. For an incident wave which consists of a superposition of a plane wave and a point source, we have computed the total wave field using the scheme presented in this paper, see Fig. 4.11 for the results (exact/simulated field and reconstruction one). In our computations, we do not need to decompose the incident wave into the superposition of incident plane waves. Here we show results for an obstacle with sound-soft boundary, however, the theory does not depend on the boundary type.

Secondly, we give some error estimate on the minimum norm solution of (1.7) from the properties of $H_n^{(1)}$, $n = 0, 1, 2, ...$, which is crucial to the choice of regularizing parameter ε. Comparing with the analogy in [14], the convergence rate for the approximate scattered wave given here is a little faster due to an improved strategy for choosing ε.

Finally, we note that the minimum norm solution to the integral equation of the first kind with the fundamental function as the right-hand side also gives a possible way to the construction of the Runge approximation for the Green function, which solves one of the problems remaining in [9] of probe method, see [2] for the numerical implementations.

We will present all results in the two-dimensional case. However, all our arguments can be applied in the three-dimensional case as well — with straightforward modifications.

2. Potential Theory and Minimum Norm Solution

In this section, to introduce our notations and for later use we review some well-known results on the potential theory for the Helmholtz equation and on the minimum norm solution for the integral equation of the first kind.

Introduce the operators

$$\mathbf{K}'\psi(x) = 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\psi(y)ds(y), \quad x \in \partial D, \tag{2.1}$$

$$\mathbf{S}\psi(x) = 2\int_{\partial D} \Phi(x,y)\psi(y)ds(y), \quad x \in \partial D, \tag{2.2}$$

for a density function $\psi(\cdot) \in C(\partial D)$. The following result solves the impedance scattering problem in the case where $-k^2$ is not a Dirichlet eigenvalue for the scattering domain D. It can be found in [1].

Lemma 2.1. *Assume $-k^2$ is not a Dirichlet eigenvalue for the Laplacian in the domain D. For any $f \in C(\partial D)$ the integral equation*

$$\psi(x) - (\mathbf{K}'\psi)(x) - ik\sigma(x)(\mathbf{S}\psi)(x) = -2f(x), \quad x \in \partial D, \tag{2.3}$$

has a unique solution $\psi \in C(\partial D)$. If we choose

$$f(x) = -\frac{\partial u^i(x)}{\partial \nu(x)} - ik\sigma(x)u^i(x), \quad x \in \partial D, \tag{2.4}$$

then the scattered wave field

$$u^s(x) = \int_{\partial D} \Phi(x,y)\psi(y)ds(y), \quad x \in \mathcal{R}^2 \setminus \overline{D}, \tag{2.5}$$

with the far-field pattern

$$u^\infty(\hat{x}, d) = \gamma \int_{\partial D} e^{-ik\hat{x}\cdot y}\psi(y)ds(y), \quad \hat{x} \in \Omega, \tag{2.6}$$

solves the impedance scattering problem (1.2), where $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$.

For known scatterer D, choose a domain G such that $\overline{D} \subset G \subset \mathcal{R}^2$. The *Herglotz wave operator* $\mathbf{H}$ is defined by

$$(\mathbf{H}g)(x) = \int_\Omega e^{ikx\cdot\xi} g(\xi) ds(\xi), \quad x \in \partial G, \tag{2.7}$$

for $g \in L^2(\Omega)$. We consider $\mathbf{H}$ as an operator from $L^2(\Omega)$ into $L^2(\partial G)$.

For any fixed $z \in \mathcal{R}^2 \setminus \overline{G}$, denote by $g_\varepsilon(z,\cdot) \in L^2(\Omega)$ the minimum norm solution to the integral equation

$$(\mathbf{H}g)(z,\cdot) = \Phi(\cdot, z) \tag{2.8}$$

with the discrepancy ε, i.e.

$$\|(\mathbf{H}g)(z,\cdot) - \Phi(\cdot, z)\|_{L^2(\partial G)} \le \varepsilon. \tag{2.9}$$

Then the following result is true.

Lemma 2.2. *Assume that $-k^2$ is not a Dirichlet eigenvalue for the Laplacian in the domain G. Then*

1. *there exists a unique minimum norm solution $g_\varepsilon(z,\cdot) \in L^2(\Omega)$;*
2. *$\|g_\varepsilon(z,\cdot)\|_{L^2(\Omega)}$ depends continuously on ε and is unbounded as $\varepsilon \to 0$;*
3. *$g_\varepsilon(z,\cdot) \in L^2(\Omega)$ depends weakly continuously on $z \in \mathcal{R}^2 \setminus \overline{G}$, that is, it follows for any $\psi \in L^2(\Omega)$ that*

$$\lim_{z\to z_0} \langle g_\varepsilon(z,\cdot), \psi\rangle = \langle g_\varepsilon(z_0,\cdot), \psi\rangle. \tag{2.10}$$

The first two conclusions are obviously true by standard arguments on the minimum norm solution. The third one follows from the facts that $\Phi(\cdot, z)$ is continuous with respect to z and that the minimum norm solution depends weakly continuously on the right-hand side ([5], Chapter 16, Problem 16.2).

3. Approximation of Scattered Wave from Far-field

In this section, we give an approximation of $u^s(x)$ from the far-field pattern for the 2-D scatterer with impedance boundary by the points source method where the solution g_x of (1.7) is calculated by the minimum norm solution to (2.8).

A similar result has been given in [14] for a 3-D scattering object with the Dirichlet boundary. However, our proof is different from that in [14]. We find that the intended result can be directly obtained by applying potential theory, rather than by using the reciprocity principle.

Secondly, we analyze the error of the approximation by estimating the behavior of $\|g_\varepsilon(z,\cdot)\|$. As a bound for $\|g_\varepsilon(z,\cdot)\|$ we verify similar estimates as in the 3-D case ([14]). However, we obtain an improved convergence rate for the approximation of the scattered wave by giving a sharper strategy for the choice of the parameter ε.

Given the minimum norm solution $g_\varepsilon(z,\cdot)$ for all $z \in M$ with some set $M \subset \mathcal{R}^2 \setminus \overline{G}$, we define an operator $\mathbf{A}_\varepsilon : L^2(\Omega) \to L^\infty(M)$ by

$$(\mathbf{A}_\varepsilon \phi)(z) := \frac{1}{\gamma} \int_\Omega g_\varepsilon(z,\xi)\phi(-\xi) ds(\xi), \quad z \in \mathcal{R}^2 \setminus \overline{G}. \tag{3.1}$$

From the conclusion (3) in Lemma 2.2, we know the operator $\mathbf{A}_\varepsilon$ maps $L^2(\Omega)$ into $C(\mathcal{R}^2 \setminus \overline{G})$. Based on this operator, we can construct the approximate scattered wave outside G from the noisy data of the far-field pattern.

Theorem 3.1. *We assume that the noisy data $u_\delta^\infty(\hat{x})$ of the far-field pattern $u^\infty(\hat{x})$ satisfy*

$$\|u_\delta^\infty(\cdot) - u^\infty(\cdot)\|_{L^2(\Omega)} \leq \delta. \tag{3.2}$$

Then, for any $z \in \mathcal{R}^2 \setminus \overline{G}$ and $\varepsilon > 0$, it follows that

$$|u^s(z) - (\mathbf{A}_\varepsilon u_\delta^\infty)(z)| \leq c\varepsilon + \frac{1}{\gamma}\|g_\varepsilon(z,\cdot)\|_{L^2(\Omega)}\delta, \tag{3.3}$$

where the constant $c = c(G, k, \sigma, D) > 0$.

Proof. From Lemma 2.1 and the definitions of $\mathbf{H}$ and $\mathbf{A}_\varepsilon$, it follows that

$$\begin{aligned} u^s(z) &= \int_{\partial D} [\Phi(y,z) - (\mathbf{H}g_\varepsilon)(z,y)]\psi(y)ds(y) + \\ &\quad \int_{\partial D}\int_\Omega e^{iky.\xi}\psi(y)ds(y)g_\varepsilon(z,\xi)ds(\xi) \\ &= \int_{\partial D} [\Phi(y,z) - (\mathbf{H}g_\varepsilon)(z,y)]\psi(y)ds(y) + \frac{1}{\gamma}\int_\Omega g_\varepsilon(z,\xi)u^\infty(-\xi)ds(\xi) \\ &= \int_{\partial D} [\Phi(y,z) - (\mathbf{H}g_\varepsilon)(z,y)]\psi(y)ds(y) + (\mathbf{A}_\varepsilon u^\infty)(z). \end{aligned}$$

From this relation we get

$$\begin{aligned} u^s(z) - (\mathbf{A}_\varepsilon u_\delta^\infty)(z) &= \int_{\partial D} [\Phi(y,z) - (\mathbf{H}g_\varepsilon)(z,y)]\psi(y)ds(y) \\ &\quad + \mathbf{A}_\varepsilon(u^\infty - u_\delta^\infty)(z). \end{aligned} \tag{3.4}$$

For the next step we use that $\Phi(\cdot,z) - (\mathbf{H}g_\varepsilon)(z,\cdot)$ solves the Helmholtz equation in $G \supset \overline{D}$ for any fixed $z \in \mathcal{R}^2 \setminus \overline{G}$. Using the continuous dependence of the interior solution on the boundary values (i.e. the well-posedness of this problem) we obtain a constant $c = c(G)$ such that

$$\|\Phi(\cdot,z) - (\mathbf{H}g_\varepsilon)(z,\cdot)\|_{L^2(\partial D)} \leq c\|\Phi(\cdot,z) - (\mathbf{H}g_\varepsilon)(z,\cdot)\|_{L^2(\partial G)}. \tag{3.5}$$

It follows from (3.4) and the Cauchy-Schwarz inequality that

$$\begin{aligned} |u^s(z) - (\mathbf{A}_\varepsilon u_\delta^\infty)(z)| &\leq \sqrt{meas(\partial D)}\|\psi\|_{C(\partial D)}\|\Phi(\cdot,z) - (\mathbf{H}g_\varepsilon)(z,\cdot)\|_{L^2(\partial D)} \\ &\quad + \frac{1}{\gamma}\|g_\varepsilon(z,\cdot)\|_{L^2(\Omega)}\|u^\infty(\cdot) - u_\delta^\infty(\cdot)\|_{L^2(\Omega)} \\ &\leq c\varepsilon + \frac{1}{\gamma}\|g_\varepsilon(z,\cdot)\|_{L^2(\Omega)}\delta, \end{aligned} \tag{3.6}$$

where we have used (2.6),(2.9),(3.2) and (3.5).

From this result, we know that if we approximate $u^s(z)$ from u_δ^∞ by $\mathbf{A}_\varepsilon u_\delta^\infty$, the error given by (3.3) depends on two independent parameters ε, δ. Although $\|g_\varepsilon(z,\cdot)\|_{L^2(\Omega)}$ is unbounded as $\varepsilon \to 0$, according to the conclusion (2) in Lemma 2.2, we will show that it is possible to choose $\varepsilon = \varepsilon(\delta) > 0$ such that

$$\varepsilon(\delta) \to 0, \quad \|g_{\varepsilon(\delta)}(z,\cdot)\|_{L^2(\Omega)}\delta \to 0 \tag{3.7}$$

as $\delta \to 0$. The existence of such a strategy for $\varepsilon = \varepsilon(\delta)$ depends both on the estimate on $\|g_\varepsilon(z,\cdot)\|_{L^2(\Omega)}$ and on the position of z.

Theorem 3.2. *Let $z \in \mathcal{R}^2 \setminus \overline{B(R,x_0)}$ and choose $G = B(R,x_0)$ in (2.9), where $B(R,x_0)$ is a circle with center x_0 and radius R. Then there exist positive constants a, b, c, d depending on R and $|z - x_0|$ such that*

$$\|g_\varepsilon(z,\cdot)\|_{L^2(\Omega)} \leq \frac{d}{(c\varepsilon)^{b\ln(-a\ln(c\varepsilon))}}, \quad \epsilon > 0. \tag{3.8}$$

Proof. Without loss of generality, we can choose an appropriate coordinate system such that $x_0 = 0$. For $x \in \partial B(R,0), z \in \mathcal{R}^2 \setminus \overline{B(R,0)}$, introduce the polar coordinates $(\rho,\theta),(r,\psi)$, that is,

$$x = (x_1,x_2) = (\rho\cos\theta, \rho\sin\theta), \quad z = (z_1,z_2) = (r\cos\psi, r\sin\psi)$$

with $\rho = |x| = R, r = |z| > R$. Then from Graf's addition theorem ([10], Ch.3, §18) we obtain

$$\Phi(x,z) = \frac{i}{4}H_0^{(1)}(k|x-z|) = \frac{i}{4}\sum_{n=-\infty}^{\infty} H_n^{(1)}(kr)J_n(k\rho)e^{in(\theta-\psi)} \tag{3.9}$$

for any $|x| < |z|$. We use the notation

$$u_N(x,z) := \sum_{n=-N}^{N} a_n(z)J_n(k\rho)e^{in\theta} \tag{3.10}$$

with the coefficients

$$a_n(z) := \frac{i}{4}H_n^{(1)}(kr)e^{-in\psi}. \tag{3.11}$$

Since $J_{-n}(\cdot) = (-1)^n J_n(\cdot)$ and $H_{-n}^{(1)}(\cdot) = (-1)^n H_n^{(1)}(\cdot)$, we have

$$\Phi(x,z) - u_N(x,z) = \frac{i}{4}\sum_{n=N+1}^{\infty} H_n^{(1)}(kr)J_n(k\rho)[e^{-in(\psi-\theta)} + e^{in(\psi-\theta)}]. \tag{3.12}$$

Therefore it follows that

$$\begin{aligned}\|\Phi(\cdot,z) - u_N(\cdot,z)\|^2_{L^2(\partial B_R(0))} &\le \frac{1}{16}\sum_{n=N+1}^{\infty}\int_0^{2\pi} |H_n^{(1)}(kr)J_n(k\rho)|^2 4R d\theta \\ &= \frac{\pi R}{2}\sum_{n=N+1}^{\infty} |H_n^{(1)}(kr)|^2|J_n(kR)|^2, \end{aligned}\tag{3.13}$$

with $r = |z| > R$. However, the asymptotic of the Hankel function tells us that

$$H_n^{(1)}(t) = \frac{(2n-1)!!}{it^{n+1}}\left[1 + O\left(\frac{1}{n}\right)\right], \quad J_n(t) = \frac{t^n}{(2n+1)!!}\left[1 + O\left(\frac{1}{n}\right)\right],$$

uniformly on compact sets of $(0,\infty)$ as $n \to \infty$, which implies

$$|H_n^{(1)}(kr)||J_n(kR)| = \frac{1}{kr}\left(\frac{R}{r}\right)^n \frac{1}{2n+1}\left[1 + O\left(\frac{1}{n}\right)\right], \tag{3.14}$$

uniformly for r, R in any compact set of $(0,\infty)$. From equations (3.13) and (3.14), with the abbreviation

$$q := \frac{R}{r} = \frac{R}{|z|} < 1, \tag{3.15}$$

there exists a constant C_0 such that

$$\begin{aligned}\|\Phi(\cdot,z) - u_N(\cdot,z)\|^2_{L^2(\partial B_R(0))} &\le C_0\frac{R}{r}\frac{1}{rk^2}\sum_{n=N+1}^{\infty}\frac{q^{2n}}{n^2} \\ &\le \frac{C_0}{k^2|z|}\frac{1}{(N+1)^2}\sum_{n=N+1}^{\infty} q^{2n} \le \frac{C_1}{|z|}\frac{q^{2(N+1)}}{1-q^2},\end{aligned}$$

where the constant $C_1 = \frac{C_0}{4k^2}$ does not depend on $R, r = |z|$ and n. Since $\Phi(\cdot,z) - u_N(\cdot,z)$ solves the Helmholtz equation in $B(R,0)$ for any N and the interior Dirichlet problem is bounded from $L^2(\partial B(R,0))$ into $C(M)$ for compact subset M of $B(R,0)$, there exists a constant $\lambda = \lambda(B(R,0))$ such that

$$\begin{aligned}\|\Phi(\cdot,z) - u_N(\cdot,z)\|^2_{L^2(\partial G)} &\le \lambda^2\|\Phi(\cdot,z) - u_N(\cdot,z)\|^2_{L^2(\partial B(R,0))} \\ &\le \frac{\lambda^2 C_1^2}{|z|(1-q^2)}q^{2(N+1)}.\end{aligned}\tag{3.16}$$

Now we take N large enough such that the right-hand side of (3.15) is less than ε^2, that is, we take N as

$$N+1 := Int\left(\frac{\ln\frac{\sqrt{|z|}\sqrt{1-q^2}}{\lambda C_1}\varepsilon}{\ln q}\right), \tag{3.17}$$

where $Int(a)$ denotes the smallest integer bigger than $a \in R^+$. Then it follows that

$$\|\Phi(\cdot,z)-u_N(\cdot,z)\|_{L^2(\partial G)} \le \varepsilon. \tag{3.18}$$

Now we show there exists a function $g_N(z,\cdot)\in L^2(\Omega)$ such that

$$\mathbf{H}g_N = u_N \tag{3.19}$$

and give an estimate on $\|g_N(z,\cdot)\|_{L^2(\Omega)}$ in terms of ε. From (3.10) and (3.11), if we can solve the equation

$$\int_\Omega e^{ikx\cdot\xi}h_n(\xi)ds(\xi) = J_n(k\rho)e^{in\theta} \tag{3.20}$$

to obtain $h_n(\xi)$ for $n=-N,-N+1,\cdots,N-1,N$, then

$$g_N(z,\xi) = \sum_{n=-N}^{N} a_n(z)h_n(\xi). \tag{3.21}$$

From the Jacobi-Anger expansion $e^{ik\rho\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(k\rho)e^{in\theta}$, we get

$$J_n(k\rho) = \frac{(-1)^n i^{-n}}{2\pi}\int_0^{2\pi} e^{ik\rho\cos\theta_1}e^{in\theta_1}d\theta_1.$$

Inserting this expression into (3.19) leads to

$$\int_0^{2\pi} e^{ik\rho\cos(\theta-\theta_1)}p(\theta_1)d\theta_1 = \frac{(-1)^n i^{-n}}{2\pi}\int_0^{2\pi} e^{ik\rho\cos\theta_2}e^{in\theta_2}d\theta_2 e^{in\theta}$$

with $p(\theta_1)=h_n(\cos\theta_1,\sin\theta_1)$. Letting $\theta_2=\theta_1-\theta$ and comparing the integrands on the two sides yields

$$h_n(\xi) = \frac{(-1)^n i^{-n}}{2\pi}e^{in\theta_1},$$

with $\xi=(\cos\theta_1,\sin\theta_1)\in\Omega$. So (3.21) generates

$$g_N(z,\xi) = \frac{1}{2\pi}\left[\frac{i}{4}H_0^{(1)}(kr)+\frac{i}{2}\sum_{n=1}^{N}H_n^{(1)}(kr)(-1)^n i^{-n}\cos n(\psi-\theta_1)\right]. \tag{3.22}$$

However, Stirling's formula and the asymptotic of $H_n^{(1)}(t)$ yields

$$H_n^{(1)}(t) = \frac{1}{2^n i t^{n+1}}\frac{(2n)!}{n!}\left[1+O\left(\frac{1}{n}\right)\right] = \frac{\sqrt{2}}{it}\left(\frac{2n}{et}\right)^n(1+o(1))$$

for $n=1,2,\cdots$. Therefore, noticing $H_0^{(1)}(t)=ie^{ikt}/t$, there exists a constant $C_2>0$ such that

$$|H_n^{(1)}(t)| \le C_2\frac{1}{t}\left(\frac{n+1}{t}\right)^n,$$

uniformly in any compact set of $(0,\infty)$ for $n=0,1,2,\cdots$. This estimate tells us from (3.21) that

$$|g_N(z,\xi)| \le \frac{1}{4\pi}\sum_{n=0}^{N}|H_n^{(1)}(kr)| \le \frac{C_2}{4\pi}\frac{1}{kr}\sum_{n=0}^{N}\left(\frac{n+1}{kr}\right)^n,$$

which leads to

$$\|g_N(z,\cdot)\|_{L^2(\Omega)} \le \frac{C_2}{2\sqrt{2\pi}}\frac{1}{kr}(N+1)\left(\frac{N+1}{kr}\right)^N = \frac{C_2}{2\sqrt{2\pi}}\left(\frac{N+1}{kr}\right)^{N+1}.$$

Since the expression for N given in (3.16) implies

$$\frac{\ln\frac{\sqrt{|z|}\sqrt{1-q^2}}{\lambda C_1}\varepsilon}{\ln q} \le N+1 \le \frac{\ln\frac{\sqrt{|z|}\sqrt{1-q^2}}{\lambda C_1}\varepsilon}{\ln q}+1 \le 2\frac{\ln\frac{\sqrt{|z|}\sqrt{1-q^2}}{\lambda C_1}\varepsilon}{\ln q},$$

we obtain

$$\|g_N(z,\cdot)\|_{L^2(\Omega)} \le \frac{C_2}{2\sqrt{2\pi}}\left(\frac{-b\ln(c\varepsilon)}{kr}\right)^{-b\ln(c\varepsilon)} = \frac{C_2}{2\sqrt{2\pi}}\frac{1}{(c\varepsilon)^{b\ln(-a\ln(c\varepsilon))}}, \tag{3.23}$$

with

$$b=-\frac{2}{\ln q}>0, \quad a=\frac{b}{kr}>0, \quad c=\frac{\sqrt{|z|}\sqrt{1-q^2}}{\lambda C_1}>0.$$

By noting that $g_\varepsilon(z,\cdot)$ is the minimum norm solution, the desired result (3.8) is obtained from (3.22) and $\|g_\varepsilon(z,\cdot)\|_{L^2(\Omega)} \le \|g_N(z,\cdot)\|_{L^2(\Omega)}$ with constant $d=\frac{C_2}{2\sqrt{2\pi}}$.

In the sequel, denote by $\mathcal{H}(D)$ the convex hull of the domain D. Then for any $z\in\mathcal{R}^2\backslash\overline{\mathcal{H}(D)}$, we can always choose z_0 and R such that $\overline{D}\subset B(R,x_0)$. Therefore, using Theorem 3.1 and Theorem 3.2, we immediately obtain the estimate

$$|u^s(z)-(\mathbf{A}_\varepsilon u_\delta^\infty)(z)| \le c_1\varepsilon+\frac{1}{\gamma}\frac{d}{(c\varepsilon)^{b\ln(-a\ln(c\varepsilon)}}\delta \tag{3.24}$$

for $z\in\mathcal{R}^2\setminus\overline{\mathcal{H}(D)}$.

Now we prove that there exists a strategy of taking $\varepsilon=\varepsilon(\delta)$ such that

$$|u^s(z)-(\mathbf{A}_{\varepsilon(\delta)}u_\delta^\infty)(z)|\to 0$$

for any fixed $z\in\mathcal{R}^2\setminus\overline{\mathcal{H}(D)}$ as $\delta\to 0$. Our proof is constructive from which we also give the convergence rate. Define

$$f(\varepsilon,\delta)=c_1\varepsilon+\frac{d}{\gamma(c\varepsilon)^{b\ln(-a\ln(c\varepsilon))}}\delta. \tag{3.25}$$

It is obvious that $(c\varepsilon)^{-b\ln(-a\ln(c\varepsilon))}\to\infty$ as $\varepsilon\to 0$. However, if we choose $\varepsilon=\varepsilon(\delta)\to 0$ in some explicit special way, we can guarantee $f(\varepsilon(\delta),\delta)\to 0$ as $\delta\to 0$. This strategy is stated as

Theorem 3.3. *For any $\beta\in(0,1)$ and fixed z, if we take*

$$\varepsilon(\delta)=a\delta^{\frac{1}{b\ln(-\ln(c\delta))}}e^{-(-\ln(c\delta))^\beta}, \tag{3.26}$$

where a,b,c are positive constants, then it follows that

$$\varepsilon(\delta)\to 0, \quad f(\varepsilon(\delta),\delta)\to 0$$

as $\delta\to 0$. Furthermore, the convergence rate of $f(\varepsilon(\delta),\delta)$ is $O(\varepsilon(\delta))$.

Proof. Without loss of generality, it suffices to consider the case $a,b,c=1$. It is easy to verify that $\varepsilon(\delta)\to 0$ as $\delta\to 0$. If we set $z=\ln(-\ln\delta)$ and $t=-\ln\varepsilon$, then it follows that

$$f(\varepsilon,\delta)=F(t,z)=e^{-t}+e^{t\ln t-e^z}. \tag{3.27}$$

For $\varepsilon(\delta)$ given by (3.26), we have

$$t(z)=\frac{e^z}{z}+e^{\beta z}. \tag{3.28}$$

In order to show $F(t(z),z)\to 0$ as $z\to+\infty$, it is enough to prove $t(z)\ln t(z)-e^z\to-\infty$ as $z\to+\infty$. Since

$$\begin{aligned} t(z)\ln t(z)-e^z &= [\frac{e^z}{z}+e^{\beta z}][\ln\frac{e^z}{z}+\ln(1+\frac{z}{e^{(1-\beta)z}})]-e^z \\ &= -[\frac{e^z\ln z}{z}-ze^{\beta z}]+(\frac{e^z}{z}+e^{\beta z})\ln(1+\frac{z}{e^{(1-\beta)z}})-e^{\beta z}\ln z \end{aligned}$$

for any $\beta \in (0,1)$, we know that $t(z)\ln t(z) - e^z \to -\infty$ as $z \to +\infty$ and the order of divergence is $O(e^z \ln z/z)$. Finally, it follows from

$$\frac{t(z)}{\frac{e^z \ln z}{z}} = \frac{1}{\ln z} + \frac{z}{e^{(1-\beta)z}\ln z} \to 0, \quad z \to +\infty,$$

that the convergence rate of $F(t(z), z)$ is $O(e^{-t(z)})$, that is, the convergence rate of $f(\varepsilon(\delta), \delta)$ is $O(\varepsilon(\delta))$.

Now, the combination of Theorems 3.1, 3.2 and 3.3 yields the following error estimate on the recovery of the scattered wave from noisy far-field data:

Theorem 3.4. *Let $u^\infty_\delta(\hat{x})$ be the noisy data of far-field pattern of $u^\infty(\hat{x})$ satisfying*

$$\|u^\infty_\delta(\cdot) - u^\infty(\cdot)\|_{L^2(\Omega)} \le \delta. \tag{3.29}$$

Then for any $z \in \mathcal{R}^2 \setminus \overline{\mathcal{H}(D)}$, there exist constants C, a, b, c depending on D, k, σ such that

$$|u^s(z) - (\mathbf{A}_{\varepsilon(\delta)} u^\infty_\delta)(z)| \le C\delta^{\frac{1}{b\ln(-a\ln(c\delta))}} e^{-(-\ln\delta)^\beta}, \tag{3.30}$$

with the constant C uniformly in any compact set of $\mathcal{R}^2 \setminus \overline{\mathcal{H}(G)}$, where the operator $\mathbf{A}_\epsilon$ and $\varepsilon(\delta)$ are given in (3.1) *and* (3.26)*, respectively.*

As a direct corollary of this error estimate, we can also obtain the stability estimate of the scattered wave on the far-field pattern, namely,

Corollary 3.5. *For two obstacles D_1, D_2, if $u^\infty_i(\cdot), i = 1,2$ are the far-field patterns satisfying*

$$\|u^\infty_1(\cdot) - u^\infty_2(\cdot)\|_{L^2(\Omega)} \le \delta, \tag{3.31}$$

then given $z \in \mathcal{R}^2 \setminus \overline{\mathcal{H}(D_1 \cup D_2)}$ there exist constants C, a, b, c depending on D_1, D_2, k, σ such that the correspond scattered waves $u^s_i(z)$ $(i = 1,2)$ satisfy

$$|u^s_1(z) - u^s_2(z)| \le C\delta^{\frac{1}{b\ln(-a\ln(c\delta))}} e^{-(-\ln\delta)^\beta}. \tag{3.32}$$

Proof. This fact can be seen from

$$\begin{aligned}
& |u^s_1(z) - u^s_2(z)| \\
\le\ & |u^s_1(z) - \mathbf{A}_{\varepsilon(\delta)} u^\infty_1(z)| + |u^s_2(z) - \mathbf{A}_{\varepsilon(\delta)} u^\infty_1(z)| + |\mathbf{A}_{\varepsilon(\delta)}(u^\infty_1 - u^\infty_2)(z)| \\
\le\ & 2\varepsilon(\delta) + C\|g_{\varepsilon(\delta)}\| \, \|u^\infty_1(\cdot) - u^\infty_2(\cdot)\|_{L^2(\Omega)} \\
\le\ & 2C\varepsilon(\delta) + C\|g_{\varepsilon(\delta)}\|\delta
\end{aligned}$$

and Theorems 3.3 and 3.4.

This result is analogous to Theorem 6 in [14]. However, our stability estimate is not limited to the scattering of plane waves and it is sharper than the estimate in [14]. This is due to the new approach using potential theory and the special strategy for taking $\varepsilon(\delta)$ given in this paper.

The strategy of choosing the regularizing parameter $\varepsilon = \varepsilon(\delta)$ presented in Theorem 3.3 (also see (4.5) in numerics) gives a convergence rate of the approximated scattered wave $(\mathbf{A}_{\varepsilon(\delta)} u^\infty_\delta)(z)$ in (3.30). Of course, the form of $\varepsilon(\delta)$ is a little complicated. On the other hand, we also know that there are other strategies for choosing the regularizing parameter such as the Tikhonov *a priori* estimate or the Morozov discrepancy principle. It should be interesting to compare the convergence rate presented in this paper with that of other regularizing schemes. However, in order to obtain the convergence rate of the regularizing solution based on these priori strategies, some *a priori* information about the exact solution must be assumed. In our problem of reconstruction of the scattered wave, we should verify these assumptions (the range of far-field operator which maps the far-field data to near field) if we want to obtain some convergence rate. This comparison of convergence rates and establishment of strategies with optimal convergence rates either in a deterministic or stochastic framework is an important topic of current research and goes beyond the scope of this paper.

4. Numerical Implementations

In this section, we carry out the numerical realization of the reconstructions to test the validity of the point source method.

Our numerics consist of three steps. Firstly, we solve the direct scattering problem by potential theory (see Lemma 2.1) to obtain the scattered wave field $u^s(z)$ for $z \in \mathcal{R}^2 \setminus \overline{D}$, as well as the far-field pattern $u^\infty(\hat{x})$. Then we take this far-field with data error δ as input for the inverse problem. Finally the validity and stability of the inversion are checked by comparing the inversion result from noisy input data with $u^s(z)$ obtained by potential method.

Firstly, consider a point-source as our incident wave. More precisely, we take

$$u^i(x) = \frac{i}{4} H_0^{(1)}(k|x - x_0|) \tag{4.1}$$

for some $x_0 \in D_1$. For this special incident wave, the scattered wave is

$$u^s(x) = -\frac{i}{4} H_0^{(1)}(k|x - x_0|), \quad x \in \mathcal{R}^2 \setminus \overline{D}_1, \tag{4.2}$$

for any impedance $\sigma(x) \geq 0$, since it satisfies the boundary condition in ∂D_1 and the radiation condition at infinity. Of course the Helmholtz equation is also satisfied in $\mathcal{R}^2 \setminus \overline{D}_1$ for $x_0 \in D_1$.

We consider a special convex obstacle with the boundary

$$\partial D_1 = \{x = (1.2\cos t, 1.2\sin t), t \in [0, 2\pi]\} \tag{4.3}$$

and the boundary impedance $\sigma(x) = 0$. In fact, for the incident wave given by (4.1), the scattered wave (4.2) is independent of $\sigma(x)$. We take $k = 1.0, x_0 = (0.5, 0.5)$ and $G_1 = 1.5 \times D_1$ as our approximate domain, while

$$\partial Z_1 := 1.15 \times \partial G_1 = 1.725 \times \partial D_1 \tag{4.4}$$

is the cycle outside G_1 where we seek the scattered wave at $z(t) \in \partial Z_1$ by our inversion method.

We generate the exact scattered wave in ∂Z_1 by the single-layer potential method (Lemma 2.1). Especially, the scattered wave can also be obtained from the expression (4.2) directly for this special configuration.

By this exact solution, we can check the inversion procedure from the noisy far-field pattern data. According to the inversion scheme constructed in the above sections, the procedure contains the following steps:

Step 1: For given error level $\delta > 0$, take the regularizing parameter $\varepsilon(\delta)$ as

$$\varepsilon(\delta, z) = a(z)\delta^{\frac{1}{b(z)\ln(-\ln(c(z)\delta))}} e^{-(-\ln(c(z)\delta))^\beta} \tag{4.5}$$

with suitable parameters $a(z), b(z), c(z), \beta$.

Step 2: Find the minimum norm solution $g_{\varepsilon(\delta)}(z, \xi)$ to the integral equation $(\mathbf{H}g)(\cdot, z) = \Phi(z, \cdot)$ with discrepancy $\varepsilon(\delta, z)$. This can be done by the standard argument ([5]).

Step 3: Construct the approximate scattered wave by $u_\delta^s(z) = \mathbf{A}_{\varepsilon(\delta,\mathbf{z})} u_\delta^\infty$ for $z \in \mathcal{R}^2 \setminus \overline{D}_1$ from the noisy input data $u_\delta^\infty(\hat{x})$, where operator $\mathbf{A}_\varepsilon$ is given in (3.1). Especially, for $z \notin \mathcal{H}(G_1)$, we can obtain the convergence rate (3.30).

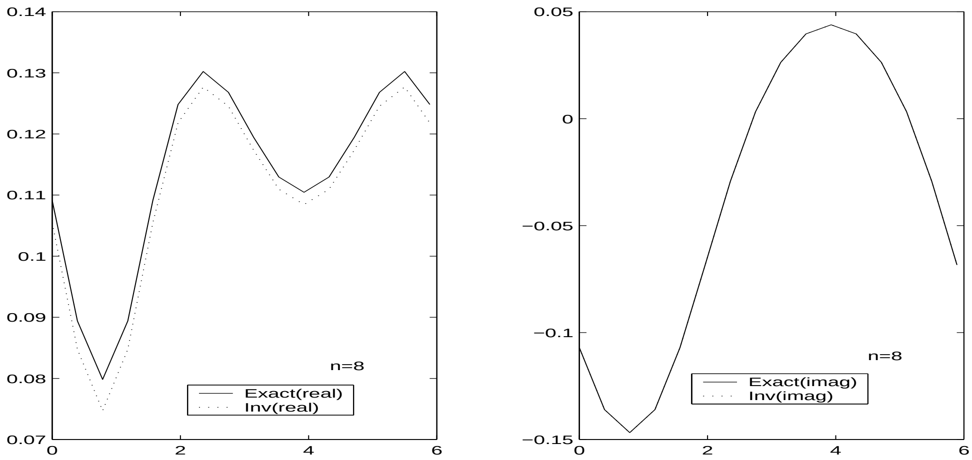

Fig. 4.1. Recovery for scattered wave with $k = 1.0, \delta = 0$, $n = 8$.

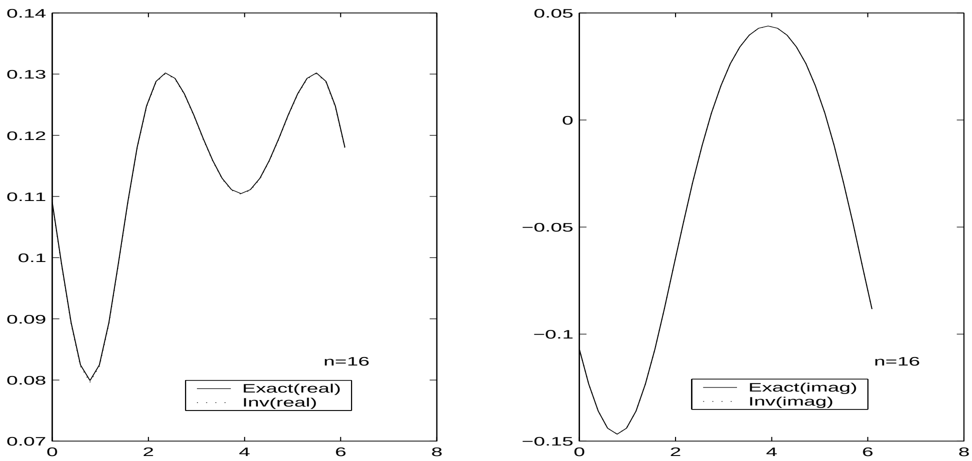

Fig. 4.2. Recovery for scattered wave with $k = 1.0, \delta = 0$, $n = 16$.

Based on the above steps, we can construct the scattered wave. We divide $[0, 2\pi]$ by $t_j = j \times \frac{\pi}{n}$ for $j = 0, 1, \cdots, 2n-1$. The results for $n = 8, 16$ with exact far-field pattern as the inversion input data are shown in Fig. 4.1 and Fig. 4.2. Notice that Fig. 4.2 contains the exact and inversion results. Obviously, if we use the exact data, then $n = 16$ can recover the scattered wave very well. In our numerics, the parameters $a(z), b(z), c(z)$ remains unchanged for all $z \in \partial Z_1$ and are obtained by trial and error.

Now we consider the inversion procedure for the noisy data. If we take the noisy far-field with error level $\delta = 0.05$ in the following way:

$$u_\delta^\infty(\hat{x}_i) = (1 + \delta) u^\infty(\hat{x}_i), \tag{4.6}$$

then the results with $n = 16$ are given in Fig. 4.3.

However, if we make the perturbation oscillatory, that is, we put the error in the form

$$u_\delta^\infty(\hat{x}_i) = (1 + (-1)^i \delta) u^\infty(\hat{x}_i), \tag{4.7}$$

then the results are not so good. The case with $k = 1.0, \delta = 0.01$ is shown in Fig. 4.4. The other numerical results for which we take $k = 2, \delta = 0.02$ in (4.7) are given in Fig. 4.5. These

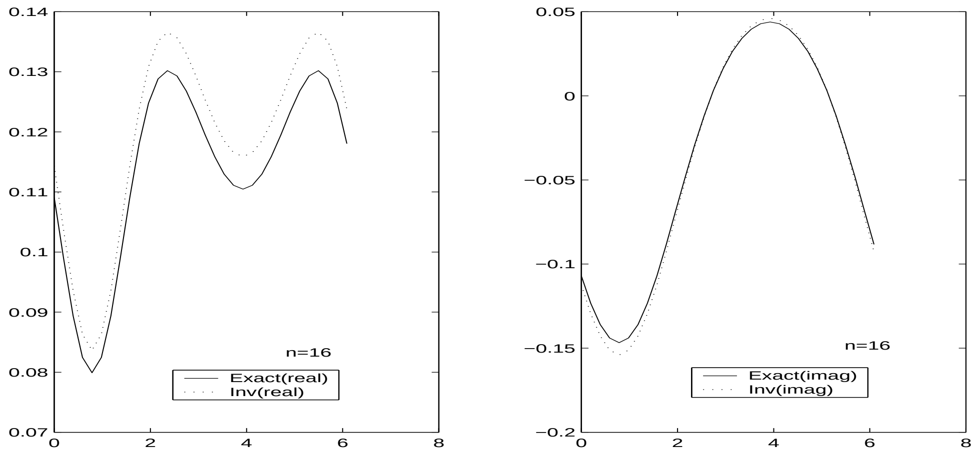

Fig. 4.3. Recovery for scattered wave with $k = 1.0, \delta = 0.05$, $n = 16$.

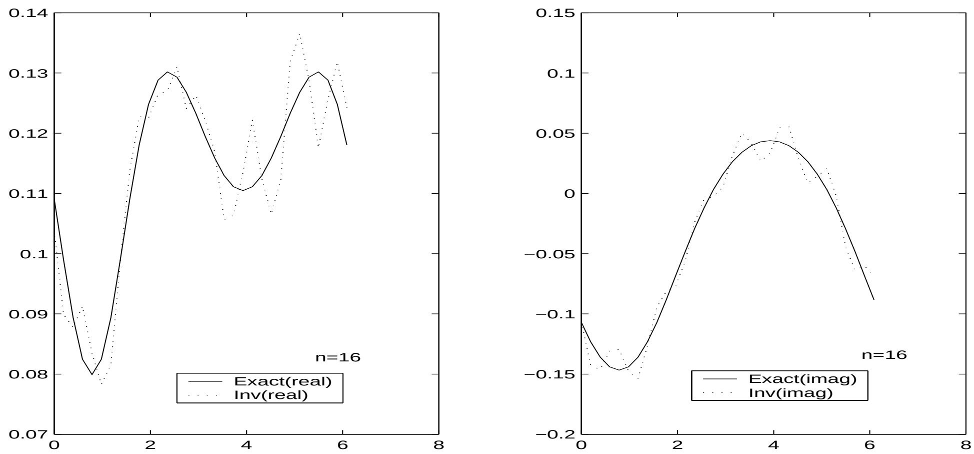

Fig. 4.4. Recovery for scattered wave with $k = 1.0, \delta = 0.01$, $n = 16$.

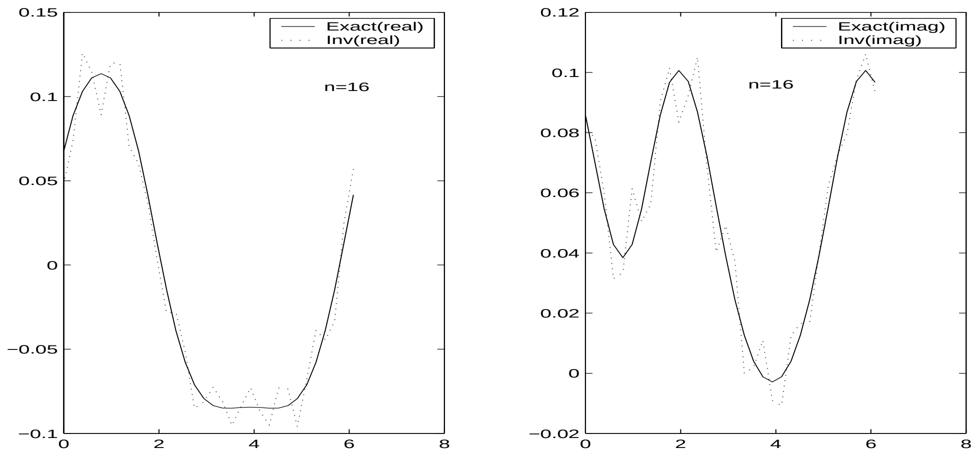

Fig. 4.5. Recovery for scattered wave with $k = 2.0, \delta = 0.02$, $n = 16$.

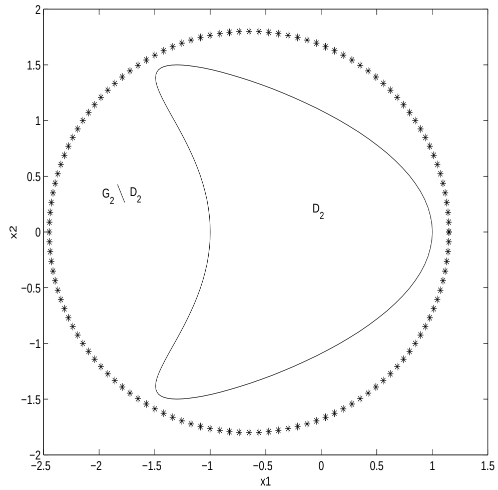

Fig. 4.6. Configure of a non-convex obstacle

two figures show that the inversion results are still satisfactory for the oscillatory error input data, except for some points near the extreme points of the scattered wave.

Remark 4.1. For the convex obstacle such as a circle, some numerics have been done with the exact far-field data ([8]). However, here we use the point-source as the incident wave to recover the corresponding scattered wave. The advantage of this model is that we can obtain the analytic expression of the scattered wave from which we can estimate the validity of our inversion method efficiently.

Table 1: $u^s(z)$ and u^∞ for $2n = 32$

t_i	$u^s(z(t_i))$	$u^\infty(\hat{x}(t_i))$
0	-0.71621860-0.67126570i	-0.84636120+1.03309100i
$\pi/2$	0.10035350+0.04220168i	-0.32021850-0.18389410i
π	0.49686100-0.19647750i	-0.54582840+0.19172820i
$3\pi/2$	-0.26164190+0.29148880i	0.12488810 -0.06168363i

Table 2: $u^s(z)$ and u^∞ for $2n = 64$

t_i	$u^s(z(t_i))$	$u^\infty(\hat{x}(t_i))$
0	-0.71621820-0.67126550i	-0.84636090+1.03309100i
$\pi/2$	0.10035340+0.04220182i	-0.32021860-0.18389440i
π	0.49686100-0.19647740i	-0.54582840+0.19172800i
$3\pi/2$	-0.26164190+0.29148880i	0.12488810-0.06168361i

Our second example is to consider a kite-shaped obstacle with the boundary

$$\partial D_2 = \{(x_1, x_2) := (\cos t + 0.65\cos(2t) - 0.65, 1.5\sin t), t \in [0, 2\pi]\}.$$

Here we take $u^i(x) = e^{ikd\cdot x}$ with incident direction d as the incident wave. In this case, we cannot obtain the analytic expression for the scattered wave in general. We have to simulate the scattered wave as well as its far-field pattern by the potential method.

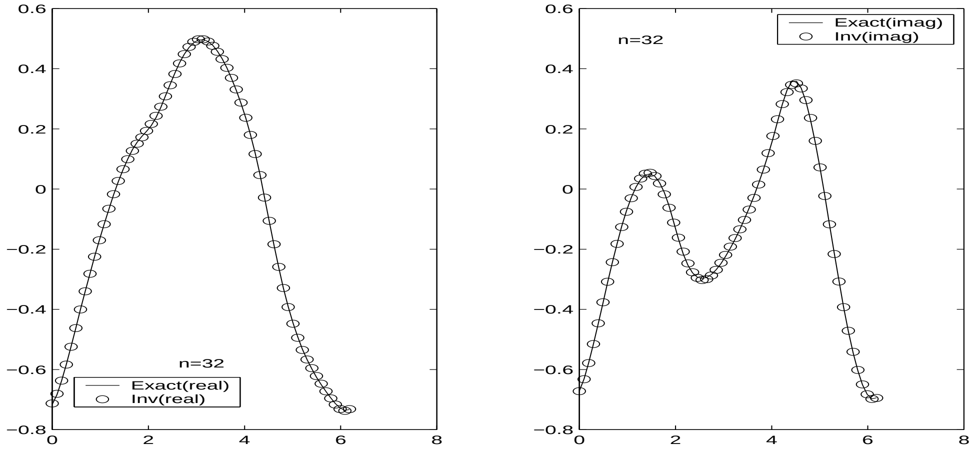

Fig. 4.7. Inversions for $z \in \partial Z$ with $n = 32, \delta = 0.05$

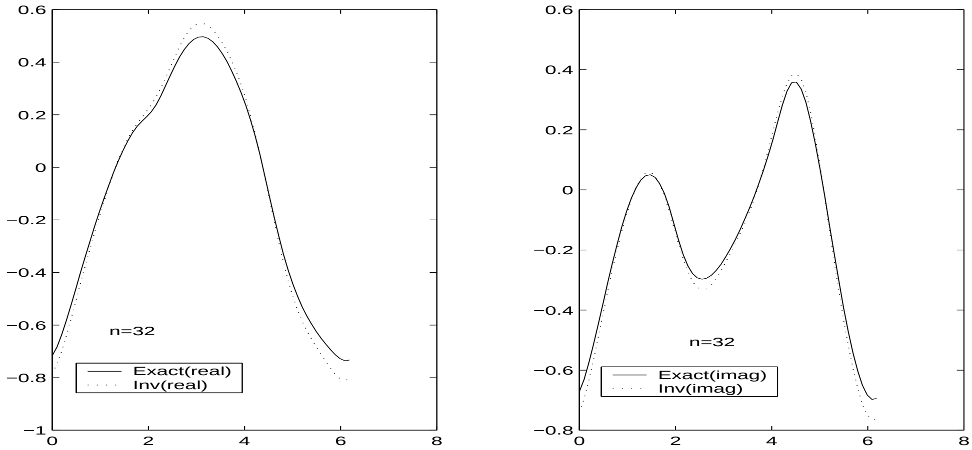

Fig. 4.8. Inversions for $z \in \partial Z$ with $n = 32, \delta = 0.1$

For the domain D_2 given here, it is easy to check that the domain G_2 bounded by

$$\partial G_2 = \{(x_1, x_2) := (-0.65 + 1.8\cos t, 1.8\sin t), t \in [0, 2\pi]\} \tag{4.8}$$

contains D_2 inside it (see Fig. 4.6). The boundary impedance on ∂D_2 is taken as

$$\sigma(x) = \frac{3 + x_1 x_2}{(3 + x_2)^2}, \quad x \in \partial D_2. \tag{4.9}$$

For this non-convex obstacle, we want to construct the scattered wave fields outside D_2 from the noisy data. According to our theoretical results, the scattered wave outside G_2 should be recovered with some convergence rate since G_2 itself is convex. To give numerical examples which relate to the theoretical results above we are interested in the reconstruction of $u^s(z)$ for $z \in \mathcal{R}^2 \setminus G_2$.

For $z \in \mathcal{R}^2 \setminus G_2$, we consider z lying in the circle $\partial Z_2 = \{x := 1.15 \times \partial G_2\}$. The values of $u^s(z)$ at four typical points $z = z(t)$ obtained by the potential method are listed in Table 1 and Table 2, where $[0, 2\pi]$ is divided into $2n = 32, 64$ subintervals respectively, when we solve the direct problem and the minimum norm solution.

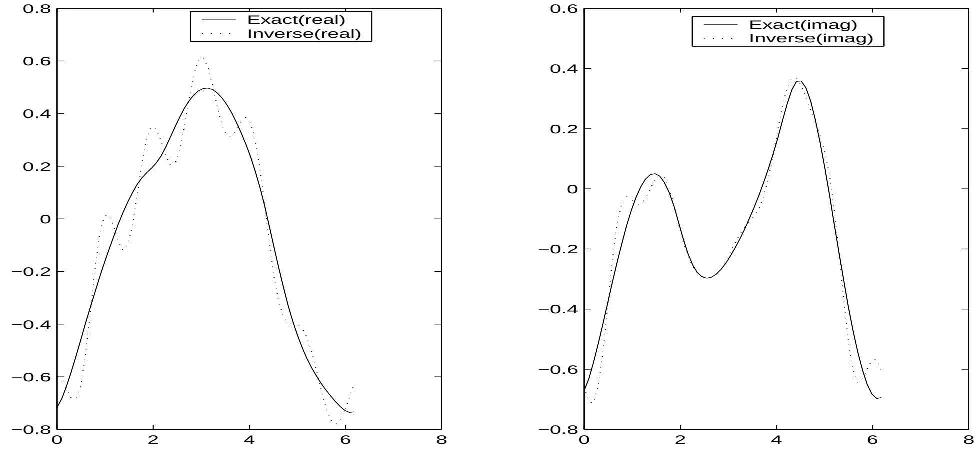

Fig. 4.9. Inversion results with random error level $\delta = 5\%$

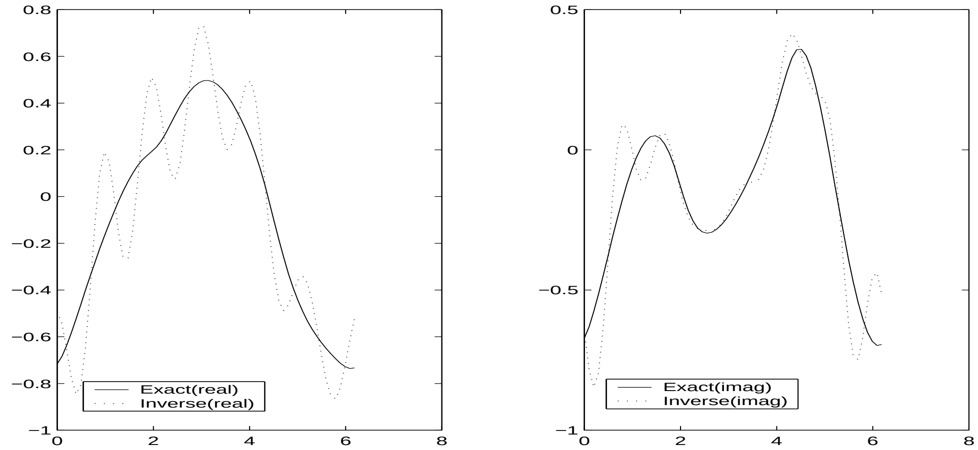

Fig. 4.10. Inversion results with random error level $\delta = 10\%$

Notice that, in our numerics, there are two regularization parameters. The first one is $\varepsilon = \varepsilon(\delta)$, which is applied to regularize the ill-posedness of determining $u^s(z)$ from its noisy far-field pattern with error level $\delta > 0$. In this step, the minimum norm solution $g_{\varepsilon(\delta)}$ is the regularizer, with $\varepsilon(\delta)$ depending on unknown constants $a(z), b(z), c(z)$. Without loss of generality, we consider $\varepsilon(\delta) < \|\Phi(\cdot, z)\|$ here. On the other hand, when we determine the minimum norm solution $g_{\varepsilon(\delta)}$ in step 2 for given $\varepsilon(\delta)$, we also introduce the other regularization parameter $\alpha = \alpha(\varepsilon(\delta))$. This is the theoretical story of determining $g_{\varepsilon(\delta)}$. Since the second step for determining $\alpha = \alpha(\varepsilon)$ has the estimate

$$\alpha(\varepsilon) \in \left(0, \frac{2\pi measu(\partial G)\varepsilon}{\|\Phi(\cdot, z)\| - \varepsilon}\right)$$

and $\varepsilon(\delta)$ depends on some unknown constants $a(z), b(z), c(z), \beta$ in (4.5), our numerics for determining $g_{\varepsilon(\delta)}$ combines the above two steps. That is, for $z \in \partial Z_2$, we take $\varepsilon(\delta)$ by trial and error for different choices of $a(z), b(z), c(z)$ in the interval $\varepsilon(\delta) \in [10^{-9}, 0.9 \times \|\Phi(\cdot, z)\|]$ and then taking $\alpha \in [10^{-9}, \frac{2\pi measu(\partial G)\varepsilon}{\|\Phi(\cdot, z)\| - \varepsilon}]$ and finally finding the minimum norm solution. Some inversion results obtained in this way with the noise level $\delta = 0, 0.05$ at special points are given Table 3. For the whole picture of inversion for points z in ∂Z_2 with error level $\delta = 0.05$ added in

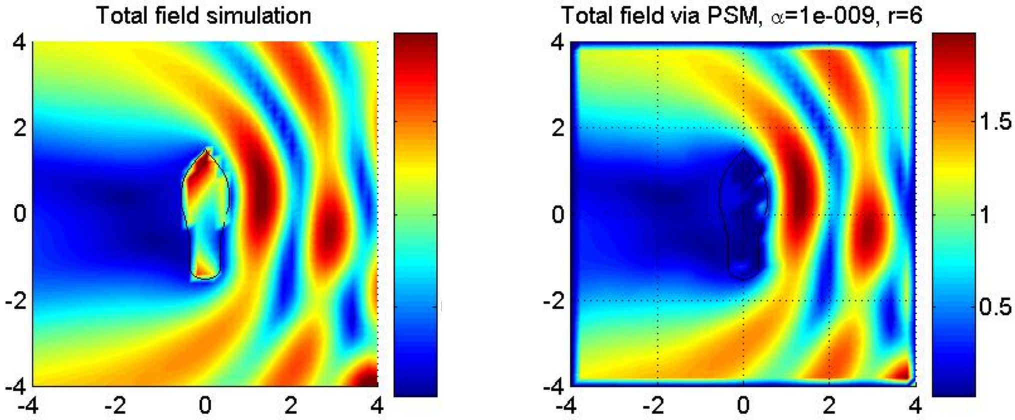

Fig. 4.11. Original total wave and reconstruction for non-plane incident wave

the form of (4.7), see Fig. 4.7. The results are so satisfactory that we cannot distinguish the inversions and true values from the figure. The difference of the two cases $\delta = 0$ and $\delta = 0.05$ can be found in Table 3. The results with noise level $\delta = 0.1$ added in the form (4.6) are shown in Fig. 4.8.

Table 3: Inversion $u^s_\delta(z)$ with $\delta = 0, 0.05$ for Special Points $z \in \partial Z$

t_j	$u^s_\delta(z(t_j)), \delta = 0$	$u^s_\delta(z(t_j)), \delta = 0.05$
0	-0.7129430-0.6721622i	-0.7082989-0.6717179i
$\pi/2$	0.0992444+0.0432346i	0.0963298+0.0392522i
π	0.4979953-0.1915083i	0.5020986-0.1829007i
$3\pi/2$	-0.2586794+0.2956636i	-0.2617419+0.2900757i

At the end of the numerical implementations, we test the numerical performance of the inversion scheme with random noisy data. Namely, rather than adding noise in the form (4.6) or (4.7), we generate the noisy data by

$$u^\infty_\delta(\hat{x}_i) = (1 + \delta \times p_s(i))u^\infty(\hat{x}_i), \tag{4.10}$$

where $p_s(i)$ for $i = 0, 1, \cdots, 2n - 1$ are random data distributed in $[-1, 1]$, which are generated by the Random(p) Subroutine for given seed value s in FORTRAN. Thus (4.10) represents the random noisy data with the relative error level δ. Two inversion results for our second example for $n = 32, s = 0.5$ with $\delta_1 = 0.05$ and $\delta_2 = 0.1$ are shown in Fig. 4.9 and Fig. 4.10. The numerical results suggest that, although there are some oscillations in the inversions with random noisy data input, our inversion scheme still generates a satisfactory result with L^2-norm error, which has been shown in our theoretical theorems.

5. Conclusions

In this paper, we propose a new regularizing scheme for the reconstruction of a scattered wave from its far-field pattern. By applying the classical potential theory, we generalize the point source method for an incident plane wave proposed by R.Potthast to general incident fields. The theoretical importance of this generalization is that we establish the relation between the classical potential theory for inverse scattering problem and the recently developed point source method. Moreover, by analyzing the asymptotic behavior of the Hankel function, we obtain an improved convergence rate of regularizing solution. Numerical examples show that this new reconstruction scheme can stably recover the scattered wave from the noisy data of far-field pattern.

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