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AN EXPANDED CHARACTERISTIC-MIXED FINITE ELEMENT METHOD FOR A CONVECTION-DOMINATED TRANSPORT PROBLEM [*1)]

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Abstract

In this paper, we propose an Expanded Characteristic-mixed Finite Element Method for approximating the solution to a convection dominated transport problem. The method is a combination of characteristic approximation to handle the convection part in time and an expanded mixed finite element spatial approximation to deal with the diffusion part. The scheme is stable since fluid is transported along the approximate characteristics on the discrete level. At the same time it expands the standard mixed finite element method in the sense that three variables are explicitly treated: the scalar unknown, its gradient, and its flux. Our analysis shows the method approximates the scalar unknown, its gradient, and its flux optimally and simultaneously. We also show this scheme has much smaller time-truncation errors than those of standard methods. A numerical example is presented to show that the scheme is of high performance.

Mathematics subject classification: 65N30, 65M25.
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1. Introduction

Given an open bounded domain $\Omega \subset R^2$ with a smooth boundary Γ and a time interval $(0, T]$, we consider the following convection-diffusion equation

$$\left\{ \begin{array}{lll} (a) & \frac{\partial c}{\partial t} + u(x) \cdot \nabla c - \nabla \cdot (a(x) \nabla c) = f(x,t), & in\ \Omega \times (0,T), \\ (b) & c(x,0) = c_0(x), & in\ \ \Omega, \end{array} \right. \tag{1}$$

where
1) c(x,t) denotes, for example, the concentration of a possible substance;
2) u(x) represents the velocity of the flow;
3) ∇ and $\nabla \cdot$ denote the gradient and the divergence operators respectively;
4) a(x) is sufficiently smooth and there exist constants a_1 and a_2 such that

$$0 < a_1 \le a(x) \le a_2 < +\infty; \tag{2}$$

5) f denotes a source term.

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This equation governs such phenomena as the flow of heat within a moving fluid, the transport of dissolved nutrients or contaminants within the groundwater, and the transport of a surfactant or tracer within an incompressible oil in a petroleum reservoir.

Because of molecular diffusion, a(x) is uniformly positive. Although this implies that the equation is uniformly parabolic, in many applications the Peclet number is quite high. Thus convection dominates diffusion, the equation is nearly hyperbolic in nature. The concentration often develops sharp fronts that are nearly shocks.

It is well known that strictly parabolic discretization schemes applied to the problem do not work well when it is convection dominated. It is especially difficult to approximate well the sharp fronts and conserve the material or mass in the system.

Effective discretization schemes should concentrate on the hyperbolic nature of the equation. Many such schemes have been developed, such as the explicit method of characteristics, upstream-weighted finite difference schemes [2], the streamline diffusion method [4], the least-squares mixed finite element method [8], the modified method of characteristics-Galerkin finite element procedure (MMOC-Galerkin)[12,13,14,15].

In this paper we propose a mixed method, called the expanded characteristic mixed finite element method. It is similar to MMOC-Galerkin for convection dominated transport problems in that we approximate the hyperbolic part of the equation along the characteristics, we use, however, the expanded mixed finite element method ([1],[16]) to discretize the diffusion part. This formulation expands the standard mixed formulation in the sense that three variables are explicitly treated; i.e., the scalar unknown, its gradient and its flux(the coefficient times the gradient). It is suitable for the case where the coefficient of the differential equations is a small tensor and does not need to be inverted.

An outline of this paper is as follows. In Section 2, we define an approximation to the characteristics and the expanded characteristic-mixed finite element formulation of the problem. We give the proof of the existence and uniqueness of the discrete problem in Section 3. In Section 4, we give some Lemmas which are important to our error analysis. In Section 5, the optimal order estimates for $c-c_h, \sigma-\sigma_h, \lambda-\lambda_h$ in $L^2(\Omega)$ are presented. The last Section is devoted to a numerical example.

2. The Expanded Characteristic-mixed Finite Element Method

We begin this section by introducing some notations.

We denote by $W^{k,p}(S)$ the standard Sobolev space of k-differential functions in $L^p(S)$. Let $\|\cdot\|_{k,p,S}$ be its norm and $\|\cdot\|_{k,S}$ be the norm of $H^k(S)=W^{k,2}(S)$ or $H^k(S)^2$, where we omit S if $S=\Omega$. When $k=0$, we let $L^2(\Omega)$ denote the corresponding space defined on Ω, its norm written as $\|\cdot\|$.

We also use the following spaces that incorporate time dependence. Let $[a,b]\subset[0,T]$, X be a Sobolev space, and $f(x,t)$ be suitably smooth on $\Omega\times[a,b]$. Also, we define $L^p(a,b;X)$ and $\|f\|_{L^p(a,b;X)}$ as follows

$$L^p(a,b;X)=\{f:\int_a^b\|f(\cdot,t)\|_X^p dt<\infty\},$$

$$\|f\|_{L^p(a,b;X)}=(\int_a^b\|f(\cdot,t)\|_X^p dt)^{\frac{1}{p}},$$

where if $p=\infty$, the integral is replaced by the essential supreme.

In this paper we assume that $u(x)$ satisfies

$$|u(x)|+|\nabla\cdot u(x)|\leq K,\qquad \forall x\in\Omega, \tag{3}$$

here and throughout this paper K denotes different constants in different places.

Under the above assumptions,we begin to discretize the problem. Let

$$\psi(x,t) = (1+|u|^2)^{\frac{1}{2}} \tag{4}$$

and let the characteristic direction associated with the operator $c_t + u \cdot \nabla u$ be denoted by $\tau = \tau(x)$, where

$$\frac{\partial}{\partial \tau(x)} = \frac{1}{\psi(x,t)}\frac{\partial}{\partial t} + \frac{u}{\psi(x,t)} \cdot \nabla, \tag{5}$$

then equation(1a) can be put in the form

$$\psi(x,t)\frac{\partial c}{\partial \tau} - \nabla \cdot (a(x)\nabla c) = f(x,t), \qquad (x,t) \in \Omega \times (0,T]. \tag{6}$$

Let $\lambda = -\nabla c, \sigma = -a(x)\nabla c = a(x)\lambda$, then (6) can be rewritten as

$$\begin{cases} (a) & \psi(x,t)\frac{\partial c}{\partial \tau} + \mathrm{div}\sigma = f, \\ (b) & \lambda + \nabla c = 0, \\ (c) & \sigma - a(x)\lambda = 0. \end{cases} \tag{7}$$

Define the spaces:

$$H(\mathrm{div};\Omega) = \{v \in (L^2(\Omega))^2 : \bigtriangledown \cdot v \in L^2(\Omega)\},$$

$$V = \{v \in H(\mathrm{div};\Omega) : v = 0 \quad on \quad \Gamma\},$$

$$W = L^2(\Omega), \qquad\qquad \Lambda = (L^2(\Omega))^2.$$

then the expanded characteristic-mixed variational problem corresponding to (1) is to find $(\sigma,\lambda,c) : (0,T] \to V \times \Lambda \times W$ such that

$$\begin{cases} (a) & (\psi\frac{\partial c}{\partial \tau}, w) + (\mathrm{div}\sigma, w) & = & (f,w), & \forall w \in W, \\ (b) & (\lambda, v) - (c, \mathrm{div} v) & = & 0, & \forall v \in V, \\ (c) & (a\lambda, \mu) - (\sigma, \mu) & = & 0, & \forall \mu \in \Lambda, \\ (d) & c(x,0) & = & c_0(x), & \forall x \in \Omega, \end{cases} \tag{8}$$

this form will be discretized below.

Let T_h be a quasi-regular polygonalization of Ω, its elements are denoted by E. $\forall E \in T_h$, $V_h(E) \times W_h(E)$[5,9–14] denote Raviart-Thomas space or Brezzi-Douglas-Marini space i.e. $V_h = (P_k(E))^n, \quad W_h = (P_{k-1}(E))(k \geq 1)$ Then we define

$$\Lambda_h = \{\mu \in \Lambda; \mu|_E \in V_h(E), \qquad \forall E \in T_h\},$$

$$V_h = \{v \in V; v|_E \in V_h(E), \qquad \forall E \in T_h\},$$

$$W_h = \{w \in W; w|_E \in W_h(E), \qquad \forall E \in T_h\}.$$

In the procedure to be used, we shall consider a time step $\Delta t > 0$ and approximate the solution at times $t^n = n\Delta t$, the characteristic derivative will be approximated basically in the following manner:

Let

$$\overline{x} = x - u(x,t^n)\Delta t,$$

and then we have the following approximation

$$\begin{aligned} \psi(x,t^n)\frac{\partial c}{\partial \tau}|_{t_n} \quad &\approx \quad \psi(x,t^n)\frac{c(x,t^n)-c(\overline{x},t^{n-1})}{\sqrt{(x-\overline{x})^2+(\Delta t)^2}} \\ &= \quad \frac{c(x,t^n)-c(\overline{x},t^{n-1})}{\Delta t}, \end{aligned}$$

so our expanded characteristics-mixed finite element method is the determination of the map $(\sigma_h, \lambda_h, c_h) : \{t^0, t^1, \cdots, t^N\} \to V_h \times \Lambda_h \times W_h$ satisfying the relations

$$\left\{ \begin{array}{lrcll} (a) & (\frac{c_h^n - \bar{c}_h^{n-1}}{\Delta t}, w_h) + (\mathrm{div}\sigma_h^n, w_h) & = & (f^n, w_h), & \forall w_h \in W_h, \\ (b) & (\lambda_h^n, v_h) - (c_h^n, \mathrm{div} v_h) & = & 0, & \forall v_h \in V_h, \\ (c) & (a\lambda_h^n, \mu_h) - (\sigma_h^n, \mu_h) & = & 0, & \forall \mu_h \in \Lambda_h, \\ (d) & c_h^0 & = & \widetilde{c}_0, & \forall x \in \Omega, \end{array} \right. \tag{9}$$

where $c_h^n = c_h(t^n), \bar{x} = x - u_h^{n-1}(x)\Delta t, \quad \bar{c}_h^{n-1} = c_h^{n-1}(\bar{x}) = c_h^{n-1}(x - u_h^{n-1}(x)\Delta t)$, $\tilde{c}_0$ are defined in (11) for $t = 0$.

3. The Existence and Uniqueness of the Solution of the Discrete Problem

In this Section we give the proof of the uniqueness of the solution of the discrete problem (9).

Theorem 1. *Under the assumption of* (2), *there exists a unique solution* $(\sigma_h, \lambda_h, c_h) \in V_h \times \Lambda_h \times W_h$ *to the expanded characteristic-mixed finite element procedure* (9).

Proof. The linear system generated by (9) is square, so the existence of the solution is implied by its uniqueness. Let c_h^{n-1}, f, be zero, then $\bar{c}_h^{n-1}$ are zero too. Then from (9) we have

$$\left\{ \begin{array}{lrcll} (a) & (\frac{c_h^n}{\Delta t}, w_h) + (\mathrm{div}\sigma_h^n, w_h) & = & 0, & \forall w_h \in W_h, \\ (b) & (\lambda_h^n, v_h) - (c_h^n, \mathrm{div} v_h) & = & 0, & \forall v_h \in V_h, \\ (c) & (a\lambda_h^n, \mu_h) - (\sigma_h^n, \mu_h) & = & 0, & \forall \mu_h \in \Lambda_h, \end{array} \right. \tag{10}$$

Choosing $w_h = c_h^n$ in (10a), $v_h = \sigma_h^n$ in (10b), $\mu_h = \lambda_h^n$ in (10c) and adding them together gives

$$\frac{1}{\Delta t}\|c_h^n\|^2 + (a\lambda_h^n, \lambda_h^n) = 0,$$

according to (2), we get $c_h^n = \lambda_h^n = 0$, then with (10c) we have $(\sigma_h^n, \mu_h) = 0$, choosing $\mu_h = \sigma_h^n$ gives $\sigma_h^n = 0$, the existence and uniqueness of the solution of the discrete problem is obtained.

In order to analyze the convergence of the method, it is convenient to introduce the expanded mixed elliptic projection associated with our equations.
Let $(\tilde{\sigma}, \tilde{\lambda}, \tilde{u}) : [0, T] \to V_h \times \Lambda_h \times W_h$ be given by the relations

$$\left\{ \begin{array}{lrlll} (a) & (a(\lambda - \tilde{\lambda}), \mu) - (\sigma - \tilde{\sigma}, \mu) & = 0, & \forall \mu \in \Lambda_h, & 0 \le t \le T, \\ (b) & (\lambda - \tilde{\lambda}, v) - (c - \tilde{c}, \mathrm{dviv}) & = 0, & \forall v \in V_h, & 0 \le t \le T, \\ (c) & (\mathrm{div}(\sigma - \tilde{\sigma}), w) & = 0, & \forall w \in W_h, & 0 \le t \le T. \end{array} \right. \tag{11}$$

Let $e_h^n = c_h^n - \widetilde{c}^n, \rho^n = \widetilde{c}^n - c^n, \varepsilon_h^n = \lambda_h^n - \widetilde{\lambda}^n, \xi^n = \widetilde{\lambda}^n - \lambda^n, \zeta_h^n = \sigma_h^n - \widetilde{\sigma}^n, \eta^n = \widetilde{\sigma}^n - \sigma^n$, the following estimates is indicated in [1],

$$\begin{array}{ll} \|\xi\| = \|\lambda - \tilde{\lambda}\| \le Kh^r\{\|\lambda\|_r + \|\sigma\|_r\}, & 1 \le r \le k+1, \\ \|\xi_t\| = \|\lambda_t - \tilde{\lambda}_t\| \le Kh^r\{\|\lambda_t\|_r + \|\sigma_t\|_r\}, & 1 \le r \le k+1, \end{array} \tag{12}$$

$$\begin{array}{ll} \|\eta\| = \|\sigma - \tilde{\sigma}\| \le Kh^r\{\|\sigma\|_r + \|\lambda\|_r\}, & 1 \le r \le k+1, \\ \|\eta_t\| = \|\sigma_t - \tilde{\sigma}_t\| \le Kh^r\{\|\sigma_t\|_r + \|\lambda_t\|_r\}, & 1 \le r \le k+1, \end{array} \tag{13}$$

$$\begin{array}{l} \|\rho\| = \|c - \tilde{c}\| \le \left\{ \begin{array}{ll} Kh\|c\|_3, & k = 1, \\ Kh^r\|c\|_r, & 2 \le r \le k. \end{array} \right. \\ \|\rho_t\| = \|c_t - \tilde{c}_t\| \le \left\{ \begin{array}{ll} Kh\|c_t\|_3, & k = 1, \\ Kh^r\|c_t\|_r, & 2 \le r \le k. \end{array} \right. \end{array} \tag{14}$$

4. Several Lemmas

To obtain the optimal error estimates, we introduce the following four basic lemmas. These four lemmas are crucial to our main arguments.

Lemma 1. [6] *For any function $\tau \in L^2(\Omega)$, there exists a function $q_\tau \in H^1(\Omega)^2$, such that*

$$\begin{cases} a) & \mathrm{div} \mathrm{q}_\tau = \tau, \\ b) & \|q_\tau\|_1 \le K\|\tau\|, \\ c) & \|q_\tau\| \le K\|\tau\|_{-1}. \end{cases} \tag{15}$$

where K is a constant.

Lemma 2. *For any $\tau \in L^2(\Omega)$, if $(\xi, \varepsilon_h) \in W_h \times \Lambda_h$ satisfying*

$$(\varepsilon_h, q_h) - (\xi, \mathrm{div} q_h) = 0, \quad \forall q_h \in V_h,$$

then, there exists a constant $K > 0$ such that

$$|(\tau, \xi)| \le K(h\|\tau\| + \|\tau\|_{-1})\|\varepsilon_h\|. \tag{16}$$

Proof. Let q_τ be the corresponding function of the given τ in (15).
Since

$$\begin{aligned} (\tau, \xi) &= (\mathrm{div} q_\tau, \xi) \\ &= (\mathrm{div}(q_\tau - \pi_h q_\tau), \xi) + (\mathrm{div} \pi_h q_\tau, \xi) \\ &= (\mathrm{div} \pi_h q_\tau, \xi) = (\varepsilon_h, \pi_h q_\tau) \\ &= (\varepsilon_h, q_\tau - \pi_h q_\tau) + (\varepsilon_h, q_\tau), \end{aligned}$$

where π_h is the R-T projection operator. So

$$\begin{aligned} |(\tau, \xi)| &\le K\{h\|\varepsilon_h\|\|q_\tau\|_1 + \|\varepsilon_h\|\|q_\tau\|\} \\ &\le K(h\|\tau\| + \|\tau\|_{-1})\|\varepsilon_h\|. \end{aligned}$$

That is the desired estimate.

Lemma 3. *Let $\eta \in L^2(\Omega)$, $\bar{\eta} = \eta(x - u(x)\Delta t)$, and $u \in W^{1,\infty}(\Omega)$. If $(\xi, \varepsilon_h) \in W_h \times \Lambda_h$ satisfying*

$$(\varepsilon_h, q_h) - (\xi, \mathrm{div} q_h) = 0, \quad \forall q_h \in V_h,$$

then

$$|(\eta - \bar{\eta}, \xi)| \le K(h + \Delta t)\|\eta\|\|\varepsilon_h\|. \tag{17}$$

Proof. Let $\tau = \eta - \bar{\eta} \in L^2(\Omega)$, lemma 2 indicates

$$|(\eta - \bar{\eta}, \xi)| \le K(h\|\eta - \bar{\eta}\| + \|\eta - \bar{\eta}\|_{-1})\|\varepsilon_h\|,$$

by [7] we obtain

$$\|\eta - \bar{\eta}\| \le k\|\eta\|\Delta t,$$

$$\|\eta - \bar{\eta}\| \le K\|\eta\|,$$

combining the above inequalities we get the conclusion.

5. Error Estimates

Under the above assumptions about Ω, c , we can derive the optimal order estimates of $(c_h - c)$, $(\sigma_h - \sigma)$ and $(\lambda_h - \lambda)$.

Theorem 2. Let $(\sigma_h,\lambda_h,c_h),(\sigma,\lambda,c)$ denote the solution of (9) and (1) respectively. Suppose $\Delta t=O(h)$, we have for sufficiently small $\Delta t>0$

$$
\begin{array}{llll}
(a) & \max\limits_{0\le n\le N}\|(\sigma_h-\sigma)(t^n)\| & \le & \left\{\begin{array}{ll} M_1h+M_0\Delta t, & k=1,\\ M_2h^r+M_0\Delta t, & 2\le r\le k.\end{array}\right.\\
(b) & \max\limits_{0\le n\le N}\|(\lambda_h-\lambda)(t^n)\| & \le & \left\{\begin{array}{ll} M_1h+M_0\Delta t, & k=1,\\ M_2h^r+M_0\Delta t, & 2\le r\le k.\end{array}\right.\\
(c) & \max\limits_{0\le n\le N}\|(c_h-c)(t^n)\| & \le & \left\{\begin{array}{ll} M_1h+M_0\Delta t, & k=1,\\ M_3h^r+M_0\Delta t, & 2\le r\le k.\end{array}\right.
\end{array}
\tag{18}
$$

where

$$M_0=K\|\frac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(0,T;L^2)},\quad M_1=K(\|c\|_{L^\infty(0,T;H^3)}+\|c_t\|_{L^2(0,T;H^3)}),$$

$$M_2=K(\|c\|_{L^\infty(0,T;H^{r+1})}+\|c_t\|_{L^2(0,T;H^{r+1})}),$$

$$M_3=K(\|c\|_{L^\infty(0,T;H^{r})}+\|c_t\|_{L^2(0,T;H^{r})}).$$

Proof. By (8), (9), (11) we get the error equation in the form

$$
\left\{\begin{array}{llcl}
(a) & (\frac{e_h^n-\bar e_h^{n-1}}{\Delta t},w_h)+(\mathrm{div}\zeta_h^n,w_h) & = & (\psi\frac{\partial c^n}{\partial\tau}-\frac{c_h^n-\bar c_h^{n-1}}{\Delta t},w_h)\\
 & & - & (\frac{\rho_h^n-\bar\rho_h^{n-1}}{\Delta t},w_h),\quad \forall w_h\in W_h,\\
(b) & (\varepsilon_h^n,v_h)-(e_h^n,\mathrm{div}v_h) & = & 0,\quad \forall v_h\in V_h,\\
(c) & (a\varepsilon_h^n,\mu_h)-(\zeta_h^n,\mu_h) & = & 0,\quad \forall \mu_h\in \Lambda_h.
\end{array}\right.
\tag{19}
$$

Taking $w_h=e_h^n,\ \ v_h=\zeta_h^n,\ \ \mu_h=\varepsilon_h^n$ in (19) gives

$$
\left\{\begin{array}{llcl}
(a) & (\frac{e_h^n-\bar e_h^{n-1}}{\Delta t},e_h^n)+(\mathrm{div}\zeta_h^n,e_h^n) & = & (\psi\frac{\partial c^n}{\partial\tau}-\frac{c_h^n-\bar c_h^{n-1}}{\Delta t},e_h^n)\\
 & & - & (\frac{\rho_h^n-\bar\rho_h^{n-1}}{\Delta t},e_h^n),\\
(b) & (\varepsilon_h^n,\zeta_h^n)-(e_h^n,\mathrm{div}\zeta_h^n) & = & 0,\\
(c) & (a\varepsilon_h^n,\varepsilon_h^n)-(\zeta_h^n,\varepsilon_h^n) & = & 0,
\end{array}\right.
$$

adding the three equalities we get

$$
\begin{array}{lcl}
(\frac{e_h^n-\bar e_h^{n-1}}{\Delta t},e_h^n)+(a\varepsilon_h^n,\varepsilon_h^n) & = & (\psi\frac{\partial c^n}{\partial\tau}-\frac{c_h^n-\bar c_h^{n-1}}{\Delta t},e_h^n)\\
 & - & (\frac{\rho_h^n-\bar\rho_h^{n-1}}{\Delta t},e_h^n),
\end{array}
\tag{20}
$$

then using the method similar to [3], the following result is obtained

$$\|\psi^n\frac{\partial c^n}{\partial\tau}-\frac{c^n-\bar u^{n-1}}{\Delta t}\|^2\le K\|\frac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;L^2)}\Delta t+\|e_h^n\|^2. \tag{21}$$

Considering the second term on the right side of (20), rewrite $\rho^n-\bar\rho^{n-1}$ as $(\rho^n-\rho^{n-1})+(\rho^{n-1}-\bar\rho^{n-1})$, then

$$|(\frac{\rho^n-\rho^{n-1}}{\Delta t},e_h^n)|\le\frac{K}{\Delta t}\int_{t^{n-1}}^{t^n}\|\rho_t\|^2ds+K\|e_h^n\|^2. \tag{22}$$

By Lemma 3 and (19b) , under the assumption $\Delta t=O(h)$, we get

$$
\begin{array}{lcl}
|(\frac{\rho^{n-1}-\bar\rho^{n-1}}{\Delta t},e_h^n)| & \le & K(\frac{h+\Delta t}{\Delta t})\|\rho^{n-1}\|\|\varepsilon_h^n\|\\
 & \le & K\|\rho\|^2_{L^\infty(0,T;L^2)}+a_1\|\varepsilon_h^n\|^2,
\end{array}
\tag{23}
$$

The left-hand side of (20) is bounded by

$$\begin{aligned} & (\tfrac{e_h^n-\bar{e}_h^{n-1}}{\Delta t}, e_h^n) + (a(x)\varepsilon_h^n, \varepsilon_h^n) \\ \geq\ & \tfrac{1}{2\Delta t}[(e_h^n, e_h^n) - (\bar{e}_h^{n-1}, \bar{e}_h^{n-1})] + a_1\|\varepsilon_h^n\|^2 \\ \geq\ & \tfrac{1}{2\Delta t}[(e_h^n, e_h^n) - (1+K\Delta t)(e_h^{n-1}, e_h^{n-1})] + a_1\|\varepsilon_h^n\|^2, \end{aligned} \tag{24}$$

where the inequality $\|\bar{e}_h^n\| \leq (1+K\Delta t)\|e_h^n\|$ ([7]) is used in the last step.

Combining (21)–(24) with (20) to give the recursive relation

$$\begin{aligned} & \tfrac{1}{2\Delta t}[(e_h^n, e_h^n) - (e_h^{n-1}, e_h^{n-1})] \\ \leq\ & K\Delta t\|\tfrac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;L^2)} \\ +\ & \tfrac{K}{\Delta t}\int_{t^{n-1}}^{t^n}\|\rho_t\|^2 ds + K\|\rho\|^2_{L^\infty(0,T;L^2)} \\ +\ & (K+1)\|e_h^n\|^2 + K\|e_h^{n-1}\|^2, \end{aligned} \tag{25}$$

multiplying (25) by $2\Delta t$, adding in time and noting that (11) implies that $e_h^0 = 0$, we obtain

$$\begin{aligned} \|e_h^n\|^2 \ \leq\ & K\Delta t^2\|\tfrac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(0,T;L^2)} + K\|\rho\|^2_{L^\infty(0,T;L^2)} \\ +\ & K\|\rho_t\|^2_{L^2(0,T;L^2)} + K\Delta t\sum_{i=1}^{n}\|e_h^i\|^2. \end{aligned}$$

Finally, by Gronwall's lemma, for sufficiently small $\Delta t > 0$,

$$\|e_h^n\| \leq K\Delta t\|\frac{\partial^2 u}{\partial\tau^2}\|^2_{L^2(0,T;L^2)} + K(\|\rho\|^2_{L^\infty(0,T;L^2)} + \|\rho_t\|^2_{L^2(0,T;L^2)}), \tag{26}$$

recall that the error $c_h - c = e_h + \rho$, (14) and (26) together imply (18c).

In order to show the estimate (18b), by (19b), we get

$$(\frac{\varepsilon_h^n - \varepsilon_h^{n-1}}{\Delta t}, v_h) - (\frac{e_h^n - e_h^{n-1}}{\Delta t}, \mathrm{div} v_h) = 0, \quad \forall v_h \in V_h. \tag{27}$$

Choosing $w_h = \frac{e_h^n-e_h^{n-1}}{\Delta t}, \mu_h = \frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\Delta t}, v_h = \zeta_h^n$ respectively in (19a), (19c) and (27) and adding to obtain

$$\begin{aligned} & (\tfrac{e_h^n-\bar{e}_h^{n-1}}{\Delta t}, \tfrac{e_h^n-e_h^{n-1}}{\Delta t}) + (\tfrac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\Delta t}, a\varepsilon_h^n) \\ =\ & (\psi^n\tfrac{\partial c^n}{\partial\tau} - \tfrac{c^n-\bar{c}^{n-1}}{\Delta t}, \tfrac{e_h^n-e_h^{n-1}}{\Delta t}) - (\tfrac{\rho^n-\rho^{n-1}}{\Delta t}, \tfrac{e_h^n-e_h^{n-1}}{\Delta t}) \\ -\ & (\tfrac{\rho^{n-1}-\bar{\rho}^{n-1}}{\Delta t}, \tfrac{e_h^n-e_h^{n-1}}{\Delta t}). \end{aligned} \tag{28}$$

The left-hand side satisfies the inequality

$$\begin{aligned} & (\tfrac{e_h^n-\bar{e}_h^{n-1}}{\Delta t}, \tfrac{e_h^n-e_h^{n-1}}{\Delta t}) + (\tfrac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\Delta t}, a\varepsilon_h^n) \\ \geq\ & \|\tfrac{e_h^n-e_h^{n-1}}{\Delta t}\|^2 + \tfrac{1}{2\Delta t}((\varepsilon_h^n, a\varepsilon_h^n) - (\varepsilon_h^{n-1}, a\varepsilon_h^{n-1})) \\ & +(\tfrac{e_h^{n-1}-\bar{e}_h^{n-1}}{\Delta t}, \tfrac{e_h^n-e_h^{n-1}}{\Delta t}). \end{aligned} \tag{29}$$

The first two terms on the right-hand side of (28) is bounded in a way analogous to that for (21), (22)

$$\begin{aligned} & |(\psi^n\tfrac{\partial u^n}{\partial\tau} - \tfrac{c^n-\bar{c}^{n-1}}{\Delta t}, \tfrac{e_h^n-e_h^{n-1}}{\Delta t}) \\ \leq\ & K\|\tfrac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;L^2)}\Delta t + \tfrac{1}{4}\|\tfrac{e_h^n-e_h^{n-1}}{\Delta t}\|^2, \end{aligned} \tag{30}$$

$$|(\frac{\rho^n-\rho^{n-1}}{\Delta t},\frac{e_h^n-e_h^{n-1}}{\Delta t})|\le\frac{K}{\Delta t}\|\rho_t\|^2_{L^2(t^{n-1},t^n;L^2)}+\frac{1}{4}\|\frac{e_h^n-e_h^{n-1}}{\Delta t}\|^2. \tag{31}$$

From (28)–(31) we get

$$\begin{aligned}
&\tfrac{1}{2}\|\tfrac{e_h^n-e_h^{n-1}}{\Delta t}\|^2+\tfrac{1}{2\Delta t}((\varepsilon_h^n,a\varepsilon_h^n)-(\varepsilon_h^{n-1},a\varepsilon_h^{n-1}))\\
\le\;& K\|\tfrac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;L^2)}\Delta t+\tfrac{K}{\Delta t}\|\rho_t\|^2_{L^2(t^{n-1},t^n;L^2)}\\
-\;& (\tfrac{\rho^{n-1}-\bar\rho^{n-1}}{\Delta t},\tfrac{e_h^n-e_h^{n-1}}{\Delta t})-(\tfrac{e_h^{n-1}-\bar e_h^{n-1}}{\Delta t},\tfrac{e_h^n-e_h^{n-1}}{\Delta t}).
\end{aligned}\tag{32}$$

Noting that $e_h^0=0$ we have $\varepsilon_h^0=0$ from (19b) and (19c). To multiply (32) by $2\Delta t$ and add them in time, we obtain

$$\begin{aligned}
&\Delta t\sum_{i=1}^n\|\tfrac{e_h^i-e_h^{i-1}}{\Delta t}\|^2+a_1\|\varepsilon_h^n\|^2\\
\le\;& K\|\tfrac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(0,T;L^2)}\Delta t^2+K\|\rho_t\|^2_{L^2(0,T;L^2)}\\
-\;& 2\sum_{i=1}^n(\tfrac{\rho^{i-1}-\bar\rho^{i-1}}{\Delta t},e_h^i-e_h^{i-1})-2\sum_{i=1}^n(\tfrac{e_h^{i-1}-\bar e_h^{i-1}}{\Delta t},e_h^i-e_h^{i-1}).
\end{aligned}\tag{33}$$

By Lemma 3, and (19b)

$$\begin{aligned}
&\sum_{i=1}^n(\tfrac{\rho^{i-1}-\bar\rho^{i-1}}{\Delta t},e_h^i-e_h^{i-1})\\
=\;& (\tfrac{\rho^{n-1}-\bar\rho^{n-1}}{\Delta t},e_h^n)+\sum_{i=1}^{n-1}(\tfrac{\rho^{i-1}-\rho^i-(\bar\rho^{i-1}-\bar\rho^i)}{\Delta t},e_h^i)\\
\le\;& K(\tfrac{h}{\Delta t}+1)\|\rho^{n-1}\|\|\varepsilon_h^n\|+K(\tfrac{h}{\Delta t}+1)\sum_{i=1}^{n-1}\|\rho^{i-1}-\rho^i\|\|\varepsilon_h^i\|\\
\le\;& K\|\rho\|^2_{L^\infty(0,T;L^2)}+\tfrac{a_1}{4}\|\varepsilon_h^n\|^2\\
+\;& K\|\rho_t\|^2_{L^2(0,T;L^2)}+K\Delta t\sum_{i=1}^{n-1}\|\varepsilon_h^i\|^2.
\end{aligned}\tag{34}$$

Similarly,

$$\begin{aligned}
&\sum_{i=1}^n(\tfrac{e_h^{i-1}-\overline{e}_h^{i-1}}{\Delta t},e_h^i-e_h^{i-1})\\
=\;& (\tfrac{e_h^{n-1}-\overline{e}_h^{n-1}}{\Delta t},e_h^n)+\sum_{i=1}^{n-1}(\tfrac{e_h^{i-1}-e_h^i-(\overline{e}_h^{i-1}-\overline{e}_h^i)}{\Delta t},e_h^i)\\
\le\;& K\|e_h^{n-1}\|^2+\tfrac{a_1}{4}\|\varepsilon_h^n\|^2+K\sum_{i=1}^{n-1}\|e_h^i-e_h^{i-1}\|\|\varepsilon_h^i\|\\
\le\;& K\|e_h^{n-1}\|^2+\tfrac{a_1}{4}\|\varepsilon_h^n\|^2\\
+\;& K\Delta t\sum_{i=1}^{n-1}\|\varepsilon_h^i\|^2+\tfrac{\Delta t}{2}\sum_{i=1}^{n-1}\|\tfrac{e_h^i-e_h^{i-1}}{\Delta t}\|^2.
\end{aligned}\tag{35}$$

Substituting (34), (35) into (33), we get

$$\begin{aligned}
&\Delta t \sum_{i=1}^{n} \|\tfrac{e_h^i - e_h^{i-1}}{\Delta t}\|^2 + a_1 \|\varepsilon_h^n\|^2 \\
\leq\ & K\Delta t^2 \|\tfrac{\partial^2 c}{\partial \tau^2}\|^2_{L^2(I;L^2)} + K(\|\rho\|^2_{L^\infty(0,T;L^2)} + \|\rho_t\|^2_{L^2(0,T;L^2)}) \\
+\ & K\|e_h^{n-1}\|^2 + K\Delta t \sum_{i=1}^{n-1} \|\varepsilon_h^i\|^2.
\end{aligned}$$

By Gronwall's Lemma, it follows that

$$\begin{aligned}
&(\Delta t \sum_{i=1}^{n} \|\tfrac{e_h^i - e_h^{i-1}}{\Delta t}\|^2)^{\frac{1}{2}} + \|\varepsilon_h^n\| \\
\leq\ & K(\|\rho\|_{L^\infty(0,T;L^2)} + \|\rho_t\|_{L^2(0,T;L^2)}) + K\|e_h^{n-1}\| + K\|\tfrac{\partial^2 c}{\partial \tau^2}\|_{L^2(0,T;L^2)}\Delta t.
\end{aligned}$$

Note that $\lambda_h^n - \lambda^n = \varepsilon_h^n + \xi^n$, by (18c), (12) and (14) and the triangle inequality, we obtain (18b). Choosing $\mu_h = \zeta_h \in V_h \subset \Lambda_h$ in (19c) we get $(a\varepsilon_h^n, \zeta_h^n) = (\zeta_h^n, \zeta_h^n)$, so

$$\begin{aligned}
\|\zeta_h^n\| &\leq K\|\varepsilon_h^n\| \\
&\leq K(\|\rho\|_{L^\infty(0,T;L^2)} + \|\rho_t\|_{L^2(0,T;L^2)}) \\
&+ K\|e_h^{n-1}\| + K\|\tfrac{\partial^2 c}{\partial \tau^2}\|_{L^2(0,T;L^2)}\Delta t.
\end{aligned}$$

Note that $\sigma_h^n - \sigma^n = \zeta_h^n + \eta^n$, by (18c), (13) and the triangle inequality, we obtain (18a).
This completes the proof of the theorem.

6. Numerical Example

To test the new mixed finite element approximation scheme presented in this paper, we implemented the algorithm (12) to the following problem:

$$\begin{cases}
(a) & \frac{\partial c}{\partial t} + \nabla c - \nu \nabla \cdot (\nabla c) = f(x,t), & in\ [0,1], \\
(b) & c(1,t) = c(0,t) = 0, & t \in (0,1), \\
(c) & c(x,0) = sin\pi x, & x \in [0,1],
\end{cases} \tag{36}$$

Choosing $\nu = \frac{1}{48}$, taking $k = 1$ in the finite element spaces. After using MATLAB program to calculate the approximate velocity c_h^n, the gradient λ_h^n and the flux σ_h^n, then we calculate the discrete norm of L^2, which are given in the following tables 1, 2, 3 respectively. We also calculate the convergence order which are given in table 4. In addition, we draw the plots of these quantities in figure 1, 2, 3. From the result of our numerical test, we can see our scheme is indeed efficient.

Table 1. $\|c - c_h\|_{0,h}$

$h = \Delta t$	t=0.1	0.2	0.3	0.4	0.5
0.01	0.0009	0.0010	0.0011	0.0012	0.0013
0.005	0.0005	0.0005	0.0006	0.0006	0.0007
0.0025	0.2382E-3	0.2633E-3	0.2910E-3	0.3216E-3	0.3555E-3
0.00125	0.1163E-3	0.1307E-3	0.1448E-3	0.1605E-3	0.1776E-3
$h = \Delta t$	t=0.6	0.7	0.8	0.9	1.0
0.01	0.0015	0.0016	0.0018	0.0020	0.0022
0.005	0.0008	0.0009	0.0009	0.0010	0.0012
0.0025	0.3928E-3	0.4342E-3	0.4798E-3	0.5303E-3	0.5861E-3
0.00125	0.1950E-3	0.2154E-3	0.2345E-3	0.2627E-3	0.2911E-3

Table 2. $\|\lambda - \lambda_h\|_{0,h}$

$h = \Delta t$	t=0.1	0.2	0.3	0.4	0.5
0.01	0.0383	0.0423	0.0468	0.0517	0.0572
0.005	0.0193	0.0213	0.0235	0.0260	0.0287
0.0025	0.0097	0.0107	0.0118	0.0130	0.0144
0.00125	0.0049	0.0054	0.0059	0.0065	0.0072
$h = \Delta t$	t=0.6	0.7	0.8	0.9	1.0
0.01	0.0632	0.0698	0.0772	0.0853	0.0942
0.005	0.0317	0.0351	0.0388	0.0429	0.0474
0.0025	0.0159	0.0176	0.0194	0.0215	0.0237
	0.0079	0.0089	0.0098	0.0108	0.0119

Table 3. $\|\sigma - \sigma_h\|_{0,h}$

$h = \Delta t$	t=0.1	0.2	0.3	0.4	0.5
0.01	0.0309	0.0342	0.0378	0.0417	0.0461
0.005	0.0154	0.0170	0.0188	0.0208	0.0230
0.0025	0.0077	0.0085	0.0094	0.0104	0.0115
0.00125	0.0038	0.0042	0.0047	0.0052	0.0057
$h = \Delta t$	t=0.6	0.7	0.8	0.9	1.0
0.01	0.0510	0.0563	0.0623	0.0688	0.0760
0.005	0.0254	0.0281	0.0310	0.0343	0.0379
0.0025	0.0127	0.0140	0.0155	0.0171	0.0189
0.00125	0.0064	0.0070	0.0077	0.0085	0.0094

Table 4. Convergence Order

Time	order for $\|c - c_h\|_{0,h}$			order for $\|\lambda - \lambda_h\|_{0,h}$			order for $\|\sigma - \sigma_h\|_{0,h}$		
t	$\frac{0.01}{0.005}$	$\frac{0.005}{0.0025}$	$\frac{0.0025}{0.00125}$	$\frac{0.01}{0.005}$	$\frac{0.005}{0.0025}$	$\frac{0.0025}{0.00125}$	$\frac{0.01}{0.005}$	$\frac{0.005}{0.0025}$	$\frac{0.0025}{0.00125}$
0.1	0.8480	1.0698	1.0343	0.9887	0.9925	1.0764	1.0047	1.0000	1.0189
0.2	1.0000	0.9252	1.0104	0.9898	0.9932	1.0410	1.0085	1.0000	1.0171
0.3	0.8475	1.0439	1.0070	0.9938	0.9939	1.0000	1.0077	1.0000	1.0000
0.4	1.0000	0.8997	1.0027	0.9917	1.0000	1.0000	1.0035	1.0000	1.0000
0.5	0.8931	0.9775	1.0012	0.9950	0.9950	1.0000	1.0031	1.0000	1.0216
0.6	0.9069	1.0262	1.0103	0.9954	0.9955	1.0091	1.0057	1.0000	1.0216
0.7	0.8301	1.0516	1.0113	0.9918	0.9959	1.0165	1.0026	1.0051	1.0000
0.8	1.0000	0.9075	1.0328	0.9925	1.0000	1.0150	1.0070	1.0000	1.0093
0.9	1.0000	0.9151	1.0134	0.9916	0.9966	1.0067	1.0042	1.0042	1.0085
1.0	0.8745	1.0338	1.0096	0.9908	1.0000	1.0061	1.0038	1.0038	1.0387

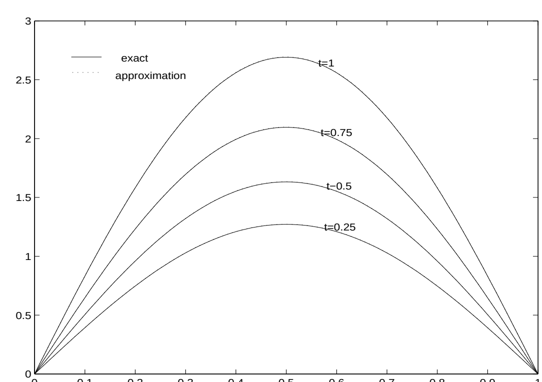

Figure 1: Unknown

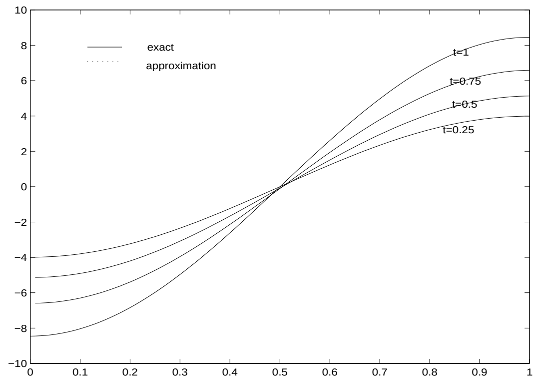

Figure 2: Gradient

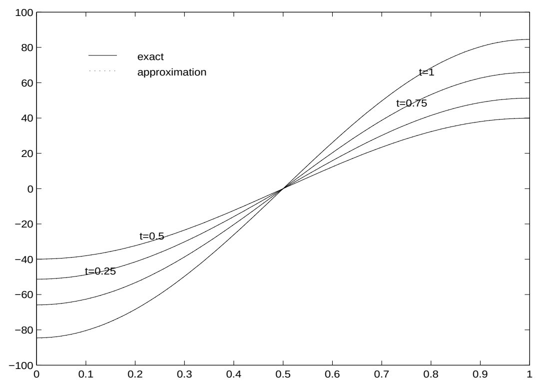

Figure 3: Flux

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