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ON THE CONVERGENCE OF IMPLICIT DIFFERENCE SCHEMES FOR HYPERBOLIC CONSERVATION LAWS*[1)]

Hua-zhong Tang
(*School of Mathematical Sciences, Peking University, Beijing 100871, China*)

Hua-mo Wu
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

This paper is to treat implicit difference approximations to hyperbolic conservation laws with non-convex flux. The convergence of the approximate solution toward the entropy solution is established for the general weighted implicit difference schemes, which include some well-known implicit and explicit difference schemes.

Key words: Conservation laws, weighted implicit schemes, entropy solution.

1. Introduction

We are are interested in the following Cauchy problem for scalar conservation laws

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, \ (x,t) \in \mathcal{R} \times \mathcal{R}^+, \\ u(0,x) = u_0(x), \ x \in \mathcal{R}, \end{cases} \tag{1.1}$$

where the initial data $u_0 \in BV(\mathcal{R})$ and the flux function $f \in C^2(\mathcal{R})$.

It is well known that this problem may not always have a smooth global solution even if the initial data u_0 is adequate smooth [11-14]. Thus, we consider its weak solution so that the problem (1.1) might have a global solution allowing discontinuities, e.g. shock wave etc.

A weak solution to the problem (1.1) is a function $u \in L^\infty(\mathcal{R} \times \mathcal{R}^+)$ satisfying:

$$\iint_{\mathcal{R}\times\mathcal{R}^+} \left(u\varphi_t + f(u)\varphi_x\right) \, dxdt + \int_{\mathcal{R}} u_0(x)\varphi(x,0) \, dx = 0, \tag{1.2}$$

for all $\varphi \in C^1(\mathcal{R} \times \mathcal{R}^+)$, with compact support in $\mathcal{R} \times \mathcal{R}^+$.

On the other hand, the week solutions to problem (1.1) are not necessarily unique. A physically relevant solution (also called the entropy solution) is characterized by a entropy condition

$$\iint_{\mathcal{R}\times\mathcal{R}^+} \left(U(u)\varphi_t + F(u)\varphi_x\right) \, dxdt \le 0, \tag{1.3}$$

for all positive test functions $\varphi \in C^1(\mathcal{R} \times \mathcal{R}^+)$, with compact support in $\mathcal{R} \times \mathcal{R}^+$, where the entropy function $U(u) \in C^2(\mathcal{R})$ is convex, i.e. $U''(u) > 0$, and the entropy flux function $F(u)$ satisfies

$$F'(u) = U'(u)f'(u).$$

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Numerical methods have been derived to approximate conservation laws in (1.1) in last two decades years, and have been applied to numerical simulations of many problems appeared in science and engineering. However, studies of convergence of approximate solutions to the entropy solution are still open for most of these numerical methods. Although there exist some related works (see references [1,2,15-23]), either some quantities depending on the space mesh size are always introduced in their investigating processes, or convex flux is only considered. In general, the difference schemes only depend on the ratio of the mesh sizes. Moreover, the introduction of these quantities may be improper for practical applications.

In this paper we study the convergence of the approximate solutions of (1.1) obtained by a class of weighted implicit schemes. They include some well-known implicit or explicit difference schemes for hyperbolic conservation laws with non-convex flux.

The paper is organized as follows. In section 2, we present construction of general weighted implicit difference schemes for one-dimensional scalar conservation laws in different form. In section 3, the convergence of the general weighted implicit difference schemes is study. Finally we give some remarks in the last section.

2. Weighted Implicit Difference Schemes

We consider the problem of construction of a general weighed implicit scheme (see [24]). For the sake of simplicity, the case of uniform grids is only considered. Let Δt and Δx be given positive numbers. Set $x_j = j\Delta x, x_{j+\frac{1}{2}} = \frac{1}{2}(x_j + x_{j+1})$ and $t^n = n\Delta t$. In the following, we also use some difference notations: $\Delta_- u_j = u_j - u_{j-1}$, $\Delta_+ u_j = u_{j+1} - u_j$, $\Delta_{j+\frac{1}{2}} u = u_{j+1} - u_j$, and $\delta_x^2 u_j = u_{j+1} - 2u_j + u_{j-1}$.

The approximate solution $u_\Delta(t,x) = u_j^n$ for $(t,x) \in [t^n, t^{n+1}) \times (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, is given by the following three-point weighted implicit difference scheme in conservative form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{\Delta x}(\hat{h}_{j+\frac{1}{2}} - \hat{h}_{j-\frac{1}{2}}) = 0, \tag{2.1a}$$

where

$$\hat{h}_{j+\frac{1}{2}} = \theta h^{n+1}_{j+\frac{1}{2}} + (1-\theta) h^n_{j+\frac{1}{2}}, \ h_{j+\frac{1}{2}} = h(u_j, u_{j+1}), \tag{2.1b}$$

θ is positive parameter, $0 \le \theta \le 1$, and numerical flux function $h_{j+\frac{1}{2}}$ is Lipschitz continuous. We require the numerical flux function to be consistent with flux $f(u)$ in the following sense

$$h(u,u) = f(u), \tag{2.2}$$

and the initial data is projected onto the space of piecewise constant functions by the restriction

$$u_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)dx, \quad I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \tag{2.3}$$

for $j \in Z$.

We say that difference scheme (2.1) is consistent with entropy condition (1.3) in an inequality of the following kind is satisfied:

$$U_j^{n+1} \le U_j^n - \lambda(\hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}}), \quad \lambda = \frac{\Delta t}{\Delta x}, \tag{2.4}$$

where $U_j^n = U(u_j^n)$, and $\hat{H}(u,v)$ is a numerical entropy flux consistent with entropy flux $F(u)$, i.e.,

$$\hat{H}(u,u) = F(u).$$

Here and below, we also assume that numerical flux $h_{j+\frac{1}{2}}$ can be written in viscous form

$$h_{j+\frac{1}{2}} = \frac{1}{2}\big(f(u_j) + f(u_{j+1}) - \frac{1}{\lambda}Q_{j+\frac{1}{2}}(u_{j+1} - u_j)\big), \tag{2.5}$$

where $Q_{j+\frac{1}{2}} = Q(u_j, u_{j+1}; \lambda)$ is the coefficient of numerical viscosity. For example, for Lax–Friedrichs scheme, $Q_{j+\frac{1}{2}} = 1$; for well–known Lax–Wendroff scheme, $Q_{j+\frac{1}{2}} = (\lambda a_{j+\frac{1}{2}})^2$, etc.

On the other hand, (2.1) can also be written in incremental form as

$$-\lambda(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) = C^{-}_{j+\frac{1}{2}}(u_{j+1} - u_j) - C^{+}_{j-\frac{1}{2}}(u_j - u_{j-1}), \tag{2.6}$$

with

$$C^{\pm}_{j+\frac{1}{2}} = \frac{\lambda}{2}\left(Q_{j+\frac{1}{2}} \pm \frac{\Delta_{j+\frac{1}{2}}f}{\Delta_{j+\frac{1}{2}}u}\right).$$

Before to end this section, we now quote Helly's Theorem, which will be used in next section.

Theorem 2.1. (Helly's Theorem) *Let the sequence of functions $\{u_n(x)\}_0^\infty$ be of uniformly bounded variation in $a \le x \le b$ and such that*

$$| u_n(x) | < A, \quad n \in \mathcal{N} \cup \{0\},$$

for some constant A. There then exists a set of integers

$$n_0 < n_1 < n_2 < \cdots$$

and a function $u(x)$ of bounded variation in $a \le x \le b$ such that

$$\lim_{k\to 0} u_{n_k}(x) = u(x), \quad \forall x \in [a,b].$$

That is, given a sequence of functions which are uniformly bounded and of uniformly bounded variation on an interval, it is possible to extract a subsequence which converges to a function of bounded variation in L^1.

Finally, the Lax-Wendroff Theorem is quoted for difference schemes written in a conservation form, which will also be required in next section.

Theorem 2.2. (The Lax-Wendroff Theorem) *Consider a difference scheme consistent with (1.1) in form*

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{\Delta x}(\bar{h}_{j+\frac{1}{2}} - \bar{h}_{j-\frac{1}{2}}) = 0,$$

where the numerical flux

$$\bar{h}_{j+\frac{1}{2}} = \bar{h}(u_{j-k+1}, ..., u_{j+k}), \ k \in \mathcal{N}, \quad \bar{h}(u, ..., u) = f(u).$$

Suppose that, as Δt, Δx tend to zero, the solution $u_\Delta(x,t)$ produced by the above conservative scheme, if applied at every x, converges boundedly almost everywhere to some function $u(x,t)$. Then $u(x,t)$ is a weak solution of (1.1).

3. Convergence of Difference Scheme

3.1. Stability Properties of Difference Scheme

Proposition 3.1. *Suppose that u_0 lies in $L^\infty(\mathcal{R}) \cap BV_{loc}(\mathcal{R})$ and that the CFL-like condition*

$$\lambda \parallel f'(u) \parallel \le \lambda Q(u,v;\lambda) \le \frac{1}{1-\theta} \tag{3.1}$$

is satisfied. Then the family of approximations u_Δ generated by the difference scheme (2.1) from initial data (2.2) is L^∞-bounded, i.e.,

$$\parallel u^{n+1} \parallel_{L^\infty} \le \parallel u^n \parallel_{L^\infty} \tag{3.2}$$

and is TV–bounded, i.e.,

$$TV(u^{n+1}) \le TV(u^n), \ TV(u^n) = \sum_{j\in\mathcal{Z}} | \Delta_{j+\frac{1}{2}} u^n | . \tag{3.3}$$

Proof. In order to prove the above results, we will consider the following three cases.

(I) $\theta = 0$. Here the difference scheme (2.1) is an explicit scheme, i.e.,

$$u_j^{n+1} = u_j^n - \lambda(h^n_{j+\frac{1}{2}} - h^n_{j-\frac{1}{2}}), \tag{3.4}$$

Using its incremental form, it is easy to verify the result, under condition (3.1). We refer the readers to [6,24].

(II) $\theta = 1$. Introduce a parameter β, $0 < \beta \ll 1$, then the scheme (2.1) can be rewritten as

$$u_j^{n+1} = \frac{\beta}{1+\beta} u_j^n + \frac{1}{1+\beta} \hat{u}, \tag{3.5a}$$

where

$$\hat{u} = u_j^{n+1} - \beta\lambda(h_{j+\frac{1}{2}}^{n+1} - h_{j-\frac{1}{2}}^{n+1}). \tag{3.5b}$$

Similar to Step (I), we have

$$\| \hat{u} \|_{L^\infty} \leq \| u^{n+1} \|_{L^\infty}, \quad TV(\hat{u}) \leq TV(u^{n+1}), \tag{3.6a}$$

from (3.5b), and

$$\begin{aligned} \| u^{n+1} \|_{L^\infty} &\leq \frac{\beta}{1+\beta} \| u^n \|_{L^\infty} + \frac{1}{1+\beta} \| \hat{u} \|_{L^\infty}, \\ TV(u^{n+1}) &\leq \frac{\beta}{1+\beta} TV(u^n) + \frac{1}{1+\beta} TV(\hat{u}), \end{aligned} \tag{3.6b}$$

from (3.5a), because coefficients in (3.5a) are all non–negative, and do not depend on j. Combining inequality (3.6a) with (3.6b), we can obtain the result.

(III) $0 < \theta < 1$. The scheme (2.1) then can be rewritten in a splitting form as follow:

$$\frac{\bar{u} - u_j^n}{(1-\theta)\Delta t} + \frac{1}{\Delta x}(h_{j+\frac{1}{2}}^n - h_{j-\frac{1}{2}}^n) = 0, \tag{3.7a}$$

$$\frac{u_j^{n+1} - \bar{u}}{\theta \Delta t} + \frac{1}{\Delta x}(h_{j+\frac{1}{2}}^{n+1} - h_{j-\frac{1}{2}}^{n+1}) = 0, \tag{3.7b}$$

i.e., (2.1) is split into an explicit scheme (3.7a) and a fully implicit scheme (3.7b). Therefore, for (3.7a), under the condition (3.1) we have

$$\| \bar{u} \|_{L^\infty} \leq \| u^n \|_{L^\infty}, \quad TV(\bar{u}) \leq TV(u^n), \tag{3.8}$$

from (3.7a). In the following, it needs to show

$$\| u^{n+1} \|_{L^\infty} \leq \| \bar{u} \|_{L^\infty}, \quad TV(u^{n+1}) \leq TV(\bar{u}). \tag{3.9}$$

To do it, we introduce a new variable $\hat{u}$ and a parameter β, $0 \leq \beta \leq \frac{1-\theta}{\theta}$, and rewrite equation (3.7b) as

$$\begin{aligned} u_j^{n+1} &= \frac{\beta}{1+\beta} \bar{u} + \frac{1}{1+\beta} \hat{u} \\ \hat{u} &= u_j^{n+1} - \theta\beta\lambda(h_{j+\frac{1}{2}}^{n+1} - h_{j-\frac{1}{2}}^{n+1}). \end{aligned} \tag{3.10}$$

It is not difficult to show that under the condition (3.1) we have

$$\| \hat{u} \|_{L^\infty} \leq \| u^{n+1} \|_{L^\infty}, \quad TV(\hat{u}) \leq TV(u^{n+1}), \tag{3.11}$$

and

$$\begin{aligned} \| u^{n+1} \|_{L^\infty} &\leq \frac{\beta}{1+\beta} \| \bar{u} \|_{L^\infty} + \frac{1}{1+\beta} \| \hat{u} \|_{L^\infty}, \\ TV(u^{n+1}) &\leq \frac{\beta}{1+\beta} TV(\bar{u}) + \frac{1}{1+\beta} TV(\hat{u}). \end{aligned} \tag{3.12}$$

Combining inequality (3.11) with (3.12), the result can be obtained.

Proposition 3.2. *The scheme of monotone type (2.1-2.2) admits an unique solution* $u_j^{n+1} \in L^\infty(\mathcal{R})$.

Proof. Let us introduce the operator $T_\lambda : L^\infty(\mathcal{R}) \to L^\infty(\mathcal{R})$ defined by

$$T_\lambda(u)_j = u_j + \theta\lambda(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}), \ j \in \mathcal{Z},$$

then the scheme (2.1) may be written as

$$T_\lambda(u^{n+1})_j = G(u^n),\ G(u^n) = u_j^n - (1-\theta)\lambda(h_{j+\frac{1}{2}}^n - h_{j+\frac{1}{2}}^n).$$

To show the existence of a solution of the scheme (2.1), it suffices to prove that the operator T_λ is invertible. To do this, we will show that there exists a positive constant C such that

$$\| T'_\lambda(u)v \|_{L^\infty} \geq C \| v \|_{L^\infty}, \quad u \in L^\infty(\mathcal{R}).$$

From the definition of the operator T_λ and the numerical flux $h_{j+\frac{1}{2}}$, we have

$$[T'_\lambda(u)v]_j = v_j + \theta\lambda(a_{j+\frac{1}{2}}v_j + b_{j+\frac{1}{2}}v_{j+1}$$
$$-a_{j-\frac{1}{2}}v_{j-1} - b_{j-\frac{1}{2}}v_j),$$

where $a = \frac{\partial h(u,v)}{\partial u}$, $b = \frac{\partial h(u,v)}{\partial v}$. Let

$$T'_\lambda(u)v = c,$$

with

$$c_j = [1 + \theta\lambda a_{j+\frac{1}{2}} - \theta\lambda b_{j-\frac{1}{2}}]v_j - \theta\lambda a_{j-\frac{1}{2}}v_{j-1} + \theta\lambda b_{j+\frac{1}{2}}v_{j+1},$$

thus

$$| c_j | \geq [1 + \theta\lambda a_{j+\frac{1}{2}} - \theta\lambda b_{j-\frac{1}{2}}] \mid v_j \mid - \theta\lambda a_{j-\frac{1}{2}} \mid v_{j-1} \mid + \theta\lambda b_{j+\frac{1}{2}} \mid v_{j+1} \mid .$$

Taking the sup over $j \in \mathcal{Z}$ in the last inequality, we get

$$\| c_j \|_{L^\infty} \geq \| v_j \|_{L^\infty},$$

that is

$$\| T'_\lambda(u)v \|_{L^\infty} \geq \| v \|_{L^\infty}, \quad u \in L^\infty(\mathcal{R}),$$

then there exists an unique solution of (2.1) in $L^\infty(\mathcal{R})$.

Using **Theorem 2.1**, **Theorem 2.2** and the previous two propositions, we have

Theorem 3.1. *Suppose that u_0 lies in $L^\infty(\mathcal{R}) \cap BV_{loc}(\mathcal{R})$ and that the CFL–like condition (3.1) is satisfied. Then the family of approximations u_Δ generated by the difference scheme (2.1) from the initial data (2.2) contains a subsequence which converges in $L^1_{loc}(\mathcal{R} \times \mathcal{R}^+)$ towards a weak solution of (1.1) as $\Delta x \to 0$.*

3.2. The Entropy Condition

Proposition 3.3. *Let the CFL–like condition (3.1) be satisfied and u_j^n be approximate solution given by the weighted implicit scheme (2.1), then they satisfy the following numerical entropy inequality*

$$U(u_j^{n+1}) \leq U(u_j^n) - \lambda(\hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}}), \tag{3.13a}$$

where

$$\hat{H}_{j+\frac{1}{2}} = \theta H_{j+\frac{1}{2}}^{n+1} + (1-\theta)H_{j+\frac{1}{2}}^n,$$
$$H_{j+\frac{1}{2}} = \frac{1}{2}\big[F(u_j) + F(u_{j+1}) - \frac{1}{\lambda}Q(u_j, u_{j+1}; \lambda)(U(u_{j+1}) - U(u_j))\big], \tag{3.13b}$$

where $Q(u_j, u_{j+1}; \lambda)$ is defined in (2.3).

Proof. In order to prove the above results, we will also consider the following three cases.

(I) $0 < \theta < 1$. Multiplying (3.7a) and (3.7b) by $U'(u_j^n)$ and $U'(u_j^{n+1})$, respectively, and using elementary equality

$$U'(a)(g(b) - g(a)) = G(b) - G(a) - \int_a^b (g(b) - g(s))U''(s)\ ds,$$

where $G(u) = \int^u g'(s)U'(s)\, ds$, we have

$$
\begin{aligned}
U'(u_j^n)\cdot LHS_{(3.7a)} &= \frac{1}{(1-\theta)\Delta t}\big(U(\bar u) - U(u_j^n)\big) + \frac{1}{2\Delta x}\left[\Delta_+ F(u_j^n) + \Delta_- F(u_j^n)\right] \\
&-\frac{Q^n_{j+\frac12}}{2\Delta t}[\Delta_+ U(u_j^n)] + \frac{Q^n_{j-\frac12}}{2\Delta t}[\Delta_- U(u_j^n)] - \frac{1}{(1-\theta)\Delta t}\int_{u_j^n}^{\bar u}(\bar u - \xi)U''(\xi)\, d\xi \\
&-\frac{1}{2\Delta x}\int_{u_j^n}^{u_{j+1}^n}(f(u_{j+1}^n) - f(\xi))U''(\xi)\, d\xi + \frac{1}{2\Delta x}\int_{u_j^n}^{u_{j-1}^n}(f(u_{j-1}^n) - f(\xi))U''(\xi)\, d\xi \\
&+\frac{Q^n_{j+\frac12}}{2\Delta t}\int_{u_j^n}^{u_{j+1}^n}(u_{j+1}^n - \xi)U''(\xi)\, d\xi + \frac{Q^n_{j+\frac12}}{2\Delta t}\int_{u_j^n}^{u_{j-1}^n}(u_{j-1}^n - \xi)U''(\xi)\, d\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
U'(u_j^{n+1})\cdot LHS_{(3.7b)} &= \frac{1}{\theta\Delta t}(U(u_j^{n+1}) - U(\bar u)) + \frac{1}{2\Delta x}[\Delta_+ F(u_j^{n+1}) + \Delta_- F(u_j^{n+1})] \\
&-\frac{Q^{n+1}_{j+\frac12}}{2\Delta t}\Delta_+ U(u_j^{n+1}) + \frac{Q^{n+1}_{j-\frac12}}{2\Delta t}\Delta_- U(u_j^{n+1}) + \frac{1}{\theta\Delta t}\int_{u_j^{n+1}}^{\bar u}(\bar u - \xi)U''(\xi)\, d\xi \\
&-\frac{1}{2\Delta x}\int_{u_j^{n+1}}^{u_{j+1}^{n+1}}(f(u_{j+1}^{n+1}) - f(\xi))U''(\xi)\, d\xi + \frac{1}{2\Delta x}\int_{u_j^{n+1}}^{u_{j-1}^{n+1}}(f(u_{j-1}^{n+1}) - f(\xi))U''(\xi)\, d\xi \\
&+\frac{Q^{n+1}_{j+\frac12}}{2\Delta t}\int_{u_j^{n+1}}^{u_{j+1}^{n+1}}(u_{j+1}^{n+1} - \xi)U''(\xi)\, d\xi + \frac{Q^{n+1}_{j-\frac12}}{2\Delta t}\int_{u_j^{n+1}}^{u_{j-1}^{n+1}}(u_{j-1}^{n+1} - \xi)U''(\xi)\, d\xi.
\end{aligned}
$$

multiplying the above two equations by $(1-\theta)$ and θ, respectively, and summing them gives

$$
\frac{1}{\Delta t}(U(u_j^{n+1}) - U(u_j^n)) + \frac{1}{\Delta x}(\hat H_{j+\frac12} - \hat H_{j-\frac12}) + LRED = 0,
$$

where $LRED$ represents the local rate of entropy dissipation, which is

$$
\begin{aligned}
LRED &= -\frac{1}{\Delta t}\int_{u_j^n}^{\bar u}(\bar u - \xi)U''(\xi)\, d\xi + \frac{1}{\Delta t}\int_{u_j^{n+1}}^{\bar u}(\bar u - \xi)U''(\xi)\, d\xi \\
&-\frac{(1-\theta)}{2\Delta x}\int_{u_j^n}^{u_{j+1}^n}(f(u_{j+1}^n) - f(\xi))U''(\xi)\, d\xi + \frac{(1-\theta)}{2\Delta x}\int_{u_j^n}^{u_{j-1}^n}(f(u_{j-1}^n) - f(\xi))U''(\xi)\, d\xi \\
&+\frac{(1-\theta)Q^n_{j+\frac12}}{2\Delta t}\int_{u_j^n}^{u_{j+1}^n}(u_{j+1}^n - \xi)U''(\xi)\, d\xi + \frac{(1-\theta)Q^n_{j-\frac12}}{2\Delta t}\int_{u_j^n}^{u_{j-1}^n}(u_{j-1}^n - \xi)U''(\xi)\, d\xi \\
&-\frac{\theta}{2\Delta x}\int_{u_j^{n+1}}^{u_{j+1}^{n+1}}(f(u_{j+1}^{n+1}) - f(\xi))U''(\xi)\, d\xi + \frac{\theta}{2\Delta x}\int_{u_j^{n+1}}^{u_{j-1}^{n+1}}(f(u_{j-1}^{n+1}) - f(\xi))U''(\xi)\, d\xi \\
&+\frac{\theta Q^{n+1}_{j+\frac12}}{2\Delta t}\int_{u_j^{n+1}}^{u_{j+1}^{n+1}}(u_{j+1}^{n+1} - \xi)U''(\xi)\, d\xi + \frac{\theta Q^{n+1}_{j-\frac12}}{2\Delta t}\int_{u_j^{n+1}}^{u_{j-1}^{n+1}}(u_{j-1}^{n+1} - \xi)U''(\xi)\, d\xi.
\end{aligned}
$$

The proof will be completed, if we can show that the local rate of entropy dissipation is non–negative under the CFL–like condition (3.1).

Actually, $LRED$ can be rewritten as

$$LRED = -\frac{1}{\Delta t}\int_{u_j^n}^{\bar{u}} \{\bar{u} - \xi + (1-\theta)\frac{\lambda}{2}[f(u_{j+1}^n) - f(u_{j-1}^n)$$

$$-Q_{j+\frac{1}{2}}^n(u_{j+1}^n - \xi) + Q_{j-\frac{1}{2}}^n(u_{j-1}^n - \xi)]\}U''(\xi)\, d\xi$$

$$+\frac{(1-\theta)}{2\Delta t}\int_{\bar{u}}^{u_{j+1}^n} \left[Q_{j+\frac{1}{2}}^n(u_{j+1}^n - \xi) - \lambda(f(u_{j+1}^n) - f(\xi))\right]U''(\xi)\, d\xi$$

$$+\frac{(1-\theta)}{2\Delta t}\int_{\bar{u}}^{u_{j-1}^n} \left[Q_{j-\frac{1}{2}}^n(u_{j-1}^n - \xi) + \lambda(f(u_{j-1}^n) - f(\xi))\right]U''(\xi)\, d\xi$$

$$+\frac{1}{\Delta t}\int_{u_j^{n+1}}^{\bar{u}} (\bar{u} - \xi)U''(\xi)\, d\xi$$

$$+\frac{\theta}{2\Delta t}\int_{u_j^{n+1}}^{u_{j+1}^{n+1}} \left[Q_{j+\frac{1}{2}}^{n+1}(u_{j+1}^{n+1} - \xi) - \lambda(f(u_{j+1}^{n+1}) - f(\xi))\right]U''(\xi)\, d\xi$$

$$+\frac{\theta}{2\Delta t}\int_{u_j^{n+1}}^{u_{j-1}^{n+1}} \left[Q_{j-\frac{1}{2}}^{n+1}(u_{j-1}^{n+1} - \xi) + \lambda(f(u_{j-1}^{n+1}) - f(\xi))\right]U''(\xi)\, d\xi.$$

Using equation (3.7a), we have

$$LRED = -\frac{1}{\Delta t}\int_{u_j^n}^{\bar{u}} (u_j^n - \xi)\left[1 - \frac{(1-\theta)\lambda}{2}(Q_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n)\right]U''(\xi)\, d\xi$$

$$+\frac{1}{\Delta t}\int_{u_j^{n+1}}^{\bar{u}} (\bar{u} - \xi)U''(\xi)\, d\xi$$

$$+\frac{(1-\theta)}{2\Delta t}\int_{\bar{u}}^{u_{j+1}^n} \left[Q_{j+\frac{1}{2}}^n(u_{j+1}^n - \xi) - \lambda(f(u_{j+1}^n) - f(\xi))\right]U''(\xi)\, d\xi$$

$$+\frac{(1-\theta)}{2\Delta t}\int_{\bar{u}}^{u_{j-1}^n} \left[Q_{j-\frac{1}{2}}^n(u_{j-1}^n - \xi) + \lambda(f(u_{j-1}^n) - f(\xi))\right]U''(\xi)\, d\xi$$

$$+\frac{\theta}{2\Delta t}\int_{u_j^{n+1}}^{u_{j+1}^{n+1}} \left[Q_{j+\frac{1}{2}}^{n+1}(u_{j+1}^{n+1} - \xi) - \lambda(f(u_{j+1}^{n+1}) - f(\xi))\right]U''(\xi)\, d\xi$$

$$+\frac{\theta}{2\Delta t}\int_{u_j^{n+1}}^{u_{j-1}^{n+1}} \left[Q_{j-\frac{1}{2}}^{n+1}(u_{j-1}^{n+1} - \xi) + \lambda(f(u_{j-1}^{n+1}) - f(\xi))\right]U''(\xi)\, d\xi.$$

It is not difficult to show that $LRED$ is non–negative under condition (3.1).

(II) $\theta = 0$ and (III) $\theta = 1$. we can similarly obtain the results.

Therefore, we have

Theorem 3.2. *Suppose that u_0 lies in $L^\infty(\mathcal{R}) \cap BV_{loc}(\mathcal{R})$, then under the CFL condition (3.1) the family of approximations u_Δ constructed by the difference scheme (2.1) from initial data (2.2) converges in $L^1_{loc}(\mathcal{R} \times \mathcal{R}^+)$ towards the entropy solution of (1.1), as $\Delta x \to 0$.*

4. Concludes

In this paper, the general weighted implicit difference approximations to one dimensional hyperbolic conservation laws with non-convex flux have been considered. We have studied the nonlinear stability properties and convergence of the approximate solution produced by first-order accurate weighted implicit schemes towards the entropy solution.

In the future, we will study convergence and nonlinear stability of the general weighted implicit difference schemes on unstructured grids for multi-dimensional conservation laws with non-convex flux and high resolution implicit schemes for conservation laws.

References

[1] F. Coquel, P. LeFloch, Conervegence of finite difference schemes for conservation laws in several space dimensions: The general theory, *SIAM J. Numer. Anal.,* **30** (1993), 675-700.

[2] F. Coquel, P. LeFloch, Conervegence of finite difference schemes for conservation laws in several space dimensions: The corrected antidiffusive flux approach, *Math. Comp.,* **57** (1991), 169-210.

[3] M. Crandall, A. Majda, Monotone difference approximations for scalar conservation laws, *Math. Comp.,* **34** (1980), 1-21.

[4] R.J. Diperna, Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Anal.,* **82** (1983), 27-70.

[5] S.K. Godunov, A finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics, *Mat. Sb.,* **47** (1959), 271-306.

[6] A. Harten, High Resolution Schemes for Hyperbolic Conservation Laws, *J. Comput. Phys.,* **49** (1983), 357-393.

[7] A. Harten, J.M. Hyman, P.D. Lax, On finite-difference approximations and entropy conditions for shocks, *Comm. Pure Appl. Math.,* **29** (1976), 297-322.

[8] G. Jennings, Discrete shocks, *Comm. Pure Appl. Math.,* **27** (1974), 25-37.

[9] G. Jiang, C.-W. Shu, On a cell entropy inequality for discontinuous Galerkin methods, *Math. Comp.,* **62** (1994), 531-538.

[10] C. Johnson, A. Szepessy, P. Hansbo, On the conservegence of shock-capturing streamline diffusion finite element methods for hyperbolic conservation laws, *Math. Comp.,* **54** (1990), 107-129.

[11] S.N.Kružkov, First order quasilinear equations in several independent variables, *Math. USSR Sbornik,* **10** (1970), 217-243.

[12] N.N. Kuznetsov, Accuracy of some approximate methods for computing the eweak solutions of a first-order quasi-linear equation, *USSR Comput. Math. and Math. Phys.,* **16** (1976), 105-119.

[13] P.D. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.,* **10** (1957), 537-566.

[14] P.D. Lax, Hyperbolic systems of conservation laws and the mathematical theory of shock waves, *SIAM,* Philadelphia, 1973.

[15] P. Lions, P. Souganidis, Convergence of MUSCL type methods for scalar conservation laws, *Acad. Sci. Paris. Ser. I math.,* **311** (1990), 408-463.

[16] B.J. Lucier, On nonlocal monotone difference schemes for scalar conservation laws, *Math. Comp.,* **47** (1986), 19-36.

[17] H. Nessyahu, E. Tadmor, Non-oscillatory central differencing for hyperbolic conservation laws, *J. Comput. Phys.,* **87** (1990), 408-463.

[18] S. Osher, On convergence of generalized MUSCL schemes, *SIAM J. Numer. Anal.,* **22** (1985), 947-961.

[19] S. Osher, S. Chakravarthy, High resolution schemes and entropy condition, *SIAM J. Numer. Anal.,* **21** (1984), 955-984.

[20] S. Osher, E. Tadmor, On the convergence of difference approximations to scalar conservation laws, *Math. Comp.,* **50** (1988), 19-51.

[21] R. Sanders, On convergence of monotone finite difference schemes with variable spatial differencing, *Math. Comp.,* **40** (1983), 91-106.

[22] E.Tadmor, Numerical viscosity and the entropy condition for conservative difference schemes, *Math. Comp.,* **43** (1984), 369-381.

[23] H.-N. Yang, On wavewise entropy inequalities for high-resolution schemes. I: The semidiscrete case, *Math. Comp.,* **65** (1996), 45-67.

[24] H.C. Yee, Construction of explicit and implicit symmetric TVD schemes and their applications, *J. Comput. Phys.,* **68** (1987), 151-179.