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# A PRECONDITIONER FOR COUPLING SYSTEM OF NATURAL BOUNDARY ELEMENT AND COMPOSITE GRID FINITE ELEMENT*1)

Qi-ya Hu
(*ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

De-hao Yu
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

**Abstract**

In this paper, based on the natural boundary reduction advanced by Feng and Yu, we discuss a coupling BEM with FEM for the Dirichlet exterior problems. In this method the finite element grids consist of fine grid and coarse grid so that the singularity at the corner points can be handled conveniently. In order to solve the coupling system by the preconditioning conjugate gradient method, we construct a simple preconditioner for the "stiffness" matrix. Some error estimates of the corresponding approximate solution and condition number estimate of the preconditioned matrix are also obtained.

*Key words*: Natural boundary reduction, Composite grid, Error estimate, Preconditioner, Condition number.

## 1. Introduction

The coupling of boundary elements and finite elements is of great importance for the numerical treatment of boundary value problems posed on unbounded domains. It permits us to combine the advantages of boundary elements for treating domains extended to infinity with those of finite elements in treating the complicated bounded domains.

The standard procedure of coupling the boundary element and finite element methods is described as follows. First, the (unbounded) domain is divided into two subregions, a bounded inner region and an unbounded outer one, by introducing an auxiliary common boundary. Next, the problem is reduced to an equivalent one in the bounded region. There are many ways to accomplish this reduction (refer to [2]-[6], [9]).

The natural boundary reduction method proposed by Feng and Yu [4] has obvious advantages over the usual boundary reduction methods: the coupling bilinear form preserve automatically the symmetry and coerciveness of the original bilinear form, so not only the analysis of the discrete problem is simplified, but also the optimal error estimates and the numerical stability are restored (see [4] and [14]).

It is well known that the analytic solution of the Dirichlet exterior problem is in general singular at the corner points. The fast adaptive composite grid (iteration) method advanced by

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McCormick (refer to [1], [7] and [8]) is very effective in dealing with this kind of local singularity. However, it can not be applied directly to the case of unbounded domain.

In the present paper we combine the composite grid method with the coupling method of natural boundary element and finite element to handle the corner singularity of the Dirichlet exterior problems. Under suitable assumptions we obtain the optimal error estimates of the corresponding approximate solutions. The underlying linear system is difficult to solve directly due to the complicated structure (which is neither sparse nor band). Instead, we use the preconditioning conjugate gradient (PCG) method by constructing a kind of simple preconditioner for the coupled "stiffness" matrix. We show that condition number of the preconditioned matrix is independent of the (coarse and fine) mesh sizes. Moreover, we give numerical examples to illustrate our theoretical results.

## 2. The Natural Boundary Reduction

We consider the following model exterior Dirichlet problem in two dimensions:

$$\begin{cases} -\Delta u = f & in \;\; \Omega^c = \mathbf{R}^2\backslash(\Omega\cup\Gamma), \\ u = g & on \;\; \partial\Omega, \end{cases} \tag{2.1}$$

subject to the asymptotic conditions

$$u(x,y) = \beta + O(\frac{1}{r}) \quad as \;\; r = \sqrt{x^2+y^2} \to \infty,$$

with $\beta$ be a constant, where $\Omega$ is a Lipschitz bounded domain. Assume that the given functions $f$ and $g$ satisfy (see [6]): $supp\ f \subset \Omega_b$ and $f \in H^{-1}(\Omega_b)$ with some $\Omega_b$ being a bounded domain and containing $\Omega$; $g \in H^{\frac{1}{2}}(\partial\Omega)$.

The variational form of the boundary value problem (2.1) is: to find $u \in \bar{H}^1(\Omega^c)$, such that

$$D(u,v) = (f,v), \quad \forall\ v \in \bar{H}^1_0(\Omega^c), \tag{2.2}$$

where

$$\bar{H}^1(\Omega^c) = \{v : \frac{v}{\sqrt{(r^2+1)}\cdot \ln(r^2+2)}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega^c)\},$$

$$\bar{H}^1_0(\Omega^c) = \{v : v \in \bar{H}^1(\Omega^c), v|_{\partial\Omega} = 0\}$$

and

$$D(u,v) = (\nabla u, \nabla v), \;\; \forall\ u,v \in \bar{H}^1(\Omega^c), \tag{2.3}$$

with $(\cdot,\cdot)$ be the $L^2$ innerproduct on $\Omega^c$.

Let $\Omega_0 \subset \Omega_b$ is a circle disc (with the radius $R$) containing $\Omega$. Set $\Omega_1 = \Omega^c \cap \Omega_0$ and $\Omega_2 = \Omega_0^c = \mathbf{R}^2\backslash\Omega_0$. We assume that the ratio of the area of $\Omega_1$ over the area of $\Omega$ is not small.

Let $\Gamma$ denote the boundary of $\Omega_0$. It follows by Green formula that

$$D_1(u,v) = (f,v)_{\Omega_1} - \int_\Gamma \tfrac{\partial u}{\partial n} v ds, \quad \forall v \in \bar{H}^1_0(\Omega^c). \tag{2.4}$$

Let $G(p,p')$ denote the Green function of the Laplace operator on the domain $\Omega_2$, which satisfies

$$\begin{cases} -\Delta G(p,p') = \delta(p-p'), \forall\ p,p' \in \Omega_2, \\ G(p,p')|_{p\in\Gamma} = 0, \;\; \forall \;\; p' \in \Omega_2. \end{cases}$$

Set $v = G(p,p')$ in the second Green formula

$$\int\int_{\Omega_2} (v\Delta u - u\Delta v) dp' = \int_\Gamma (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}) ds,$$

we obtain (refer [13])

$$u(p) = \int\int_{\Omega_2} f\cdot G(p,p')dp' - \int_\Gamma \frac{\partial}{\partial n'} G(p,p')\cdot u(p')dp', \;\; \forall\ p \in \Omega_2.$$

Here $n$ and $n'$ denote respectively the exterior normal vectors of $\Gamma$ ( which is regarded as the boundary of $\Omega_2$) at the points $p$ and $p'$.
Thus

$$\begin{aligned}\frac{\partial u}{\partial n}(p) = & \int\int_{\Omega_2} f(p')\frac{\partial}{\partial n}G(p,p')dp' \\ - & \int_\Gamma \frac{\partial^2}{\partial n\partial n'}G(p,p')\cdot u(p')dp',\ \forall\ p\in\Gamma.\end{aligned} \tag{2.5}$$

Set

$$\frac{\partial}{\partial n}G(p,p') = G_n^{(2)}(p,p'),\ p,p'\in\Gamma,$$

and

$$-\int_\Gamma \frac{\partial^2}{\partial n\partial n'}G(p,p')\cdot u(p')dp' = K_2u(p),\ p\in\Gamma.$$

Then (2.5) can be written as

$$\frac{\partial u}{\partial n}(p) = \int\int_{\Omega_2} f(p')G_n^{(2)}(p,p')dp' + K_2u(p),\ p\in\Gamma. \tag{2.6}$$

Substituting (2.6) into (2.4), we obtain the coupling variational problem of (2.2): to find $u\in H_g^1(\Omega_1)$ such that

$$D_1(u,v) + < K_2u, v>_\Gamma = \int\int_{\Omega_1} fvdxdy - < w_f, v>_\Gamma,\ \forall\ v\in H_0^1(\Omega_1), \tag{2.7}$$

where $<\cdot,\cdot>_\Gamma$ denote the $L^2$ innerproduct on $\Gamma$,

$$w_f(p) = \int\int_{\Omega_2} f(p')G_n^{(2)}(p,p')dp',\ p\in\Gamma,$$

$$H_g^1(\Omega_1) = \{v : v\in H^1(\Omega_1), v|_{\partial\Omega} = g\},$$

and

$$H_0^1(\Omega_1) = \{v : v\in H^1(\Omega_1), v|_{\partial\Omega} = 0\}.$$

**Remark 2.1.** It has been shown in [13] that the operator $K_2 : H^{\frac{1}{2}}(\Gamma)\to H^{-\frac{1}{2}}(\Gamma)$ is just the Dirichlet-Neumann operator (Steklov-Poincare operator) for $\Omega_2$. Thus, it is symmetric and semi-positive definite with respective to the innerproduct $<\cdot,\cdot>_\Gamma$. This means that the coupling bilinear form

$$A(u,v) = D_1(u,v) + < K_2u, v>_\Gamma$$

is symmetric, bounded and coercive in $H_0^1(\Omega_1)$. In particular, the variational problem (2.7) has unique solution $u\in H_g^1(\Omega_1)$.

## 3. Composite Grid Discretization

Without loss of generality, we assume that: (i) the domain $\Omega$ is a polygon; (ii) $g\equiv 0$. Let the auxiliary boundary $\Gamma$ be divided into $m$ circular arcs with the same length. Moreover, let the domain $\Omega_1$ be divided into some quasi-uniform triangular or quarilateral elements with the diameter $H$ ($\approx 2\pi R/m$), such that the finite element nodes on $\Gamma$ coincide with the $m$ dividing points on $\Gamma$. The corresponding piecewise linear finite element space is denoted by $S_H(\Omega_1)\subset H_g^1(\Omega_1) = H_0^1(\Omega_1)$. Because the analytic solution $u$ is in general singular nearby the concave angle points of $\Omega_1$, even if the given functions $f$ and $g$ are smooth enough on their definition domains $\Omega^c$ and $\partial\Omega$, the finite-dimensional subspace $S_H(\Omega_1)$ can not provide a "good" approximation of $u$ unless the mesh size $H$ is very small. Let $\Omega_3$ is a subdomain of $\Omega_1$,such that $\overline{\Omega}_3$ containes the concave angle points of $\Omega_1$. We assume that $\Omega_3$ is just the union set of some elements of $\Omega_1$. Set

$$H_0^1(\Omega_3) = \{v : v\in H^1(\Omega_1), supp\ v\subset\Omega_3\}.$$

We make a refining division to $\Omega_3$, such that the diameter of finer elements is $h < H$. Let $S_h^0(\Omega_3)\subset H_0^1(\Omega_3)$ be the corresponding piecewise linear finite element space. We define the composite grid space $S_{h,H}\subset H_g^1(\Omega_1) = H_0^1(\Omega_1)$ by $S_{h,H} = S_H(\Omega_1) + S_h^0(\Omega_3)$.

The discrete variational problem of (2.7) is: to find $u_{h,H}\in S_{h,H}$ such that

$$A(u_{h,H}, v) = \int\int_{\Omega_1} fvdxdy - < w_f, v >_\Gamma,\ \forall\ v\in S_{h,H}\cap H_0^1(\Omega_1). \tag{3.1}$$

For this approximation,we have the following error estimates.

**Theorem 3.1.** *Assume that $f\in L^2(\Omega^c)$ and $g\in H^{\frac{1}{2}}(\partial\Omega)$. Then, there is a decomposition $u = \hat{u} + \tilde{u}$, such that $\hat{u}\in H^2(\Omega_1)\cap H_0^1(\Omega_1)$ and $\tilde{u}\in H_0^1(\Omega_3)\cap H^{1+\alpha}(\Omega_3)$ with $0 < \alpha < 1$. Moreover, we have*

$$(\|u_{h,H} - u\|_{1,\Omega_1}^2 + \|u_{h,H} - u\|_{\frac{1}{2},\Gamma}^2)^{\frac{1}{2}} \le C(h^\alpha\|\tilde{u}\|_{1+\alpha,\Omega_3} + H\|\hat{u}\|_{2,\Omega_1}), \tag{3.2}$$

*and*

$$\|u_{h,H} - u\|_{0,\Omega_1} \le C(h^{2\alpha}\|\tilde{u}\|_{1+\alpha,\Omega_3} + H^2\|\hat{u}\|_{2,\Omega_1}). \tag{3.3}$$

*Proof.* It can be proved as in [14] that

$$\begin{aligned} &\|u_{h,H} - u\|_{1,\Omega_1}^2 + \|u_{h,H} - u\|_{\frac{1}{2},\Gamma}^2 \\ &\le \inf_{v\in S_{h,H}}\{\|v - u\|_{1,\Omega_1}^2 + \|v - u\|_{\frac{1}{2},\Gamma}^2\}. \end{aligned} \tag{3.4}$$

It is well known that the analytic solution $u$ can be decomposed into $u = \bar{u} + u_s$ with $\bar{u}\in H^2(\Omega_1)\cap H_0^1(\Omega_1)$ and $u_s$ satisfying $u_s\in H_0^1(\Omega_1)\cap H^{1+\alpha}(\Omega_3)$ and $u_s\in H^2(\Omega_1^0)$ for any $\Omega_1^0 \subset\subset \Omega_1$. Here, the value of $\alpha$,satisfying $0 < \alpha < 1$, depends on the degrees of the concave angles of $\Omega_1$. By the unit resolution Theorem, we can show that there are $u_{s1}\in H_0^1(\Omega_3)\cap H^{1+\alpha}(\Omega_3)$ and $u_{s2}\in H^2(\Omega_1)\cap H_0^1(\Omega_1)$ such that $u_{s1} + u_{s2} = u_s$.

Let $P_h : H_0^1(\Omega_1)\to S_h^0(\Omega_3)$ and $P_H : H_0^1(\Omega_1)\to S_H(\Omega_1)$ denote the sequences of orthogonal projection operators with respect to the innerproduct $D_1(\cdot,\cdot)$. Set

$$v = P_h u_{s1} + P_H(\bar{u} + u_{s2})\in S_{h,H}.$$

Then, by the triangular inequality and the Trace Theorem, we obtain

$$\begin{aligned} \|v - u\|_{1,\Omega_1}^2 = &\ \|(P_h - I)u_{s1} + (P_H - I)(\bar{u} + u_{s2})\|_{1,\Omega_1}^2 \\ \le &\ 2(\|(P_h - I)u_{s1}\|_{1,\Omega_3}^2 + \|(P_H - I)(\bar{u} + u_{s2})\|_{1,\Omega_1}^2) \\ \le &\ C[(h^\alpha\|u_{s1}\|_{1+\alpha,\Omega_3})^2 + (H\|\bar{u} + u_{s2}\|_{2,\Omega_1})^2] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|v - u\|_{\frac{1}{2},\Gamma}^2 = &\ \|(P_H - I)(u_{s2} + \bar{u})\|_{\frac{1}{2},\Gamma}^2 \le C\|(P_H - I)(u_{s2} + \bar{u})\|_{1,\Omega_1}^2 \\ \le &\ C(H\|u_{s2} + \bar{u}\|_{2,\Omega_1})^2. \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we deduce (3.2).

Similarly, we can prove (3.3).

**Remark 3.1.** Theorem 3.1 indicates that the fine mesh size $h$ and the coarse mesh size $H$ should satisfy $h^\alpha\approx H$.

## 4. A Preconditioner for the Discrete System

It is obvious that the stiffness matrix of the bilinear form $A(\cdot,\cdot)$ is neither sparse nor band. Thus, it is difficult to solve the discrete problem (3.1) in the direct way. Fortunately, this stiffness matrix is symmetric and positive definite,so (3.1) can be solved by the PCG method. Now we construct a kind of preconditioner for this bilinear form.

For convenience' sake, we define the operators $A, \bar{A} : S_{h,H} \to S_{h,H}$ by

$$(A\varphi, \psi) = D_1(\varphi, \psi) + < K_2\varphi, \psi >_\Gamma, \forall\varphi, \psi \in S_{h,H}.$$

$$(\bar{A}\varphi, \psi) = D_1(\varphi, \psi), \varphi \in S_{h,H}, \forall\psi \in S_{h,H},$$

Let $A_1 : S_H(\Omega_1) \to S_h(\Omega_1)$ and $A_3 : S_h^0(\Omega_3) \to S_h^0(\Omega_3)$ denote the restrictions of the operator $\bar{A}$,which satisfy

$$(A_1\varphi_1, \psi_1) = (\bar{A}\varphi_1, \psi_1), \varphi_1 \in S_H(\Omega_1), \forall\psi_1 \in S_H(\Omega_1),$$

and

$$(A_3\varphi_3, \psi_3) = (\bar{A}\varphi_3, \psi_3), \varphi_3 \in S_h(\Omega_3), \forall\psi_3 \in S_h(\Omega_3).$$

It is clear that the operators $A_1$ and $A_3$ are symmetric and positive definite with respect to the $L^2$ innerproduct.

We define the preconditioner of the operator $A$ as

$$B = A_1^{-1}Q_1 + A_3^{-1}Q_3, \tag{4.1}$$

where $Q_1 : S_{h,H} \to S_H(\Omega_1)$ and $Q_3 : S_{h,H} \to S_h^0(\Omega_3)$ are the $L^2$ orthogonal projection operators.

**Theorem 4.1.** *There exists a constant $C$ independent of $m$, $h$ and $H$, such that*

$$cond(BA) \le C \tag{4.2}$$

In order to prove Theorem 4.1 we need two Lemmas.

**Lemma 4.1.** *There exists a constant $C_0$ independent of $m$, $h$ and $H$, such that for any $v \in S_{h,H}$ there is a decomposition $v = v_1 + v_2$ with $v_1 \in S_H(\Omega_1)$ and $v_2 \in S_h^0(\Omega_3)$, which satisfy*

$$|v_1|_{1,\Omega_1}^2 + |v_2|_{1,\Omega_3}^2 \le C_0|v|_{1,\Omega_1}^2. \tag{4.3}$$

*Proof.* Set $S_H(\Omega_3) = S_H(\Omega_1)|_{\Omega_3}$ and

$$S_H^0(\Omega_3) = \{v : v \in S_H(\Omega_3), v|_{\partial\Omega_3} = 0\}.$$

Let $v_H \in S_H(\Omega_3)$ be the discrete Harmonic extension of $v|_{\partial\Omega_3}$, which satisfies

$$\begin{cases} D_1(v_H, \psi) = 0, & \forall\psi \in S_H^0(\Omega_3), \\ v_H|_{\partial\Omega_3} = v|_{\partial\Omega_3}. \end{cases}$$

Then there is a constant $\bar{C} \ge 1$ such that

$$|v_H|_{1,\Omega_3}^2 \le \bar{C}|v_H|_{\frac{1}{2},\partial\Omega_3}^2 = \bar{C}|v|_{\frac{1}{2},\partial\Omega_3}^2. \tag{4.4}$$

We define $v_1 \in S_H(\Omega_1)$ as follows:

$$v_1 = \begin{cases} v_H, & in \ \ \Omega_3, \\ v, & on \ \bar{\Omega}_1 \backslash \Omega_3. \end{cases}$$

Set $v_2 = v - v_1$. It is clear that $v_2 \in S_H^0(\Omega_3) \subset S_h^0(\Omega_3)$.

From (4.4) and the Trace Theorem, we have

$$\begin{aligned} |v_1|_{1,\Omega_1}^2 &= |v_1|_{1,\Omega_3}^2 + |v_1|_{1,\Omega_1\backslash\Omega_3}^2 \\ &= |v_H|_{1,\Omega_3}^2 + |v|_{1,\Omega_1\backslash\Omega_3}^2 \\ &\le \bar{C}|v|_{\frac{1}{2},\partial\Omega_3}^2 + |v|_{1,\Omega_1\backslash\Omega_3}^2 \\ &\le \tilde{C}|v|_{1,\Omega_3}^2 + |v|_{1,\Omega_1\backslash\Omega_3}^2 \\ &\le \tilde{C}|v|_{1,\Omega_1}^2. \end{aligned} \tag{4.5}$$

Furthermore

$$|v_2|_{1,\Omega_1}^2 \le 2(|v|_{1,\Omega_1}^2 + |v_1|_{1,\Omega_1}^2) \le \hat{C}|v|_{1,\Omega_1}^2,$$

which,together with (4.5),yields (4.3).

**Remark 4.1.** In Lemma 4.1, we do not assume that the ratio $H/h$ is bounded (compare [1]). Besides, our proof is very concise (compare [8]).

**Lemma 4.2.** *For any $\phi \in S_H(\Omega_1)$, we have*

$$< K_2\phi, \phi >_\Gamma \le |\phi_H|_{1,\Omega_1}^2, \tag{4.6}$$

*where $\phi_H \in S_H(\Omega_1)$ is the discrete Harmonic extension of $\phi|_\Gamma$.*

*Proof.* Let $G_0(p, p')$ denote the Green function of the Laplace operator on the domain $\Omega_0$. We define the operator $K_1 : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ by

$$-\int_\Gamma \frac{\partial^2}{\partial n \partial n'} G_0(p, p') \cdot v(p')dp' = K_1 v(p), \ p \in \Gamma.$$

Then, we have $K_1v = K_2v$ for all $v \in H^{\frac{1}{2}}(\Gamma)$. We extend the coarse mesh division on $\Omega_1$ to $\Omega$ in the natural way. The corresponding piecewise linear finite element space is denoted by $S_H(\Omega_0) \subset H^1(\Omega_1)$. For $\phi \in S_H(\Omega_1)$,let $\phi_{H0} \in S_H(\Omega_0)$ be the discrete Harmonic extension of $\phi|_\Gamma$ on $\Omega_0$. It has been shown in [13] that $K_1$ is just the Dirichlet-Neumann operator associated with the domain $\Omega_0$. Thus, by the definition of the Dirichlet-Neumann operator (see [13] and [14]), we obtain

$$< K_2\phi, \phi >_\Gamma = < K_1\phi, \phi >_\Gamma = |\phi_{H0}|^2_{1,\Omega_0}. \tag{4.7}$$

Let $\phi_{HH} \in S_H(\Omega_0)$ denote the zero extension of $\phi_H$. By the minimal energy property of the discrete harmonic extension, we have

$$|\phi_{H0}|^2_{1,\Omega_0} \leq |\phi_{HH}|^2_{1,\Omega_0} = |\phi_H|^2_{1,\Omega_1},$$

which, togther with (4.7), yields (4.6).

*Proof of Theorem 4.1.*

It is clear that

$$(\bar{A}\varphi, \varphi) \leq (A\varphi, \varphi), \forall \varphi \in S_{h,H}. \tag{4.8}$$

Set

$$S^0_H(\Omega_1) = \{v : v \in S_H(\Omega_1), v|_\Gamma = 0\}$$

and

$$S^0_{h,H} = S^0_h(\Omega_3) + S^0_H(\Omega_1).$$

For $\varphi \in S_{h,H}$, let $\varphi_{h,H} \in S_{h,H}$ denote the discrete harmonic extension of $\varphi|_\Gamma$, which is defined by

$$\begin{cases} (\nabla\varphi_{h,H}, \nabla\psi) = 0, \quad \forall\ \psi \in S^0_{h,H}, \\ \varphi_{h,H}|_{\partial\Omega} = \varphi|_{\partial\Omega},\ \varphi_{h,H}|_\Gamma = \varphi|_\Gamma. \end{cases}$$

Set $\varphi_p = \varphi - \varphi_{h,H}$ $(\in S^0_{h,H})$. Then, $(A\varphi, \varphi)$ and $(\bar{A}\varphi, \varphi)$ can be written respectively as

$$(A\varphi, \varphi) = |\varphi_p|^2_{1,\Omega_1} + |\varphi_{h,H}|^2_{1,\Omega_1} + < K_2\varphi, \varphi >_\Gamma \tag{4.9}$$

and

$$(\bar{A}\varphi, \varphi) = |\varphi_p|^2_{1,\Omega_1} + |\varphi_{h,H}|^2_{1,\Omega_1}. \tag{4.10}$$

From the proof of Lemma 4.1, we know that there is a function $\varphi_1 \in S_H(\Omega_1)$ satisfying $\varphi_2 = \varphi_{h,H} - \varphi_1 \in S^0_h(\Omega_3)$, such that

$$|\varphi_1|^2_{1,\Omega_1} \leq \tilde{C} |\varphi_{h,H}|^2_{1,\Omega_1}.$$

Let $\varphi_H \in S_H(\Omega_1)$ denote the discrete harmonic extension of $\varphi|_\Gamma$. Namely, $\varphi_H$ satisfies

$$\begin{cases} (\nabla\varphi_H, \nabla\psi) = 0, \quad \forall\ \psi \in S^0_H(\Omega_1), \\ \varphi_H|_{\partial\Omega} = \varphi|_{\partial\Omega}, \quad \varphi_H|_\Gamma = \varphi_\Gamma. \end{cases}$$

Since (note that $\varphi_2|_\Gamma = 0$)

$$\varphi_1|_\Gamma = \varphi_{h,H}|_\Gamma = \varphi|_\Gamma = \varphi_H|_\Gamma,$$

we obtain

$$|\varphi_H|^2_{1,\Omega_1} \leq |\varphi_1|^2_{1,\Omega_1} \leq \tilde{C} |\varphi_{h,H}|^2_{1,\Omega_1},$$

by (4.9), (4.10) and Lemma 4.2, this leads to

$$(A\varphi, \varphi) \leq (\tilde{C} + 1)(\bar{A}\varphi, \varphi), \forall \varphi \in S_{h,H}.$$

Therefore, using (4.8) and the Cauchy inequality, we can prove

$$(\tilde{C} + 1)^{-1}(\varphi, A\varphi) \leq (\bar{A}^{-1}A\varphi, A\varphi) \leq (\varphi, A\varphi), \forall \varphi \in S_{h,H}. \tag{4.11}$$

On the other hand, we have

$$B\bar{A} = P_H + P_h.$$

Thus, it can be verified in the standard way that (refer to [10] and [11], note Lemma 4.1)

$$C_0^{-1}(\varphi, \bar{A}\varphi) \leq (B\bar{A}\varphi, \bar{A}\varphi) \leq (\varphi, \bar{A}\varphi), \forall \varphi \in S_{h,H}. \tag{4.12}$$

Now we prove (4.2).

In fact, for any $v \in S_{h,H}$ we have

$$(BAv, Av) = (B\bar{A}(\bar{A}^{-1}Av), \bar{A}(\bar{A}^{-1}Av)).$$

Hence, by (4.12) and (4.11) we obtain (set $\varphi = \bar{A}^{-1}Av$)

$$(BAv, Av) \le (\bar{A}^{-1}Av, Av) \le (v, Av). \tag{4.13}$$

In an analogous way, we can prove

$$(BAv, Av) \ge C_0^{-1}(\tilde{C}+1)^{-1}(v, Av),$$

which, together with (4.13) give the desired result ($C = C_0(\tilde{C}+1)$).

**Remark 4.2.** Theorem 4.1 indicates that the PCG algorithbm of the coupled system (3.1) with the preconditioner $B$ has fast convergence speed which is independent of the mesh sizes ($h$ and $H$).

## 5. Implementation

For the coupling method of finite element and the usual boundary element, a boundary integral equation need to be solved. Instead, we need only to calculate some singular integrations.

At first, we give the stiffness matrix of the hypersingular integral operator $K_2$.

Let $\{\phi_i\}_{i=1}^m$ be the set of the piecewise linear nodal basis functions defined on $\Gamma$. It can be verified directly that (see [12], [14])

$$\begin{aligned} < K_2\phi_i, \phi_j >_\Gamma = & -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\phi_i(\theta)\phi_j(\theta')}{\sin^2\frac{\theta-\theta'}{2}}d\theta d\theta' \\ = & \frac{4m^2}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3}\sin^4\frac{k\pi}{m}\cos\frac{2k(i-j)\pi}{m}. \end{aligned}$$

$$(i, j = 1, \cdots, m)$$

Since the series in the above expression possesses fast convergence speed, it can be replaced by a suitable finite sum. It is obvious that the stiffness matrix of $K_2$ is a circulant matrix, so only small calculation and memory space are needed.

Secondly, we discuss how to calculate the singular integration $< w_f, \phi >_\Gamma$. Here, $\phi$ denote a piecewise linear nodal basis function defined on $\Omega_1$. It is clear that $< w_f, \phi >_\Gamma = 0$ for any internal node. Thus, we consider only the boundary nodes (at $\Gamma$).

Let $f(r,\theta)$ and $\phi(R,\theta')$ denote respectively the polar coordinates forms of $f(x,y)$ and $\phi(x,y)$. Then

$$< w_f, \phi >_\Gamma = \frac{R}{2\pi}\int_0^{2\pi}[\phi(R,\theta')\int_0^{2\pi}\int_R^{\infty}\frac{r^2-R^2}{R^2+r^2-2Rr\cos(\theta-\theta')}rf(r,\theta)drd\theta]d\theta'.$$

Using the transformation $r = \frac{1}{t}$, we obtain

$$< w_f, \phi >_\Gamma = \frac{R}{2\pi}\int_0^{2\pi}[\phi(R,\theta')\int_0^{2\pi}\int_0^{R^{-1}}\frac{1-R^2t^2}{1+R^2t^2-2Rt\cos(\theta-\theta')}F(t,\theta)dtd\theta]d\theta',$$

where

$$F(t,\theta) = f(t^{-1},\theta)t^{-3}.$$

Since the integrant is singular when $r = R$ and $\theta = \theta'$, this integration should not be calculated by the standard Gauss integration formula. Thus, we write $< w_f, \phi >_\Gamma$ as the form

$$\begin{aligned} < w_f, \phi >_\Gamma = & \frac{R}{2\pi}\int_0^{2\pi}\{\phi(R,\theta')\int_0^{2\pi}\int_0^{R^{-1}}\frac{1-R^2t^2}{1+R^2t^2-2Rt\cos(\theta-\theta')}[F(t,\theta)-F(R,\theta)]dtd\theta\}d\theta' \\ + & \frac{R}{2\pi}\int_0^{2\pi}\{\phi(R,\theta')\int_0^{2\pi}[F(R,\theta)\int_0^{R^{-1}}\frac{1-R^2t^2}{1+R^2t^2-2Rt\cos(\theta-\theta')}dt]d\theta\}d\theta' \\ = & I_1 + I_2. \end{aligned}$$

Under suitable assumptions the function $F(t,\theta)$ is differentable on the integration domain $[0, R^{-1}]\times[0,2\pi]$,so $I_1$ is a normal integration.

Now we consider $I_2$. It can be verified that

$$\begin{aligned} \frac{1-R^2t^2}{1+R^2t^2-2Rt\cos(\theta-\theta')} = & \frac{2\sin^2(\theta-\theta')}{[Rr-\cos(\theta-\theta')]^2+\sin^2(\theta-\theta')} - 1 \\ - & \frac{1}{R}\cos(\theta-\theta')\frac{2R[Rr-\cos(\theta-\theta')]}{[Rr-\cos(\theta-\theta')]^2+\sin^2(\theta-\theta')}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{R^{-1}}\frac{1-R^2t^2}{1+R^2t^2-2Rt\cos(\theta-\theta')}dt = & \frac{2\sin(\theta-\theta')}{R}[\arctan(\tan\frac{\theta-\theta'}{2}) + \arctan(\frac{1}{\tan(\theta-\theta')})] \\ - & \frac{1}{R} - \frac{1}{R}\cos(\theta-\theta')\ln 4\sin^2\frac{\theta-\theta'}{2}. \end{aligned}$$

In above expression, only the last term involve a weakly singular function, which can be written in the series form (see [14])

$$\ln 4\sin^2\frac{\theta-\theta'}{2} = -2\sum_{n=1}^{\infty}\frac{1}{n}\cos\frac{n}{2}(\theta-\theta'),\ \theta,\theta'\in[0,2\pi].$$

Therefore, we can obtain a formula to calculate $I_2$.

Finally, we consider implementation of the preconditioner $B$.

Let $A_{HH}$ and $A_{hh}$ denote respectively the stiffness matrices of the operators $A_1$ and $A_3$. It can be verified directly that the block-matrix form of the operator $B$ is (refer to [10])

$$\begin{pmatrix} 0 & 0 \\ 0 & A_{HH}^{-1} \end{pmatrix} + \begin{pmatrix} A_{hh}^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

which has the same order with the stiffness matrix of the operator $A$. We must point out that the above sum-matrix can not be replaced by the followwing block-diagonal matrix

$$\begin{pmatrix} A_{hh}^{-1} & 0 \\ 0 & A_{HH}^{-1} \end{pmatrix},$$

because the order of this matrix is higher than that of the stiffness matrix of the operator $A$(note that $S_h^0(\Omega_3)\cap S_H(\Omega_1)\neq\{0\}$).

It is expensive to calculate exactly the matrices $A_{hh}^{-1}$ and $A_{HH}^{-1}$, so we describe the action of the preconditioner $B$ in variational form.

For $\bar{g}\in L^2(\Omega_1), \bar{u} = B\bar{g}\in S_{h,H}$ can be calculated as follows:

$1^0$ Solving $\bar{u}_h\in S_h^0(\Omega_3)$ and $\bar{u}_H\in S_H(\Omega_1)$ in parallel

$$(\nabla\bar{u}_h, \nabla v) = (\bar{g}, v),\quad \forall\ v\in S_h^0(\Omega_3);$$

$$(\nabla\bar{u}_H, \nabla v) = (\bar{g}, v),\quad \forall\ v\in S_H(\Omega_1).$$

$2^0$ Set $\bar{u} = \bar{u}_h + \bar{u}_H$.

**Remark 5.1.** The above algorithm indicates that only the subproblems in $S_h^0(\Omega_3)$ and $S_H(\Omega_1)$ need to be solved (independently).

**Remark 5.2.** The preconditioning algorithm introduced in this paper is additive,so our results can be extended directly to the case of nonexact local solver (refer to [10]). Since we use the natural boundary reduction method, not only the theoretical analysis but also the numerical algorithm are simpler than that of the other boundary reduction methods (compare [6] and [9]).

**Remark 5.3.** For the Dirichlet exterior problems without corner singularity, Yu [12] advanced a kind of D-N alternating method associated with the coupling of FEM and the natural BEM. However, this method can not be applied directly to the case discussed in this paper.

## 6. Numerical Examples

To illustrate the theoretical results stated in Section 3 and Section 4,we consider

$$\begin{cases} -\Delta u = f, & in\ \Omega^c, \\ u = g, & on\ \partial\Omega, \end{cases} \tag{6.1}$$

where $\Omega = [-1,0]\times[-1,0]$;$f$ and $g$ are given functions such that its exact sulution is $u(x,y) = \frac{(x^2+y^2)^{\frac{1}{3}}}{(x+\frac{1}{2})^2+(y+\frac{1}{2})^2}$.

$$(f(x,y) = -u(x,y)\{\frac{2/3}{(x^2+y^2)[(x+\frac{1}{2})^2+(y+\frac{1}{2})^2]} + \frac{8/3}{(x+\frac{1}{2})^2+(y+\frac{1}{2})^2} - \frac{8/9}{x^2+y^2}\})$$

It is clear that the analytic solution $u$ is singular at the corner point (0,0) ($\alpha = \frac{2}{3}$). This problem is solved by the method introduced in Section 2. Here, radius of the auxiliary circle $\Gamma$ is $R = 2$. Moreover, the subdomain $\Omega_3$ is chosen as the sector with radius 1. We use quasi-uniform triangular elements. The resulting linear system is solved by the PCG algorithm with the preconditioner proposed in Section 4.

The error estimates (3.2) and (3.3) are confirmed by Table 1 (with the equivalent discrete norms).

Table 1

error estimates ($H = 4\pi/m, h = H/4$)

| m | $\|u_H - u\|_{1,\Omega_1}$ | $\|u_{h,H} - u\|_{1,\Omega_1}$ | $\|u_H - u\|_{0,\Omega_1}$ | $\|u_{h,H} - u\|_{0,\Omega_1}$ |
|---|---|---|---|---|
| 20 | 9.87D-1 | 7.25D-1 | 9.31D-1 | 4.66D-1 |
| 40 | 6.37D-1 | 3.64D-1 | 3.75D-1 | 1.20D-1 |
| 80 | 4.12D-1 | 1.83D-1 | 1.53D-1 | 3.14D-2 |
| 160 | 2.65D-1 | 9.24D-2 | 6.14D-2 | 8.09D-3 |

The numbers of iteration are given in Table 2, which can confirm Theorem 4.1. Here, the domination error with the discrete $l^2$ norm is $5.0\times10^{-5}$.

Table 2

numbers of iteration

| m | 20 | 40 | 80 | 160 |
|---|---|---|---|---|
| PCG | 14 | 14 | 15 | 14 |

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