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BIVARIATE LAGRANGE-TYPE VECTOR VALUED RATIONAL INTERPOLANTS*1)

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Abstract

An axiomatic definition to bivariate vector valued rational interpolation on distinct plane interpolation points is at first presented in this paper. A two-variable vector valued rational interpolation formula is explicitly constructed in the following form: the determinantal formulas for denominator scalar polynomials and for numerator vector polynomials, which possess Lagrange-type basic function expressions. A practical criterion of existence and uniqueness for interpolation is obtained. In contrast to the underlying method, the method of bivariate Thiele-type vector valued rational interpolation is reviewed.

Key words: Bivariate vector value, Rational interpolation, Determinantal formula.

1. Introduction

Wynn [11] proposed a method for rational interpolation of vector-valued quantities given on a set of distinct interpolation points. He used continued fractions and generalized inverses for the reciprocal of vector-valued quantities. McCleod [9] pointed out that Wynn's proof of the termination of a continued fraction representation of a rational function requires that the underlying field be algebraically closed. He provided a solution to the dilemma by noting that the algebraic operations used in Wynn's proof are valid if N is restricted to be any associative division algebra over the complex filed $A(C)$. Using Thiele fractions interpreted with generalized inverses as follows:

$$\vec{v}^{-1} = 1/\vec{v} = \vec{v}^*/|\vec{v}|^2, \quad \vec{v} \neq 0, \vec{v} \in C^d \tag{1.1}$$

Graves-Morris [7] proved that an interpolating fraction

$$\vec{R}(x) = \vec{b}_0 + \frac{x - x_0}{\vec{b}_1} + \cdots + \frac{x - x_{n-1}}{\vec{b}_n}$$

may normally be found for vector data $\{(x_i, \vec{v}_i) : i = 0, 1, \cdots, n\}$, where $\vec{v} \in C^d, \vec{b}_i \in C^d, x_i \in R$.

Gu Chuanqing [3-4] , Zhu Gongqin and Gu Chuanqing [5] showed that by means of the convergents of Thiele -type branched continued fractions for two-variable functions [10], the generalized inverse (1.1) may be used to define bivariate Thiele-type vector valued rational interpolants (see(5.1) and (5.2)) for vector data

$$\{\vec{v}_{i,j} : \vec{v}_{i,j} = \vec{v}(x_i, y_j) \in C^d, (x_i, y_j) \in \widetilde{Z}_{n,m}\} \tag{1.2}$$

where $\widetilde{Z}_{n,m} = \{(x_i, y_j) : i = 0, 1, \cdots, n, j = 0, 1, \cdots, m, x_i, y_j \in R\}$ be a rectangular grid contained in R^2 and each finite vector $\vec{v}_{i,j}$ is associate with a distinct plane interpolation point (x_i, y_j).

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Graves-Morris and Jenkins [8] presented an axiomatic approach to vector valued rational interpolation in the one-variate case. They constructed interpolants for vector-valued data so that the components of the resulting vector valued rational interpolant share a common denominator polynomial. An explicit determinantal formula for denominator polynomials was given for the denominator polynomial for vector valued rational interpolation on distinct real or complex points. In this paper, an axiomatic definition to bivariate vector-valued rational interpolation on distinct plane interpolation points is at first presented. A two-variable vector valued rational interpolation formula is explicitly constructed in the following form: the determinantal formulas for denominator scalar polynomials and for numerator vector polynomials, which possess Lagrange-type basic function expressions. A practical criterion of existence and uniqueness for interpolation is obtained. Some examples are given to illustrate the results in this paper. In the end, in contrast to the underlying method, the method of bivariate Thiele-type vector valued rational interpolation([4],[5]) is reviewed.

2. Definition

Two-variable, generalized inverse vector valued rational interpolants discussed by this paper obey some basic principles, which was at first put forward by Graves-Morris [7] in the one-variate case, as follows:

(i) If, for some fixed $k, k = 1, 2, \cdots, d$, the kth components of the vectors $\vec{v}_{i,j}$ is the only non-zero components, then the vector valued interpolant reduces to the corresponding rational fraction interpolant .

(ii) The value of the vector rational interpolant does not depend on the order in which the interpolation points are used to construct the interpolant.

(iii) There is some sense in which a specified rational interpolant is unique.

(iv) The poles of the d components of the vector interpolant normally occur at common positions in the xoy-plane.

Given the date set as (1.2), Let the interpolation set $\widetilde{Z}_{n,m}$ change to

$$Z_{n,m} = \{(x_i, y_j) : i = 0, 1, \cdots, n, j = 0, 1, \cdots, m, x_i \in C, y_j \in C\}$$

in (1.2).

Definition 2.1. *For vector data (1.2) with $Z_{n,m}$, the generalized inverse vector valued rational interpolant ($BGIRI_L$) of type [n+m/n+m] is a vector of rational function*

$$\vec{R}(x, y) = \vec{P}(x, y)/q(x, y), \tag{2.1}$$

where $\vec{P}(x, y) = (p^{(1)}(x, y), \cdots, p^{(d)}(x, y)) \in C^d$ is a complex vector polynomial, $q(x, y)$ is a scalar polynomial, satisfying the following conditions:

(i) $\partial\{\vec{P}\} = \max_{1\le k\le d} \partial\{p^{(k)}\} \le n + m, \quad \partial\{q\} = n + m,$ (2.2)

(ii) $q(x, y)|\vec{P}(x, y) \cdot \vec{P}^*(x, y),$ (2.3)

(iii) $q(x, y) = q^*(x, y),$ (2.4)

(iv) $\vec{R}(x_i, y_j) = \vec{v}_{i,j}, \quad (x_i, y_j) \in Z_{n,m}, q(x_i, y_j) \neq 0,$ (2.5)

where a superscript $*$ denotes complex conjugate and the dot product between elements used in (2.3) is usually defined by $\vec{u} \cdot \vec{v} = u^{(1)}v^{(1)} + \cdots + u^{(d)}v^{(d)}$.

3. Construction

The $ijth$ cardinal polynomial of Lagrange-type is defined by

$$l_{ij}(x, y) = \prod_{u=0,u\neq i}^{n} \frac{x - x_u}{x_i - x_u} \prod_{v=0,v\neq j}^{m} \frac{y - y_v}{y_j - y_v}, \quad (x_i, y_j) \in Z_{n,m}. \tag{3.1}$$

Denote the order of points in $Z_{n,m}$ by

$$I = \{(x_0,y_0),(x_1,y_0),(x_0,y_1),(x_2,y_0),(x_1,y_1),(x_0,y_2),\cdots,(x_n,y_m)\}.$$

Definition 3.1. *Let $(x_i,y_j) \in Z_{n,m}$, interpolation point (x_i,y_j) is said to be a real point pair if $x_i,y_j \in R$, (x_i,y_j) is said to be a complex point pair if $x_i,y_j \in C$, (x_i,y_j) is said to be a hybrid point pair if $x_i \in R, y_j \in C$, or $x_i \in C, y_j \in R$. (x_i,y_j) and (x_i^*,y_j^*) is said to be a complex conjugate pairs.*

The set is separated into two disjoint component set:

$$I_1 = \{(x_{i0},y_{j0}),\cdots,(x_{iJ},y_{jJ})\}, I_2 = \{(x_{iJ+1},y_{jJ+1}),\cdots,(x_{in+m},y_{jn+m})\}. \tag{3.2}$$

where the set I_1 consists of interpolation points whose complex conjugates are not in I, the set I_2 consists of real point pairs , hybrid point pairs, and complex conjugate pairs. Either I_1 or I_2 may be empty. For the sake of simpleness, let $\sum_{i,j} = \sum_{i=0}^{n}\sum_{j=0}^{m}$ in Theorem 3.1.

Theorem 3.1. *Let $\vec{R}(x,y) = \vec{P}(x,y)/q(x,y)$ be a $BGIRI_L$ of type [n+m/n+m] for data (1.2) with $Z_{n,m}$. Then hold*

$$q(x,y) = \begin{vmatrix} H_{00,00} & H_{00,10} & H_{00,01} & H_{00,20} & H_{00,11} & \cdots & H_{00,nm} \\ H_{10,00} & H_{10,10} & H_{10,01} & H_{10,20} & H_{10,11} & \cdots & H_{10,nm} \\ H_{01,00} & H_{01,10} & H_{01,01} & H_{01,20} & H_{01,11} & \cdots & H_{01,nm} \\ H_{20,00} & H_{20,10} & H_{20,01} & H_{20,20} & H_{20,11} & \cdots & H_{20,nm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{nm-1,00} & H_{nm-1,10} & H_{nm-1,01} & H_{nm-1,20} & H_{nm-1,11} & \cdots & H_{nm-1,nm} \\ l_{00}(x,y) & l_{10}(x,y) & l_{01}(x,y) & l_{20}(x,y) & l_{11}(x,y) & \cdots & l_{nm}(x,y) \end{vmatrix} \tag{3.3}$$

and

$$\vec{P}(x,y) = \sum_{i,j} l_{ij}(x,y) q_{ij} \vec{v}_{ij}, \quad q_{ij} = q(x_i,y_j), \tag{3.4}$$

where

$$H_{st,uv} = \sum_{i,j} l_{ij}^*(x_s,y_t) l_{uv}(x_i^*,y_j^*) \vec{v}_{ij}^* \cdot (\vec{v}_{st} - \vec{v}_{uv}), \quad (x_i,y_j) \in I_1, \tag{3.5}$$

and

$$H_{st,uv} = (\frac{\partial^2}{\partial x \partial y} l_{uv}(x_s,y_t)) \widetilde{v}_{st}^* \cdot (\vec{v}_{st} - \vec{v}_{uv})$$

$$+ \sum_{i,j} (\frac{\partial^2}{\partial x \partial y} l_{ij}^*(x_s,y_t)) l_{uv}(x_i^*,y_j^*) \vec{v}_{ij}^* \cdot (\vec{v}_{uv} - \vec{v}_{st}), \quad (x_s,y_t) \in I_2, \tag{3.6}$$

with $\widetilde{v}_{st}^* = \vec{P}^*(x_s,y_t)/q_{st}$.

Proof. By interpolation property (2.5) of $BGIRI_L$, express $\vec{P}(x,y)$ as (3.4). Let

$$q(x,y) = \sum_{i,j} l_{ij}(x,y) q_{ij}, \quad q_{ij} = q(x_i,y_j), \tag{3.7}$$

From divisibility hypothesis (2.4), define a polynomial $Q(x,y)$ of degree n+m by

$$S(x,y) = q(x,y)Q(x,y) = \vec{P}(x,y) \cdot \vec{P}^*(x,y), \tag{3.8}$$

where

$$Q(x,y) = \sum_{i,j} l_{ij}(x,y) Q_{ij}, \quad Q_{ij} = Q(x_i,y_j). \tag{3.9}$$

From (3.8) and (2.5), hold

$$Q_{ij} = \vec{v}_{ij}\vec{P}^*(x_i, y_j), \quad (x_i, y_j) \in Z_{nm} \tag{3.10}$$

Because $q(x,y), Q(x,y)$ are real analytic, have

$$q^*_{st} = \sum_{i,j} l_{ij}(x^*_s, y^*_t)q_{ij}, \quad (x_s, y_t) \in Z_{nm}, \tag{3.11}$$

$$Q^*_{st} = \sum_{i,j} l_{ij}(x^*_s, y^*_t)Q_{ij}, \quad (x_i, y_j) \in Z_{nm}. \tag{3.12}$$

Substituting (3.10) into (3.12) and taking its conjugate, get

$$\vec{v}_{ij} \cdot \vec{P}^*(x_i, y_j) = Q_{st} = \sum_{i,j} \vec{v}^*_{ij}\vec{P}^*(x_i, y_j)l^*_{ij}(x_s, y_t), \quad (x_s, y_t) \in Z_{nm}. \tag{3.13}$$

Now substituting (3.4) into (3.13) and using (3.11),(3.12), it is derived that

$$\begin{aligned} &\vec{v}_{st} \cdot \sum_{i,j} l^*_{ij}(x_s, y_t) \sum_{u,v} q_{uv}l_{uv}(x^*_i, y^*_j)\vec{v}^*_{ij} \\ &= \sum_{i,j} l^*_{ij}(x_s, y_t)\vec{v}^*_{ij} \cdot \sum_{u,v} l_{uv}(x^*_i, y^*_j)q_{uv}\vec{v}_{uv}, \quad (x_s, y_t) \in Z_{nm}. \end{aligned} \tag{3.14}$$

By means of (3.14), obtain the following the linear equations

$$\sum_{u,v} H_{st,uv}q_{uv} = 0, \quad (x_s, y_t) \in I_1. \tag{3.15}$$

the coefficient of q_{uv} in (3.15) is $H_{st,uv}$, as given by (3.5) .

Although (3.15) hold for all $(x_s, y_t) \in Z_{nm}$, it turns out that (3.15) is null for $(x_s, y_t) \in I_2$. Differentiate (3.8) with respect of x first, giving

$$\begin{aligned} &(\tfrac{\partial}{\partial x}Q(x,y))q(x,y) + Q(x,y)(\tfrac{\partial}{\partial x}q(x,y)) \\ &= (\tfrac{\partial}{\partial x}\vec{P}(x,y))\vec{P}^*(x,y) + P(x,y)(\tfrac{\partial}{\partial x}\vec{P}(x,y)), \end{aligned} \tag{3.16}$$

then differentiate (3.16) with respect of y, giving

$$\begin{aligned} &(\tfrac{\partial^2}{\partial x\partial y}Q(x,y))q(x,y) + (\tfrac{\partial}{\partial x}Q(x,y))(\tfrac{\partial}{\partial y}q(x,y)) \\ &+(\tfrac{\partial}{\partial y}Q(x,y))(\tfrac{\partial}{\partial x}q(x,y)) + Q(x,y)(\tfrac{\partial^2}{\partial x\partial y}q(x,y)) \\ &= (\tfrac{\partial^2}{\partial x\partial y}\vec{P}(x,y))\vec{P}^*(x,y) + (\tfrac{\partial}{\partial x}\vec{P}(x,y))(\tfrac{\partial}{\partial y}\vec{P}^*(x,y)) \\ &+\tfrac{\partial}{\partial y}\vec{P}(x,y))(\tfrac{\partial}{\partial y}\vec{P}^*(x,y)) + P(x,y)(\tfrac{\partial^2}{\partial x\partial y}\vec{P}^*(x,y)). \end{aligned} \tag{3.17}$$

By putting $(x_s, y_t) \in I_2$ in (3.17), making

$$\begin{aligned} &(\tfrac{\partial}{\partial x}Q(x_s,y_t))(\tfrac{\partial}{\partial y}q(x_s,y_t)) + (\tfrac{\partial}{\partial y}Q(x_s,y_t))(\tfrac{\partial}{\partial x}q(x_s,y_t)) \\ &= (\tfrac{\partial}{\partial x}\vec{P}(x_s,y_t))(\tfrac{\partial}{\partial y}\vec{P}^*(x_s,y_t)) + (\tfrac{\partial}{\partial y}\vec{P}(x_s,y_t))(\tfrac{\partial}{\partial x}\vec{P}^*(x_s,y_t)), \end{aligned} \tag{3.18}$$

and substituting from (3.10), it is derived that

$$\begin{aligned} &\textstyle\sum_{i,j}(\frac{\partial^2}{\partial x\partial y}l^*_{ij}(x_s,y_t))\vec{v}^*_{ij} \cdot \sum_{u,v} l_{uv}(x^*_i,y^*_j)q_{uv}\vec{v}_{uv} \\ &\textstyle+\vec{v}_{st} \cdot \vec{v}^*_{st} \sum_{u,v}(\frac{\partial^2}{\partial x\partial y}l_{uv}(x_s,y_t))q_{uv} \\ &\textstyle= \sum_{u,v}(\frac{\partial^2}{\partial x\partial y}l_{uv}(x_s,y_t))q_{uv}\vec{v}_{uv} \cdot \vec{v}^*_{st} \\ &\textstyle+\vec{v}_{st} \cdot \sum_{i,j}(\frac{\partial^2}{\partial x\partial y}l^*_{ij}(x_s,y_t))\vec{v}^*_{ij} \sum_{u,v} l_{uv}(x^*_i,y^*_j)q_{uv}, (x_s,y_t) \in I_2. \end{aligned} \tag{3.19}$$

By means of (3.19), obtain the following the linear equations

$$\sum_{u,v} H_{st,uv} q_{uv} = 0, \quad (x_s, y_t) \in I_2, \tag{3.20}$$

the coefficient of q_{uv} in (3.20) is $H_{st,uv}$, as given by (3.6).

The equations (3.15), the equations (3.20) (except the last equation , i.e. $(x_s, y_t) \neq (x_n, y_m)$) and (3.7) form a system of $n+m+1$ non-homogeneous equations for $q_{00}, q_{10}, \cdots, q_{nm}$ as shown by

$$q(x,y) = \begin{bmatrix} H_{00,00} & H_{00,10} & \cdots & H_{00,nm} \\ H_{10,00} & H_{10,10} & \cdots & H_{10,nm} \\ \vdots & \vdots & \ddots & \vdots \\ H_{nm-1,00} & H_{nm-1,10} & \cdots & H_{nm-1,nm} \\ l_{00}(x,y) & l_{10}(x,y) & \cdots & l_{nm}(x,y) \end{bmatrix} \begin{bmatrix} q_{00} \\ q_{10} \\ \vdots \\ q_{nm-1} \\ q_{nm} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q(x,y) \end{bmatrix}. \tag{3.21}$$

It is now showed that the equations (3.15) and (3.20) are consistent. Note that it is interpolated for $S(x,y)$ in (3.8) at $2J+2$ distinct plane points in order to derive (3.14) from (3.8) via (3.11),(3.12) and (3.13). It is interpolated for $S(x,y)$ and $(\partial^2/\partial x\partial y)S(x,y)$ at $2N-2J$ ($N = nm+n+m+1$) further distinct plane points in order to get (3.19) from (3.8) via (3.17),(3.18). In fact, it is chosen for $Q(x,y)$ to make (3.18) hold. Hence, it is interpolated for $S(x,y)$ or $(\partial^2/\partial x\partial y)S(x,y)$ a total of $2N+2$ times to get equations (3.14) and (3.19), or to get equations (3.15) and (3.20). As to the reason which choose to ignore the last row of the equations (3.20) is to preserve its property of having zeros on its diagonal. Solving (3.2), obtain that $q(x,y)$ is given by the determinate formula (3.3).

If all interpolation points (x_s, y_t) are real point pairs, the denominator polynomial for a $[n+m/n+m]$ $BGIRI_L$ takes simple form . By (3.6),

$$H_{st,uv} = \sum_{u,v} \left(\frac{\partial^2}{\partial x \partial y} l_{uv}(x_s, y_t)\right) |\vec{v}_{st} - \vec{v}_{uv}|^2, (x_s, y_t) \in I_2. \tag{3.22}$$

Theorem 3.2. *Let $\vec{R}(x,y) = \vec{P}(x,y)/q(x,y)$ be a $BGIRI_L$ of type $[n+m/n+m]$ for data (1.2) with $Z_{n,m}$ and $q(x,y)$ be as in (3.3). Then hold*

$$\vec{P}(x,y) = \begin{vmatrix} H_{00,00} & H_{00,10} & H_{00,01} & H_{00,20} & H_{00,11} & \cdots & H_{00,nm} \\ H_{10,00} & H_{10,10} & H_{10,01} & H_{10,20} & H_{10,11} & \cdots & H_{10,nm} \\ H_{01,00} & H_{01,10} & H_{01,01} & H_{01,20} & H_{01,11} & \cdots & H_{01,nm} \\ H_{20,00} & H_{20,10} & H_{20,01} & H_{20,20} & H_{20,11} & \cdots & H_{20,nm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{nm-1,00} & H_{nm-1,10} & H_{nm-1,01} & H_{nm-1,20} & H_{nm-1,11} & \cdots & H_{nm-1,nm} \\ \vec{v}_{00} l_{00} & \vec{v}_{10} l_{10} & \vec{v}_{01} l_{01} & \vec{v}_{20} l_{20} & \vec{v}_{11} l_{11} & \cdots & \vec{v}_{nm} l_{nm} \end{vmatrix} \tag{3.23}$$

and satisfy

$$\vec{R}(x_i, y_j) = \vec{P}(x_i, y_j)/q(x_i, y_j) = \vec{v}_{ij}, \quad (x_i, y_j) \in Z_{n,m}$$

where $l_{ij} = l_{ij}(x,y)$ is as in (3.1).

Proof. By means of (3.21) ,get easily (3.23). From (3.23) or (3.4)

$$\vec{P}(x,y) = \sum_{i,j} l_{ij}(x,y) q_{ij} \vec{v}_{ij},$$

it is held that

$$\vec{P}(x_i, y_j) = q_{ij}\vec{v}_{ij} = q(x_i, y_j)\vec{v}(x_i, y_j), \quad (x_i, y_j) \in Z_{n,m}.$$

Example 3.1. Find the [2/2] type $BGIRI_L$ $\vec{R}(x,y)=\vec{P}(x,y)/q(x,y)$ for the data

$$\vec{v}_{00}=(1,0),\vec{v}_{10}=(1,-2),\vec{v}_{01}=(0,-1),\vec{v}_{11}=(-1,1), \tag{3.24}$$

at interpolation points

$$(x_0,y_0)=(0,0),(x_1,y_0)=(1,0),(x_0,y_1)=(0,1),(x_1,y_1)=(1,1).$$

Solution. The cardinal functions are

$$l_{00}(x,y)=(x-1)(y-1),l_{10}(x,y)=x(1-y),l_{01}(x,y)=(1-x)y,l_{11}(x,y)=xy,$$

From(3.3) and (3.22), (3.23), get respectively

$$q(x,y)=\begin{vmatrix} 0 & -4 & -2 & 5 \\ 4 & 0 & -2 & 13 \\ 2 & -2 & 0 & 5 \\ l_{00} & l_{10} & l_{01} & l_{11} \end{vmatrix}=8(-14xy+10x+17y-9),$$

$$\vec{P}(x,y)=\begin{vmatrix} 0 & -4 & -2 & 5 \\ 4 & 0 & -2 & 13 \\ 2 & -2 & 0 & 5 \\ \vec{v}_{00}l_{00} & \vec{v}_{10}l_{10} & \vec{v}_{01}l_{01} & \vec{v}_{11}l_{11} \end{vmatrix}=(-112xy+80x+72y-72,112xy-16x-64y).$$

It is showed that $\vec{R}(x,y)=\vec{P}(x,y)/q(x,y)$ satisfy: $\vec{R}(x_i,y_j)=\vec{P}(x_i,y_j)/q(x_i,y_j)=\vec{v}_{ij}$ (see(3.24)).

Example 3.2. Find the denominator $q(x,y)$ of the [2/2] type for data $\vec{v}_{00},\vec{v}_{10},\vec{v}_{01},\vec{v}_{11}$ at interpolation points:

$$(x_0,y_0)=(0,2i),(x_1,y_0)=(i,2i),(x_0,y_1)=(0,1),(x_1,y_1)=(i,1).$$

Solution. The cardinal functions are

$$l_{00}(x,y)=\frac{2-i}{5}(x-i)(y-1),l_{10}(x,y)=\frac{-2+i}{5}x(y-1),$$

$$l_{01}(x,y)=\frac{-2+i}{5}(x-i)(y-2i),l_{11}(x,y)=\frac{2-i}{5}x(y-2i),$$

By (3.5) and (3.6), get

$$q(x,y)=\begin{vmatrix} 0 & H_{00,10} & H_{00,01} & H_{00,11} \\ H_{10,00} & 0 & H_{10,01} & H_{10,11} \\ H_{01,00} & H_{01,10} & 0 & H_{01,11} \\ l_{00} & l_{10} & l_{01} & l_{11} \end{vmatrix} \tag{3.25}$$

where

$$\begin{aligned}
H_{00,10} &= \tfrac{-2+i}{5}|\vec{v}_{00}-\vec{v}_{10}|^2,\\
H_{00,01} &= \tfrac{1}{5}((2+i)\vec{v}^*_{00}-8\vec{v}^*_{10}-(2+i)\vec{v}^*_{01}+2(2+i)\vec{v}^*_{11})\cdot(\vec{v}_{00}-\vec{v}_{01}),\\
H_{00,11} &= \tfrac{1}{5}((2-i)\vec{v}^*_{00}-4\vec{v}^*_{10}+(2+i)\vec{v}^*_{11})\cdot(\vec{v}_{00}-\vec{v}_{11}),\\
H_{10,00} &= \tfrac{-6+8i}{5}\vec{v}^*_{10}\cdot(\vec{v}_{10}-\vec{v}_{00}),\\
H_{10,01} &= \tfrac{16-8i}{5}\vec{v}^*_{10}\cdot(\vec{v}_{10}-\vec{v}_{01}),\\
H_{10,11} &= \tfrac{-8+4i}{5}\vec{v}^*_{10}\cdot(\vec{v}_{10}-\vec{v}_{11}),\\
H_{01,00} &= \tfrac{2-i}{5}(\vec{v}^*_{10}-2\vec{v}^*_{10}+\vec{v}^*_{01})\cdot(\vec{v}_{01}-\vec{v}_{00}),\\
H_{01,10} &= \tfrac{-2+i}{5}|\vec{v}_{01}-\vec{v}_{10}|^2,\\
H_{01,11} &= \tfrac{1}{5}(-4\vec{v}^*_{10}+(2-i)\vec{v}^*_{01}+(2+i)\vec{v}^*_{11})\cdot(\vec{v}_{01}-\vec{v}_{11}).
\end{aligned}$$

Example 3.3. Find the [2/2] type $BGIRI_L$ $\vec{R}(x,y)=\vec{P}(x,y)/q(x,y)$ for the data

$$\vec{v}_{00}=(1,-1),\vec{v}_{10}=(i,0),\vec{v}_{01}=(0,-2i),\vec{v}_{11}=(2,1), \tag{3.26}$$

at the same interpolation points as Ex.3.2.

Solution. From (3.25) and (3.23), get respectively

$$q(x,y)=\frac{12-6i}{5^4}\begin{vmatrix} 0 & -2+i & 4+8i & -2-3i \\ -7+i & 0 & 8-4i & -8-6i \\ 6+2i & -10+5i & 0 & -2-25i \\ (x-i)(y-1) & x(1-y) & -(x-i)(y-2i) & x(y-2i) \end{vmatrix}$$
$$=2\lambda((-647+541i)xy+(1332-756i)x+(-359+87i)y-434-62i))$$

$$\vec{P}(x,y)=\frac{12-6i}{5^4}\begin{vmatrix} 0 & -2+i & 4+8i & -2-3i \\ -7+i & 0 & 8-4i & -8-6i \\ 6+2i & -10+5i & 0 & -2-25i \\ \vec{v}_{00}(x-i)(y-1) & \vec{v}_{10}x(1-y) & \vec{v}_{01}(i-x)(y-2i) & \vec{v}_{11}x(y-2i) \end{vmatrix}$$
$$=2\lambda((-210-730i)xy+(1630+150i)x+(-364+52i)y-364-52i,$$
$$(320+660i)xy+(540-900i)x+(434-62i)y-384-88i))$$

where $\lambda=(12-6i)/5^4$. It is showed that $\vec{R}(x,y)=\vec{P}(x,y)/q(x,y)$ satisfy: $\vec{R}(x_i,y_j)=\vec{P}(x_i,y_j)/q(x_i,y_j)=\vec{v}_{ij}$ (see (3.26)).

4. Existence and Uniqueness

Theorem 4.1. *Let $q(x,y)$ be as in (3.3) and $\vec{P}(x,y)$ be as in (3.23), respectively. If*

$$D_{nm}=\begin{vmatrix} H_{00,00} & H_{00,10} & \cdots & H_{00,nm} \\ H_{10,00} & H_{10,10} & \cdots & H_{10,nm} \\ \vdots & \vdots & \ddots & \vdots \\ H_{nm-1,00} & H_{nm-1,10} & \cdots & H_{nm-1,nm} \end{vmatrix}\neq 0, \tag{4.1}$$

the $[n+m/n+m]$ type $BGIRI_L$ $\vec{R}(x,y)=\vec{P}(x,y)/q(x,y)$ exists and is unique.

Proof. From the construction of $q(x,y)$, obtain

$$\begin{bmatrix} H_{00,00} & H_{00,10} & \cdots & H_{00,nm-1} & H_{00,nm} \\ H_{10,00} & H_{10,10} & \cdots & H_{10,nm-1} & H_{10,nm} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{nm-1,00} & H_{nm-1,10} & \cdots & H_{nm-1,nm-1} & H_{nm-1,nm} \end{bmatrix}\begin{bmatrix} q_{00} \\ q_{10} \\ \vdots \\ q_{nm-1} \\ q_{nm} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{4.2}$$

By the condition (2.5),$q_{nm}\neq 0$, take $q_{nm}=1$ in (4.2), then get

$$\begin{bmatrix} H_{00,00} & H_{00,10} & \cdots & H_{00,nm-2} & H_{00,nm-1} \\ H_{10,00} & H_{10,10} & \cdots & H_{10,nm-2} & H_{10,nm-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{nm-1,00} & H_{nm-1,10} & \cdots & H_{nm-1,nm-2} & H_{nm-1,nm-1} \end{bmatrix}\begin{bmatrix} q_{00} \\ q_{10} \\ \vdots \\ q_{nm-2} \\ q_{nm-1} \end{bmatrix}=-\begin{bmatrix} H_{00,nm} \\ H_{10,nm} \\ \vdots \\ H_{nm-2,nm} \\ H_{nm-1,nm} \end{bmatrix}. \tag{4.3}$$

Thus the condition $D_{nm} \neq 0$ means that (4.3) exists a unique solution $q_{00}, q_{10}, \cdots, q_{nm}$ for $q(x,y)$. Similarly, By the construction of $\vec{P}(x,y)$,as in (3.23), it is found that $D_{nm} \neq 0$ means that exists a unique solution for $\vec{P}(x,y)$.

Example 4.1. In Example 3.1, it is found that

$$D_{11} = \begin{vmatrix} 0 & -4 & -2 \\ 4 & 0 & -2 \\ 2 & -2 & 0 \end{vmatrix} = 32 \neq 0,$$

so that the [2/2] type $BGIRI_L$ $\vec{R}(x,y) = \vec{P}(x,y)/q(x,y)$ exists uniquely.

5. Connection

Gu Chuanqing [3-4] , Zhu Gongqin and Gu Chuanqing [5] showed that by means of the convergents of Thiele -type branched continued fractions for two-variable functions, the generalized inverse (1.1) may be used to define bivariate vector valued rational interpolants

$$\vec{R}^{(0)}_{n,m}(x,y) = \vec{g}_{0,0}(y) + \frac{x - x_0}{\vec{g}_{1,0}(y)} + \cdots + \frac{x - x_{n-1}}{\vec{g}_{n,0}(y)} \tag{5.1}$$

where for $l = 0, 1, \cdots, n$,

$$\vec{g}_{l,0}(y) = \vec{b}_{l,0}(x_0 \cdots x_l, y_0) + \frac{y - y_0}{\vec{b}_{l,1}(x_0 \cdots x_l, y_0 y_1)} + \cdots + \frac{y - y_{m-1}}{\vec{b}_{l,m}(x_0 \cdots x_l, y_0 \cdots y_m)} \tag{5.2}$$

for vector data (1.2) with $\widetilde{Z}_{n,m}$, by using the following recursive formulas:

$$\begin{aligned}
&\vec{b}_{0,0}(x_0, y_0) = \vec{v}_{0,0}, \\
&\vec{b}_{l,0}(x_0 \cdots x_l, y_0) = (x_l - x_{l-1})/(\vec{b}_{l-1,0}(x_0 \cdots x_{l-2} x_l, y_0) - \vec{b}_{l-1,0}(x_0 \cdots x_{l-1}, y_0)), \\
&\vec{b}_{0,k}(x_0, y_0 \cdots y_k) = (y_k - y_{k-1})/(\vec{b}_{0,k-1}(x_0, y_0 \cdots y_{k-2} y_k) - \vec{b}_{0,k-1}(x_0, y_0 \cdots y_{k-1})), \\
&\vec{b}_{l,k}(x_0 \cdots x_l, y_0 \cdots y_k) = (y_k - y_{k-1})/(\vec{b}_{l,k-1}(x_0 \cdots x_l, y_0 \cdots y_{k-2} y_k) \\
&- \vec{b}_{l,k-1}(x_0 \cdots x_l, y_0 \cdots y_{k-1})), k \geq 1.
\end{aligned}$$

In (5.1), for $u = 0, 1, \cdots, n-1$, let

$$\vec{R}^{(u)}_{n,m}(x,y) = \vec{g}_{u,0}(y) + \frac{x - x_u}{\vec{R}^{(u+1)}_{n,m}(x,y)}, \quad \vec{R}^{(n)}_{n,m}(x,y) = \vec{g}_{n,0}(y)$$

and in (5.2), for $v = 0, 1, \cdots, m-1$, let

$$\vec{g}_{l,v}(y) = \vec{b}_{l,v}(x_0 \cdots x_l, y_0 \cdots y_v) + \frac{y - y_v}{\vec{g}_{l,v+1}(y)}, \quad \vec{g}_{l,m}(y) = \vec{b}_{l,m}(x_0 \cdots x_l, y_0 \cdots y_m).$$

Theorem 5.1. ([4]) *Let $\vec{b}_{l,k}(x_0 \cdots x_l, y_0 \cdots y_k), 0 \leq l \leq n, 0 \leq k \leq m$ exist and be different from zero (except for $\vec{b}_{0,0}(x_0, y_0)$). If the following condition are satisfied:*

(i) $\vec{R}^{(i+1)}_{n,m}(x_i, y_j) \neq 0, i = 0, 1, \cdots, n-1$,

(ii) $\vec{g}_{l,j+1}(y_j) \neq 0, l = 0, 1, \cdots, n, j = 0, 1, \cdots, m-1$,

(iii) $\vec{b}_{k,0}(x_0 \cdots x_{k-1} x_i, y_j)$ *for* $k = 1, \cdots, i, i = 1, 2, \cdots, n$ *and* $\vec{b}_{l,k}(x_0 \cdots x_l, y_0 \cdots y_k)$ *for* $k = 1, \cdots, j, j = 1, 2, \cdots, m$ *exist and are different from zero ,then* $\vec{R}^{(0)}_{n,m}(x,y)$ *as (4.1) and (4.2) exists such that*

$$\vec{R}^{(0)}_{n,m}(x_i, y_j) = \vec{v}_{ij}, (x_i, y_j) \in Z_{n,m}.$$

Theorem 5.2. *([4],[5]) Let $\vec{b}_{l,k}(x_0\cdots x_l, y_0\cdots y_k)\in C^d, (x_i,y_j)\in \widetilde{Z}_{n,m}, x,y{\in}R$. Define $\vec{R}^{(0)}_{n,m}(x,y)$ as in (5.1) and (5.2) by a tail-to-head rationalization using (1.1) and suppose every intermediate denominator be different from zero in the operation, then a polynomial vector $\vec{N}(x,y)$ and a real polynomial $D(x,y)$ exist such that*

(i) $\vec{R}^{(0)}_{n,m}(x,y)=\vec{N}(x,y)/D(x,y)$;

(ii) $D(x,y)\big|\|\vec{N}(x,y)\|^2$.

Definition 5.1. *A vector valued Thiele-type rational fraction*

$$\vec{R}^{(0)}_{n,m}(x,y)=\vec{N}(x,y)/D(x,y)$$

is defined to be a bivariate generalized inverse , rational interpolant ($BGIRI_T$) for vector data (1.2) with $\widetilde{Z}_{n,m}$ if $D(x,y)$ is real, $D(x,y)\big|\|\vec{N}(x,y)\|^2$ and $\vec{R}^{(0)}_{n,m}(x,y)=\vec{v}_{ij},\quad (x_i,y_j)\in\widetilde{Z}_{n,m}$.

Theorem 5.3. *([4],[5]) Let a $BGIRI_T$ $\vec{R}^{(0)}_{n,m}(x,y)=\vec{N}(x,y)/D(x,y)$ be express as (5.1) and (5.2). Then hold: if both n and m are even, $\vec{R}^{(0)}_{n,m}(x,y)$ is of $[r/r]$, otherwise $\vec{R}^{(0)}_{n,m}(x,y)$ is of type $[r/r-1]$, where $r=nm+n+m$.*

Example 5.1. Find a BGIRI $\vec{R}_{1,1}(x,y)=\vec{N}(x,y)/d(x,y)$ for the data (3.24) at interpolation points

$$(x_0,y_0)=(0,0),(x_1,y_0)=(1,0),(x_0,y_1)=(0,1),(x_1,y_1)=(1,1).$$

Solution. From (5.1),(5.2) and above recursive formulas, get

$$\vec{R}_{1,1}(x,y)=\vec{B}_{0,0}(y)+\frac{x-x_0}{\vec{B}_{1,0}(x,y)}=\frac{\vec{N}(x,y)}{D(x,y)}$$

where

$$\begin{aligned}&\vec{B}_{0,0}(y)=(1,0)+\tfrac{y}{\frac{1}{2}(-1,-1)},\quad \vec{B}_{1,0}(y)=(0,-1/2)+\tfrac{y}{\frac{2}{17}(-2,9)},\\ &\vec{N}(x,y)=(-17y^3+35y^2-4xy-23y+5,-17y^3+18y^2+18xy-6y-10x),\\ &D(x,y)=17y^2-18y+5.\end{aligned}$$

It is verified that $\vec{R}_{1,1}(x,y)=\vec{N}(x,y)/D(x,y)$ satisfy $\vec{R}_{1,1}(x_i,y_j)=\vec{v}_{ij}$ (see(3.24)).

Note that $BGIRI_T$ $\vec{R}_{1,1}(x,y)=\vec{N}(x,y)/d(x,y)$ is obviously different from $BGIRI_L$ $\vec{R}(x,y)=\vec{P}(x,y)/q(x,y)$ (see (3.24) and (3.25)). In fact , $\partial\{\vec{N}\}=3,\partial\{D\}=2$.

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