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CONVERGENCE RESULTS OF RUNGE-KUTTA METHODS FOR MULTIPLY-STIFF SINGULAR PERTURBATION PROBLEMS*1)

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Abstract

The main purpose of this paper is to present some convergence results for algebraically stable Runge-Kutta methods applied to some classes of one- and two-parameter multiply-stiff singular perturbation problems whose stiffness is caused by small parameters and some other factors. A numerical example confirms our results.

Key words: Singular perturbation problems, Runge-Kutta methods, Convergence, Multiple-stiffness.

1. Introduction

The initial value problems of ordinary differential equations in singular perturbation form often arise in many practical applications, such as chemical kinetics, automatic control et.al.(cf. [1, 7, 13-15]). The asymptotic behaviours and expansion solutions of these problems have been studied in detail by many authors(such as [1, 7, 13-15]). The initial value problems in singular perturbation form may be considered as a special class of stiff initial value problems. But it is sorry that they can't be satisfactorily covered by B-theory (cf. [3-6, 8, 11, 19]) because of their very special structures. In the recent more than ten years, many authors[8-10, 12, 16-18] have presented many important and interesting convergence results for linear multistep methods, Runge-Kutta methods, Rosenbrock methods, partitioned linearly implicit Runge-Kutta methods and general linear methods etc. applied to one-parameter singular perturbation problems(SPPs). But all these results are within the limits of the SPPs whose the right-side functions satisfy Lipschitz conditions with moderately-sized Lipschitz constants as the essential problem-characterizing parameters, we thus call these problems singly-stiff singular perturbation problems(SSPPs) because their stiffness is only caused by small parameters. For the SPPs whose stiffness is caused by small parameters and some other factors, we call them multiply- stiff singular perturbation problems(MSPPs), and the corresponding reduced problems are called stiff differential-algebraic equations(SDAEs). Some practical examples of MSPPs have been given in [21].

So far, there exists some convergence results of partitioned Runge-Kutta methods, one-leg methods and linear multistep methods for MSPPs (cf. [20, 21]). In the present paper, we will obtain some convergence results of algebraically stable Runge-Kutta methods(RKMs) for SDAEs and one-parameter MSPPs in Section 2. We will extend the results given in Section 2 to two classes of MSPPs with two parameters in Section 3. In Section 4, we will also give the convergence results of Runge-Kutta methods applied to a class of MSPPs with an algebraic constraint. In Section 5, a numerical example confirms our results.

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2. One-parameter MSPPs

Consider the SPP with one parameter

$$\begin{cases} x'(t) &= f(x,y), \qquad t\in[t_0,t_e], \\ \epsilon y'(t) &= g(x,y), \qquad 0<\epsilon<<1 \end{cases} \tag{2.1}$$

with initial values $(x(t_0),y(t_0))\in\check{G}$ admitting a smooth solution (x(t),y(t))(i.e. all derivatives of $x(t)$ and $y(t)$ up to a sufficiently high order are bounded independently of the stiffness of the problem), where $\check{G}$ is an appropriate region on $R^M\times R^N$, and the maps $f:\ \check{G}\to R^M$ and $g:\ \check{G}\to R^N$ are sufficiently smooth and satisfy

$$< f(x_1,y)-f(x_2,y),x_1-x_2 > \le m\|x_1-x_2\|^2, \quad \forall\ (x_1,y),(x_2,y)\in\check{G}, \tag{2.2a}$$

$$< g(x,y_1)-g(x,y_2),y_1-y_2 > \le -\|y_1-y_2\|^2, \quad \forall\ (x,y_1),(x,y_2)\in\check{G}, \tag{2.2b}$$

$$\|f(x,y_1)-f(x,y_2)\| \le L_1\|y_1-y_2\|, \quad \forall\ (x,y_1),(x,y_2)\in\check{G}, \tag{2.2c}$$

$$\|g(x_1,y)-g(x_2,y)\| \le L_2\|x_1-x_2\|, \quad \forall\ (x_1,y),(x_2,y)\in\check{G} \tag{2.2d}$$

with moderately-sized constants m, L_1 and L_2, where, throughout this paper, $< .,. >$ is the standard inner product in real Euclid space with the corresponding norm $\|\ .\ \|$, the matrix norm used in the following text is subject to $\|\ .\ \|$, and $\mu(.)$ denotes the logarithmic norm with respect to $< .,. >$. In the proof of the following results, we often make use of the following fact

$$\mu(f_x+\Phi)\le m+L, \quad for\ \|\ \Psi\ \|\le L, \tag{2.3}$$

where $\Psi\in R^{M\times M}$, L is moderately-sized.

A Runge-Kutta method (A,b,c) with

$$A=[a_{ij}]\in R^{s\times s}, \quad b^T=(b_1,b_2,\cdots,b_s), \quad c^T=(c_1,c_2,\cdots,c_s)$$

applied to the problem (2.1) reads

$$X_i = x_n + h\sum_{j=1}^{s} a_{ij} f(X_j,Y_j), \quad i=1,2,\cdots,s, \tag{2.4a}$$

$$\epsilon Y_i = \epsilon y_n + h\sum_{j=1}^{s} a_{ij} g(X_j,Y_j), \quad i=1,2,\cdots,s, \tag{2.4b}$$

$$x_{n+1} = x_n + h\sum_{i=1}^{s} b_i f(X_i,Y_i), \tag{2.4c}$$

$$\epsilon y_{n+1} = \epsilon y_n + h\sum_{i=1}^{s} b_i g(X_i,Y_i), \tag{2.4d}$$

with the starting values x_0 and y_0, where $h>0$ is the stepsize, x_n, y_n, X_i and Y_i are approximations to the exact solutions $x(t_n), y(t_n), x(t_n+c_ih)$ and $y(t_n+c_ih)$ respectively, and $n=0,1,\cdots,\check{N}$, $(\check{N}+1)h\le t_e-t_0$.

For any positive integer k,l and $k\times l$ matrix H, let I_l denotes an $l\times l$ unit matrix and $\bar{H} = H\otimes I_M$, $\widetilde{H} = H\otimes I_N$, $\otimes$ denotes the Kronecker product of two matrices. Then the method (2.4) can be written in more compact form

$$X = e_s\otimes x_n + h\bar{A}F(X,Y), \tag{2.5a}$$

$$\epsilon Y = \epsilon e_s\otimes y_n + h\widetilde{A}G(X,Y), \tag{2.5b}$$

$$x_{n+1} = x_n + h\bar{b}^T F(X,Y), \tag{2.5c}$$

$$\epsilon y_{n+1} = \epsilon y_n + h\widetilde{b}^T G(X,Y), \tag{2.5d}$$

where $e_s = (1,1,\cdots,1)^T\in R^s$,

$$X = (X_1^T, X_2^T, \cdots, X_s^T)^T\in R^{Ms}, \quad Y = (Y_1^T, Y_2^T, \cdots, Y_s^T)^T\in R^{Ns}, \tag{2.6a}$$

$$F(X,Y) = (f(X_1,Y_1)^T, f(X_2,Y_2)^T, \cdots, f(X_s,Y_s)^T)^T\in R^{Ms}, \tag{2.6b}$$

$$G(X,Y) = (g(X_1,Y_1)^T, g(X_2,Y_2)^T, \cdots, g(X_s,Y_s)^T)^T\in R^{Ns}. \tag{2.6c}$$

Now we introduce Butcher simplifying assumptions

$$\begin{array}{lll} B(p): & ib^T c^{i-1} = 1, & i = 1,2,\cdots,p, \\ C(q): & iAc^{i-1} = c^i, & i = 1,2,\cdots,q, \end{array}$$

where $c^i = (c_1^i, c_2^i, \cdots, c_s^i)^T$. If the method (A,b,c) satisfies $B(q)$ and $C(q)$, then it is of stage order q.

The method (A,b,c) is called algebraically stable(cf.[3,8]) if the matrix $BA + A^TB - bb^T$ is nonnegative definite, where $B = diag(b)$. The method (A,b,c) is called diagonally stable(cf.[3,8]) if there exists a positive definite diagonal matrix D such that $DA + A^TD$ is positive definite. Algebraic stability and diagonal stability are two important concepts of B-theory. Barker, Berman and Plemmons[2] have proved that diagonal stability implies that A^{-1} exists.

Throughout this paper, the constants symbolized in the $O(\cdots)$ terms are independent of the stiffness of the considered problem. To prove the following results, we introduce the following lemmas given in [19]

Lemma 2.1. *Assume the method (A,b,c) is diagonally stable. Then there exist the positive constants α_0, D_1 and D_2 which depend only on the method such that for any given $h > 0, \sigma\in R, z\in D_\sigma$ with $h\sigma\leq\alpha_0$, the matrix $\bar{I}_s - h\bar{A}z$ is invertible and*

$$\| (\bar{I}_s - h\bar{A}z)^{-1} \| \leq D_1, \quad \|h\bar{b}^T z(\bar{I}_s - h\bar{A}z)^{-1} \| \leq D_2,$$

where $D_\sigma = \{z: \ z = diag(z_1, z_2, \cdots, z_s)\in R^{Ms\times Ms}, \ z_i\in R^{M\times M}, \ \mu(z_i)\leq\sigma\}$.

Lemma 2.2. *Assume the method (A,b,c) is algebraically stable and diagonally stable. Then there exist the positive constants α_0, D_3 which depend only on the method such that for any given $\sigma\in R$, $h > 0, z\in D_\sigma$ and $h\sigma\leq\alpha_0$,*

$$\|I_M + h\bar{b}^T z(\bar{I}_s - h\bar{A}z)^{-1}\bar{e}_s \| \leq 1 + D_3 h\sigma\delta(\sigma),$$

where $\delta(\sigma) = 1$ for $\sigma > 0$ and $\delta(\sigma) = 0$ for $\sigma\leq 0$.

Theorem 2.1. *Assume the method (A,b,c) is of stage order $q\geq 1$, algebraically stable and diagonally stable, and satisfies $|\alpha| < 1$ and that the eigenvalues of A have positive real parts.*

Then when this method applied to the problem (2.1), the following global error estimates hold for $\epsilon \leq C_0 h^2, h \leq h_0, nh \leq t_e - t_0$

$$\|x_n - x(t_n)\| \leq C_1(\|x_0 - x(t_0)\| + \epsilon\|y_0 - y(t_0)\| + h^q),$$

$$\|y_n - y(t_n)\| \leq C_2(\|x_0 - x(t_0)\| + (\widetilde{\epsilon} + \alpha^n)\|y_0 - y(t_0)\| + h^q),$$

where

$$\alpha = 1 - b^T A^{-1} e_s, \quad \widetilde{\epsilon} = \epsilon(1 + \frac{1}{h}),$$

$h_0, C_i(i = 0, 1, 2)$ *depend only on the method,* m, L_1, L_2 *and some derivative bounds of the exact solutions* $(x(t), y(t))$.

Proof. Let $\Delta x_n = x(t_n) - x_n,\ \Delta y_n = y(t_n) - y_n$,

$$\check{X} = (x(t_n + c_1 h)^T, x(t_n + c_2 h)^T, \cdots, x(t_n + c_s h)^T)^T \in R^{Ms},$$

$$\check{Y} = (y(t_n + c_1 h)^T, y(t_n + c_2 h)^T, \cdots, y(t_n + c_s h)^T)^T \in R^{Ns},$$

$$\begin{aligned}\check{F}(\check{X}, \check{Y}) &= (f(x(t_n + c_1 h), y(t_n + c_1 h))^T, f(x(t_n + c_2 h), y(t_n + c_2 h))^T, \\ &\quad \cdots, f(x(t_n + c_s h), y(t_n + c_s h))^T)^T \in R^{Ms},\end{aligned}$$

$$\begin{aligned}\check{G}(\check{X}, \check{Y}) &= (g(x(t_n + c_1 h), y(t_n + c_1 h))^T, g(x(t_n + c_2 h), y(t_n + c_2 h))^T, \\ &\quad \cdots, g(x(t_n + c_s h), y(t_n + c_s h))^T)^T \in R^{Ns},\end{aligned}$$

$$\Delta X = \check{X} - X,\ \Delta Y = \check{Y} - Y,\ \Delta F = \check{F} - F,\ \Delta G = \check{G} - G.$$

We observe the conditions $B(q)$ and $C(q)$ imply

$$\check{X} = e_s \otimes x(t_n) + h\bar{A}\check{F} + O(h^{q+1}), \tag{2.7a}$$

$$\check{Y} = e_s \otimes y(t_n) + \frac{h}{\epsilon}\widetilde{A}\check{G} + O(h^{q+1}), \tag{2.7b}$$

$$x(t_{n+1}) = x(t_n) + h\bar{b}^T\check{F} + O(h^{q+1}), \tag{2.7c}$$

$$y(t_{n+1}) = y(t_n) + \frac{h}{\epsilon}\widetilde{b}^T\check{G} + O(h^{q+1}). \tag{2.7d}$$

It follows from (2.5) and (2.7) that

$$\Delta X = e_s \otimes \Delta x_n + h\bar{A}\Delta F + O(h^{q+1}), \tag{2.8a}$$

$$\Delta Y = e_s \otimes \Delta y_n + \frac{h}{\epsilon}\widetilde{A}\Delta G + O(h^{q+1}), \tag{2.8b}$$

$$\Delta x_{n+1} = \Delta x_n + h\bar{b}^T\Delta F + O(h^{q+1}), \tag{2.8c}$$

$$\Delta y_{n+1} = \Delta y_n + \frac{h}{\epsilon}\widetilde{b}^T\Delta G + O(h^{q+1}), \tag{2.8d}$$

Since diagonal stability of the method implies that A is invertible(cf.[2,6]), we can compute ΔF and ΔG from (2.8a,b)

$$\Delta F = \frac{1}{h}\bar{A}^{-1}(\Delta X - e_s \otimes \Delta x_n + O(h^{q+1})), \tag{2.9a}$$

$$\Delta G = \frac{\epsilon}{h}\widetilde{A}^{-1}(\Delta Y - e_s \otimes \Delta y_n + O(h^{q+1})). \tag{2.9b}$$

Further, it follows from (2.8) and (2.9) that

$$\Delta x_{n+1} = \alpha \Delta x_n + \bar{b}^T \bar{A}^{-1} \Delta X + O(h^{q+1}), \tag{2.10a}$$

$$\Delta y_{n+1} = \alpha \Delta y_n + \widetilde{b}^T \widetilde{A}^{-1} \Delta Y + O(h^{q+1}). \tag{2.10b}$$

We can obtain easily

$$\Delta F = F_X \Delta X + F_Y \Delta Y, \quad \Delta G = G_X \Delta X + G_Y \Delta Y, \tag{2.11}$$

where

$$F_X = blockdiag(U_1^F, U_2^F, \cdots, U_s^F), \quad F_Y = blockdiag(V_1^F, V_2^F, \cdots, V_s^F),$$
$$G_X = blockdiag(U_1^G, U_2^G, \cdots, U_s^G), \quad G_Y = blockdiag(V_1^G, V_2^G, \cdots, V_s^G),$$

where, for $i = 1, 2, \cdots, s$,

$$U_i^F = \int_0^1 f_x(X_i + \theta(x(t_n + c_i h) - X_i), y(t_n + c_i h)) d\theta,$$

$$V_i^F = \int_0^1 f_y(X_i, Y_i + \theta(y(t_n + c_i h) - Y_i)) d\theta,$$

$$U_i^G = \int_0^1 g_x(X_i + \theta(x(t_n + c_i h) - X_i), y(t_n + c_i h)) d\theta,$$

$$V_i^G = \int_0^1 g_y(X_i, Y_i + \theta(y(t_n + c_i h) - Y_i)) d\theta.$$

For (2.11) and (2.8b), we have

$$\Delta Y = \frac{h}{\epsilon}(\widetilde{I}_s - \frac{h}{\epsilon}\widetilde{A} G_Y)^{-1}(\frac{\epsilon}{h} e_s \otimes \Delta y_n + \widetilde{A} G_X \Delta X + O(\epsilon h^q)). \tag{2.12}$$

(2.12) and (2.11) inserted to (2.8a) yield

$$\begin{aligned}&(\bar{I}_s - h\bar{A}(F_X + F_Y \tfrac{h}{\epsilon}(\widetilde{I}_s - \tfrac{h}{\epsilon}\widetilde{A} G_Y)^{-1} \widetilde{A} G_X)) \Delta X \\ &= e_s \otimes \Delta x_n + h\bar{A} F_Y (\widetilde{I}_s - \tfrac{h}{\epsilon}\widetilde{A} G_Y)^{-1}(e_s \otimes \Delta y_n + O(h^{q+1})) + O(h^{q+1}).\end{aligned} \tag{2.13}$$

Since (2.2b) holds and the eigenvalues of A have positive real parts, the matrix-valued version of a theorem of von Neumann(cf.[8-10]) yields for $\epsilon \leq Const.h$

$$\| \frac{h}{\epsilon}(\widetilde{I}_s - \frac{h}{\epsilon}\widetilde{A} G_Y)^{-1} \| \leq C_3. \tag{2.14}$$

Thus, (2.12)-(2.14) and Lemma 2.1 yield

$$\| \Delta X \| \leq C(\| \Delta x_n \| + \epsilon \| \Delta y_n \| + h^{q+1} + \epsilon h^{q+1}), \tag{2.15a}$$

$$\| \Delta Y \| \leq C(\frac{\epsilon}{h}(\| \Delta y_n \| + h^{q+1}) + \| \Delta x_n \| + \epsilon \| \Delta y_n \| + h^{q+1} + \epsilon h^{q+1}). \tag{2.15b}$$

Further, it follows from (2.8c) and (2.11) that

$$\Delta x_{n+1} = \Delta x_n + h\bar{b}^T F_X \Delta X + \sigma_n, \tag{2.16}$$

where $\| \sigma_n \| \leq C(h \| \Delta Y \| + h^{q+1})$. Note the fact (2.3), from (2.13), (2.14), (2.16) and Lemma 2.1-2.2, we easily obtain

$$\| \Delta x_{n+1} \| \leq (1+O(h)) \| \Delta x_n \| + O(\epsilon \| \Delta y_n \|) + O(h^{q+1}) + O(\epsilon h^{q+1}). \tag{2.17a}$$

Inserting (2.15b) to (2.10b) yields

$$\| \Delta y_{n+1} \| \leq (\alpha + O(\widetilde{\epsilon})) \| \Delta y_n \| + O(\| \Delta x_n \|) + O(h^{q+1}) + O(\epsilon h^q). \tag{2.17b}$$

where $\widetilde{\epsilon} = \epsilon(1+\frac{1}{h})$. Further, (2.17) yields

$$\begin{pmatrix} \| \Delta x_{n+1} \| \\ \| \Delta y_{n+1} \| \end{pmatrix} \leq \begin{pmatrix} 1+O(h) & O(\epsilon) \\ O(1) & \alpha + O(\widetilde{\epsilon}) \end{pmatrix} \begin{pmatrix} \| \Delta x_n \| \\ \| \Delta y_n \| \end{pmatrix} + \Psi \begin{pmatrix} h \\ 1 \end{pmatrix}, \tag{2.18}$$

where $\Psi = O(h^q) + O(\epsilon h^q)$. By means of the same technique used in the proof of [8,pp.432-433,Lemma 2.9], we easily obtain the conclusion of Theorem 2.1.

Remark 2.1. (2.5a) and (2.5b) constitute a nonlinear system. It is not difficult to show that Lemma 5 in [10] still holds for the existence and uniqueness of the solution of the system (2.5a,b). In fact, by means of Lemma 2.1, we can show similarly as in the proof of Lemma 5 of [10] that the Jacobian of the system (2.5a,b)

$$\begin{pmatrix} \bar{I}_s - h\bar{A}F_X & O(h) \\ O(1) & \Gamma(\frac{\epsilon}{h}, X, Y) \end{pmatrix}. \tag{2.19}$$

where $\Gamma(\frac{\epsilon}{h}, X, Y) = \frac{\epsilon}{h}\bar{I}_s - \widetilde{A}G_Y$, has a bounded inverse.

The corresponding reduced equations of (2.1) is a SDAE

$$x'(t) = f(x,y), \qquad t \in [t_0, t_e], \tag{2.20a}$$

$$0 = g(x,y) \tag{2.20b}$$

whose initial values $x(t_0)$ and $y(t_0)$ are consistent if $0 = g(x(t_0), y(t_0))$. If the Jacobian $g_y(x,y)$ is invertible and bounded, then the problem (2.20) is of index 1 and the equation (2.20b) then possesses a unique solution $y = \Omega(x)$ which inserted into (2.20a) gives

$$x'(t) = f(x, \Omega(x)). \tag{2.21}$$

It is clear that (2.21) is a stiff ordinary differential equation which can be covered by B-theory. This is the state space form approach. Now we consider the ϵ-embedding approach. We suppose that A^{-1} exists and obtain from (2.5b)

$$hG = \epsilon \widetilde{A}^{-1}(Y - e_s \otimes y_n). \tag{2.22}$$

Insert this into (2.5d) and let $\epsilon = 0$ in (2.5). Then

$$X = e_s \otimes x_n + h\bar{A}F(X,Y), \tag{2.23a}$$

$$0 = G(X,Y), \tag{2.23b}$$

$$x_{n+1} = x_n + h\bar{b}^T F(X,Y), \tag{2.23c}$$

$$y_{n+1} = \alpha y_n + \widetilde{b}^T \widetilde{A}^{-1} Y. \tag{2.23d}$$

Theorem 2.2. *Suppose that the method (A, b, c) is of stage order $q \geq 1$, algebraically stable and diagonally stable and satisfies $-1 \leq \alpha < 1$. If the problem (2.20) satisfies (2.2a,c,d) and that g_y is invertible and bounded and that the initial values are consistent, then the numerical solution of (2.23) has global error*

$$x_n - x(t_n) = O(h^q), \qquad y_n - y(t_n) = O(h^q)$$

when $x_0 - x(t_0) = O(h^q)$, $y_0 - y(t_0) = O(h^q)$, $nh \leq t_e - t_0, h \leq h_0$.

Proof. Because (2.23a-c) are independent of y_n and do not change if (2.23d) is replaced by $0 = g(x_{n+1}, y_{n+1})$. $x_n - x(t_n) = O(h^q)$ follows from the fact that (2.21) is a stiff ordinary differential equation which can be covered by B-theory. For the remaining cases, we first observe that the conditions $B(q)$ and $C(q)$ imply

$$\check{Y} = e_s \otimes y(t_n) + h\widetilde{A}\check{Y}' + O(h^{q+1}), \tag{2.24a}$$

$$y(t_{n+1}) = y(t_n) + h\widetilde{b}^T \check{Y}' + O(h^{q+1}), \tag{2.24b}$$

where

$$\check{Y}' = (y'(t_n + c_1 h)^T, y'(t_n + c_2 h)^T, \cdots, y'(t_n + c_s h)^T)^T \in R^{Ns}.$$

We easily obtain from (2.24)

$$y(t_{n+1}) = \alpha y(t_n) + \widetilde{b}^T \widetilde{A}^{-1} \check{Y} + O(h^{q+1}). \tag{2.25}$$

Subtracting (2.25) from (2.23d) yields

$$\Delta y_{n+1} = \alpha \Delta y_n + \widetilde{b}^T \widetilde{A}^{-1} \Delta Y + O(h^{q+1}). \tag{2.26}$$

On the other hand, (2.23b), Lemma 2.1 and

$$\Delta X = \Delta x_n + h\bar{A}(F(\check{X}, \check{Y}) - F(X, Y)) + O(h^{q+1}), \tag{2.27}$$

$$\check{Y} = \Omega(\check{X}) = (\Omega(x(t_n + c_1 h))^T, \Omega(x(t_n + c_2 h))^T, \cdots, \Omega(x(t_n + c_s h))^T)^T$$

imply

$$\Delta X = O(h^q), \quad \Delta Y = O(h^q) \quad for \;\; h \leq h_0.$$

It follows from (2.26) that

$$\Delta y_{n+1} = \alpha \Delta y_n + \sigma_{n+1}, \qquad \sigma_{n+1} = O(h^q).$$

Repeated insertion of this formula gives

$$\Delta y_{n+1} = \alpha^n \Delta y_n + \sum_{i=1}^{s} \alpha^{n-i} \sigma_i. \tag{2.28}$$

The conclusions follows from (2.28).

3. Two-parameter MSPPs

Consider the SPP with two parameters

$$\left\{ \begin{aligned} x'(t) &= u(x, y, z), & & t \in [t_0, t_e], \\ \epsilon_1 y'(t) &= v(x, y, z), & & 0 < \epsilon_1 << 1, \\ \epsilon_1 \epsilon_2 z'(t) &= w(x, y, z), & & 0 < \epsilon_2 << 1 \end{aligned} \right. \tag{3.1}$$

with initial values $(x(t_0), y(t_0), z(t_0))\in\check{G}$ admitting a smooth solution (x(t),y(t),z(t))(i.e. all derivatives of $x(t)$, $y(t)$ and $z(t)$ up to a sufficiently high order are bounded independently of the stiffness of the problem), where $\check{G}$ is an appropriate region on $R^M\times R^N\times R^Q$, and the maps $u:\ \check{G}\rightarrow R^M$, $v:\ \check{G}\rightarrow R^N$ and $w:\ \check{G}\rightarrow R^Q$ are sufficiently smooth and satisfy

$$< u(x_1,y,z)-u(x_2,y,z), x_1-x_2 > \leq m\|x_1-x_2\|^2, \quad \forall\ (x_1,y,z),(x_2,y,z)\in\check{G}, \tag{3.2a}$$

$$< v(x,y_1,z)-v(x,y_2,z), y_1-y_2 > \leq -\|y_1-y_2\|^2, \quad \forall\ (x,y_1,z),(x,y_2,z)\in\check{G}, \tag{3.2b}$$

$$< w(x,y,z_1)-w(x,y,z_2), z_1-z_2 > \leq -\|z_1-z_2\|^2, \quad \forall\ (x,y,z_1),(x,y,z_2)\in\check{G}, \tag{3.2c}$$

$$\|u_y\|\leq L_1, \quad \|u_z\|\leq L_2, \quad \|v_x\|\leq L_3, \quad \|v_z\|\leq L_4, \quad \|w_x\|\leq L_5, \quad \|w_y\|\leq L_6, \tag{3.2d}$$

where the constants $m,\ L_i(i=1,2,\cdots,6)$ are moderately-sized.

A Runge-Kutta method (A,b,c) applied to the problem (3.1) reads

$$X_i = x_n + h\sum_{j=1}^{s} a_{ij}u(X_j,Y_j,Z_j), \quad i=1,2,\cdots,s, \tag{3.3a}$$

$$\epsilon_1 Y_i = \epsilon_1 y_n + h\sum_{j=1}^{s} a_{ij}v(X_j,Y_j,Z_j), \quad i=1,2,\cdots,s, \tag{3.3b}$$

$$\epsilon_1\epsilon_2 Z_i = \epsilon_1\epsilon_2 z_n + h\sum_{j=1}^{s} a_{ij}w(X_j,Y_j,Z_j), \quad i=1,2,\cdots,s, \tag{3.3c}$$

$$x_{n+1} = x_n + h\sum_{i=1}^{s} b_i u(X_i,Y_i,Z_i), \tag{3.3d}$$

$$\epsilon_1 y_{n+1} = \epsilon_1 y_n + h\sum_{i=1}^{s} b_i v(X_i,Y_i,Z_i), \tag{3.3e}$$

$$\epsilon_1\epsilon_2 z_{n+1} = \epsilon_1\epsilon_2 z_n + h\sum_{i=1}^{s} b_i w(X_i,Y_i,Z_i), \tag{3.3f}$$

with the starting values x_0, y_0 and z_0, where x_n, y_n, z_n, X_i, Y_i and Z_i are approximations to the exact solutions $x(t_n), y(t_n), z(t_n)$, $x(t_n+c_ih)$, $y(t_n+c_ih)$ and $z(t_n+c_ih)$ respectively, and $n=0,1,\cdots,\check{N}$, $(\check{N}+1)h\leq t_e-t_0$.

We now consider two special classes of (3.1)

$$\left\{\begin{array}{rll} x'(t) & = u(x,y,z), & t\in[0,T], \\ \epsilon_1 y'(t) & = v(x,y), & 0<\epsilon_1<<1, \\ \epsilon_1\epsilon_2 z'(t) & = w(x,y,z), & 0<\epsilon_2<<1, \end{array}\right. \tag{3.4}$$

and

$$\left\{\begin{array}{rll} x'(t) & = u(x,y,z), & t\in[0,T], \\ \epsilon_1 y'(t) & = v(x,y,z), & 0<\epsilon_1<<1, \\ \epsilon_1\epsilon_2 z'(t) & = w(x,z), & 0<\epsilon_2<<1. \end{array}\right. \tag{3.5}$$

We will obtain the following convergence results of the method (3.3) applied to the problems (3.4) and (3.5).

Theorem 3.1. *Assume the method (A,b,c) is of stage order $q\geq 1$, algebraically stable and diagonally stable, and satisfies that $|\alpha|<1$ and that the eigenvalues of A have positive real* *parts. Then when this method applied to the problem (3.4) and (3.5) respectively, the following global error estimates hold for* $\epsilon \leq C_0 h^2, h \leq h_0, nh \leq t_e - t_0$

$$\begin{cases} \|x_n - x(t_n)\| & \leq C_1(\|\Delta x_0\| + \epsilon_1 \|\Delta y_0\| + \epsilon_1\epsilon_2 \|\Delta z_0\| + O(h^q)), \\ \|y_n - y(t_n)\| & \leq C_2(\|\Delta x_0\| + (\widetilde{\epsilon}_1 + \alpha^n)\|\Delta y_0\| + \epsilon_1\epsilon_2 \|\Delta z_0\| + O(h^q)), \\ \|z_n - z(t_n)\| & \leq C_3(\|\Delta x_0\| + \widetilde{\epsilon}_1 \|\Delta y_0\| + (\alpha^n + \epsilon_2\widetilde{\epsilon}_1)\|\Delta z_0\| + O(h^q)), \end{cases} \tag{3.6}$$

for the problem (3.4), and

$$\begin{cases} \|x_n - x(t_n)\| & \leq C_4(\|\Delta x_0\| + \epsilon_1 \|\Delta y_0\| + \epsilon_1\epsilon_2 \|\Delta z_0\| + O(h^q)), \\ \|y_n - y(t_n)\| & \leq C_5(\|\Delta x_0\| + (\widetilde{\epsilon}_1 + \alpha^n)\|\Delta y_0\| + \epsilon_2\widetilde{\epsilon}_1 \|\Delta z_0\| + O(h^q)), \\ \|z_n - z(t_n)\| & \leq C_6(\|\Delta x_0\| + \epsilon_1 \|\Delta y_0\| + (\alpha^n + \epsilon_2\widetilde{\epsilon}_1)\|\Delta z_0\| + O(h^q)), \end{cases} \tag{3.7}$$

for (3.5), where

$$\widetilde{\epsilon} = \epsilon_1(1 + \frac{1}{h}), \quad \Delta x_0 = x(t_0) - x_0, \quad \Delta y_0 = y(t_0) - y_0, \quad \Delta z_0 = z(t_0) - z_0$$

and $h_0, C_i(i = 1, 2, \cdots, 6))$ *depend only on the method,* $m, L_i(i = 1, 2, \cdots, 6)$ *and some derivative bounds of the exact solutions* $(x(t), y(t), z(t))$.

Proof. The proof of Theorem 2.1 can be extended to (3.6) and (3.7) with no difficulties.

4. One-parameter MSPPs with a Constraint

For $\epsilon_2 = 0$ and $\epsilon_1 \neq 0$, the problem (3.1) becomes an one-parameter MSPP with an algebraic constraint, i.e.

$$\begin{cases} x'(t) & = u(x, y, z), \qquad t \in [0, T], \\ \epsilon_1 y'(t) & = v(x, y, z), \qquad 0 < \epsilon_1 << 1, \\ 0 & = w(x, y, z). \end{cases} \tag{4.1}$$

Let the initial values $x(t_0), y(t_0)$ and $z(t_0)$ be consistent and let w_z be invertible and bounded. Then the problem (4.1) may be also considered as a SDAE with index one, and the algebraic constraint of (4.1) possesses a unique solution $z = \Omega(x, y)$ which inserted into the other formulae of (4.1) gives

$$\begin{cases} x'(t) & = u(x, y, \Omega(x, y)), \qquad t \in [0, T], \\ \epsilon_1 y'(t) & = v(x, y, \Omega(x, y)), \qquad 0 < \epsilon_1 << 1. \end{cases} \tag{4.2}$$

When $\mu(v_y + v_z \Omega_y) \leq -1$ (which can be satisfied by (3.4) and (3.5) with $\epsilon_2 = 0$), the convergence results of the method (A,b,c) applied to the problem (4.2) have been proposed in Theorem 2.1. Thus, the convergence results of the method (A,b,c) applied to the problem (4.1) can be easily obtained. The above approach is indirect. Now we consider a direct approach. In analogy to the process which yields (2.23), we can obtain from (3.3)

$$X_i = x_n + h \sum_{j=1}^{s} a_{ij} u(X_j, Y_j, Z_j), \quad i = 1, 2, \cdots, s, \tag{4.3a}$$

$$\epsilon_1 Y_i = \epsilon_1 y_n + h \sum_{j=1}^{s} a_{ij} v(X_j, Y_j, Z_j), \quad i = 1, 2, \cdots, s, \tag{4.3b}$$

$$0 = w(X_i, Y_i, Z_i), \quad i = 1, 2, \cdots, s, \tag{4.3c}$$

$$x_{n+1} = x_n + h\sum_{i=1}^{s} b_i u(X_i, Y_i, Z_i), \tag{4.3d}$$

$$\epsilon_1 y_{n+1} = \epsilon_1 y_n + h\sum_{i=1}^{s} b_i v(X_i, Y_i, Z_i), \tag{4.3e}$$

$$z_{n+1} = \alpha z_n + \sum_{i=1}^{s}\sum_{j=1}^{s} b_i \omega_{ij} Z_j, \tag{4.3f}$$

where $A^{-1} = [\omega_{ij}]$. We obtain the following convergence results

Theorem 4.1. *Consider the method (A,b,c) which is of stage order $q\geq 1$, algebraically stable and diagonally stable, and satisfies that $|\alpha| < 1$ and that the eigenvalues of A have positive real parts. Assume the problem (4.1) satisfies (3.2a,b,d) and that w_z is invertible and bounded, and suppose $\mu(v_y + v_z\Omega_y)\leq -1$ and that the initial values are consistent. Then when this method applied to the problem (4.1), the following global error estimates hold for $\epsilon_1\leq C_0 h^2, h\leq h_0, nh\leq t_e - t_0$*

$$x_n - x(t_n) = O(h^q), \quad y_n - y(t_n) = O(h^q), \quad z_n - z(t_n) = O(h^q)$$

when $x_0 - x(t_0) = O(h^q)$, $y_0 - y(t_0) = O(h^q)$ and $z_0 - z(t_0) = O(h^q)$.

Proof. The proof of Theorem 4.1 is similar to that of Theorem 2.2.

5. Numerical Example

Consider the nonlinear one-parameter MSPP

$$x'(t) = -100x, \qquad x(0) = 1, \tag{5.1a}$$

$$y'(t) = -x^2 - y, \qquad y(0) = 200/199, \tag{5.1b}$$

$$\epsilon z'(t) = -xy - z, \qquad z(0) = 1 + \frac{1}{101\epsilon - 1} + \frac{1}{199(300\epsilon - 1)}. \tag{5.1c}$$

The true solution of the problem (5.1) is

$$x(t) = e^{-100t}, \quad y(t) = e^{-t} + \frac{1}{199}e^{-200t},$$

$$z(t) = e^{-\frac{t}{\epsilon}} + \frac{1}{101\epsilon - 1}e^{-101t} + \frac{1}{199(300\epsilon - 1)}e^{-300t}.$$

Its Jacobian matrix is

$$J = \begin{pmatrix} -100 & 0 & 0 \\ -2x & -1 & 0 \\ -y/\epsilon & -x/\epsilon & -1/\epsilon \end{pmatrix}.$$

We can easily show that the logarithmic norm $\mu(J)\approx 3.319567\times 10^2 >> 1$ for $\epsilon = 10^{-3}$ and $\mu(J)\approx 3.670711\times 10^5 >> 1$ for $\epsilon = 10^{-5}$. Surely, the problem (5.1) can't be covered by B-theory. It is easy to show that (5.1) statisfies (2.2) with $m < 0$. When $t\leq 0.3$, the solution is rapidly damped, we apply the explicit method with order 2

$$\begin{cases} \chi_{n+1} &= \chi_n + h\phi(t_{n+\frac{1}{2}}, \chi_{n+\frac{1}{2}}), \\ \chi_{n+\frac{1}{2}} &= \chi_n + \frac{h}{2}\phi(t_n, \chi_n), \end{cases} \tag{5.2}$$

where $t_{n+\frac{1}{2}} = t_n + h/2$, $h = 1.0E - 06$ and $\chi_0 = (x(0), y(0), z(0))^T$ to this problem. When $t > 0.3$, the transient phase is over, we apply two-stage Radau IIA method with order 3 and stage order 2

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

with $h = 0.01$ to the problem (5.1). Let Δx, Δy and Δz denote the absolute global errors of the components x, y and z, respectively. Let $\epsilon = 1.0E - 05$, $errk = max(\Delta x, \Delta y, \Delta z)$. we obtain the following numerical results in Table 5.1.

Table 5.1

t	0.35	0.5	1.0	1.5	2.0
errk	3.5762E-05	1.3282E-04	3.6779E-04	5.0889E-04	5.9303E-04

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