Journal of Computational Mathematics, Vol.19, No.2, 2001, 187–194.

GLOBAL SUPERCONVERGENCES OF THE DOMAIN DECOMPOSITION METHODS WITH NONMATCHING GRIDS*1)

Ping Luo
(*Department of Computer Science, Tsinghua University, Beijing 100084, China*)

Guo-ping Liang
(*Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, the global superconvergences of the domain decomposition methods with Lagrangian multiplier and nonmatching grids are proven for solving the second order elliptic boundary value problems. Moreover, L^∞ and L^2 error estimates are discussed and a defect correction scheme is presented.

Key words: Domain decomposition, Defect correction, Global superconvergence, Nonmatching grids.

1. Introduction

A nonconforming domain decomposition method with Lagrangian multipliers was proposed in [13]. The basic idea of this method is to deal with the nonconforming of nonmatching grids by introducing the Lagrangian multipliers on interfaces of subdomains and its advantages are that it allows not only the incompatibility of the internal variables on the interface between subdomains, but also the discontinuity of the boundary variables on the common vertices of subdomains. Thus one can choose different mesh size, interpolating function and type of element in different subdomains according to the different requirement of pratical problems. So this method is very flexible and well suitable to parallel computational environment.

The superconvergence estimates and error expansions for the finite element method are well studied in many papers. We refer to Chen [5], Križek and Neittanmäki [11], Lin and Xu [15], Lin and Zhu [14,27], Križek [12] and Wahlbin [25] for a detail and survey and to Lin and Zhu [14] for some techniques of high accuracy analysis.

Even so, there still remains some foundmental problems to need studying. In particularly, for high accuracy analysis of the domain decomposition with nonmatching grids, it has seldom been found in the literature. This paper just studies this problem and gives its global superconvergence estimates and defect correction.

2. Domain Decomposition and Global Superconvergence

Let our problem be to solve the following differential equation

$$\begin{cases} -\triangle u + u & = f \quad \text{in} \quad \Omega \\ \qquad\quad u & = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where Ω is a convex smooth domain and $f \in L^2(\Omega)$.

Throughout the paper, Ω is assumed a rectangle to simplify the discussion, although the results are valid for convex smooth domain (cf [14]). We first shall divide the domain Ω into rectangular subdomains Ω_j $(j = 1, \dots, n_d)$ and then subdivide subdomains Ω_j into rectangular meshes T_h^j with the two widths h_j and k_j, where $h = \max\{h_j, k_j\}$. Let A_j (according to certain

* Received June 12, 1998; Final revised September 7, 2000.
1)This research was supported by National Science Foundation grant 19971050 and the 973 grant number G1998030420.

order) be the vertices of $\partial\Omega_j$, d the diameter of Ω_j and Γ_{ij} the straight edge of $\partial\Omega_j$ with the vertices A_{j-1} and A_j. Let $\partial\Omega_j = \bigcup_i \Gamma_{ij}$ and $\Gamma = \bigcup_{i,j} \Gamma_{ij}$.

Now let us define the functional spaces

$$H(\Omega) = \prod_{j=1}^{n_d} H^1(\Omega_j) \text{ and } H(\Gamma) = \prod_{j=1}^{n_d} H^{-\frac{1}{2}}(\partial\Omega_j)$$

with the norm

$$||v||_\Omega^2 = \sum_{j=1}^{n_d} ||v||_{1,\Omega_j}^2 \text{ and } ||\mu||_\Gamma^2 = \sum_{j=1}^{n_d} ||\mu||_{-\frac{1}{2},\partial\Omega_j}^2$$

respectively, where

$$||u||_{\frac{1}{2},\partial\Omega_j} = (d^{-1}||u||_{0,\partial\Omega_j}^2 + |u|_{\frac{1}{2},\partial\Omega_j}^2)^{\frac{1}{2}},\ |u|_{\frac{1}{2},\partial\Omega_j}^2 = \int_{\partial\Omega_j}\int_{\partial\Omega_j} \frac{(u(x)-u(y))^2}{(x-y)^2} dxdy$$

and

$$||u||_{-\frac{1}{2},\partial\Omega_j} = \sup_{\substack{v\in H^{\frac{1}{2}}(\partial\Omega_j) \\ |v|\neq 0}} \frac{|\int_{\partial\Omega_j} uv|}{||v||_{\frac{1}{2},\partial\Omega_j}}.$$

Then $(v,\mu) \in \mathbf{H}$ if and only if $v \in H(\Omega)$ and $\mu \in H(\Gamma)$, where $\mathbf{H} = H(\Omega) \times H(\Gamma)$ with the norm $||(v,\mu)||_{\mathbf{H}}^2 = ||v||_\Omega^2 + ||\mu||_\Gamma^2$. Obviously, $\mathbf{H}$ is a Hilbert space.

We may introduce a bilinear form

$$B(u,\lambda;v,\mu) = \sum_{j=1}^{n_d}\{\int_{\Omega_j}(\sum_{i=1}^{2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + uv)dx - \int_{\partial\Omega_j}(v\lambda + u\mu)ds\},\ \ \forall(u,\lambda),(v,\mu)\in\mathbf{H},$$

and a functional

$$F(v,\mu) = \sum_{j=1}^{n_d}\int_{\Omega_j} fvdx.$$

Then the weak form of problem (1) is defined as follows

$$\begin{cases} \text{find} \quad (u,\lambda) & \in \mathbf{H} \text{ such that} \\ B(u,\lambda;v,\mu) & = F(v,\mu),\ \ \forall(v,\mu)\in\mathbf{H}. \end{cases} \tag{2}$$

We know by [13] that the problem (2) has one and only one solution $(u_0,\lambda_0) \in \mathbf{H}$. Let $w \in H^1(\Omega)$ be the weak solution of problem (1). Then $u_0 = w$ and $\lambda_0 = \frac{\partial w}{\partial \vec{n}}$, where $\vec{n}$ is the unit outer normal of Ω_j.

For any positive integer k,l,n,m_j, we define the finite element spaces as follows:

$$S_h(\Omega_j) = \{v \in C(\Omega_j) |\, v|_e \in Q_{m_j}(e), \forall e \in T_h^j\}$$

and

$$S_n(\partial\Omega_j) = \{\mu |\, \mu|_{\Gamma_{ij}} \in P_n(\Gamma_{ij}), \forall\Gamma_{ij} \subset \Gamma\},$$

where

$$Q_{m_j}(e) = \text{span}\{x^k y^l : \quad 0 \le k,l \le m_j\}$$

and

$$P_n(\Gamma_{ij}) = \text{span}\{x^k y^l : \quad 0 \le k+l \le n\}.$$

Let

$$S_h(\Omega) = \prod_{j=1}^{n_d} S_h(\Omega_j) \text{ and } S_n(\Gamma) = \prod_{j=1}^{n_d} S_n(\partial\Omega_j).$$

Then $(v,\mu) \in S_{h\times n}$ if and only if $v \in S_h(\Omega)$ and $\mu \in S_n(\Gamma)$, where

$$S_{h\times n} = S_h(\Omega) \times S_n(\Gamma) \subseteq \mathbf{H} = H(\Omega) \times H(\Gamma)$$

with the norm $||(v,\mu)||_{S_{h\times n}} = ||(v,\mu)||_{\mathbf{H}}$.

We define the finite element approximation of problem (2) as follows

$$\begin{cases} \text{find} \quad (u,\lambda) & \in S_{h\times n} \text{ such that} \\ B(u,\lambda;v,\mu) & = F(v,\mu), \;\; \forall (v,\mu) \in S_{h\times n}. \end{cases} \tag{3}$$

We know by [13] that the following conditions hold for $\frac{n^2 h}{d}$ sufficiently small:

$$\sup_{\substack{(v,\mu)\in S_{h\times n}\\ ||(v,\mu)||_{\mathbf{H}}\le 1}} |B(u,\lambda;v,\mu)| \ge c||(u,\lambda)||_{\mathbf{H}}, \;\; \forall (u,\lambda) \in S_{h\times n}, \tag{4}$$

$$\sup_{\substack{(u,\lambda)\in S_{h\times n}\\ ||(u,\lambda)||_{\mathbf{H}}\le 1}} |B(u,\lambda;v,\mu)| \ge c||(v,\mu)||_{\mathbf{H}}, \;\; \forall (v,\mu) \in S_{h\times n}, \tag{5}$$

$$B(u,\lambda;v,\mu) \le c||(u,\lambda)||_{\mathbf{H}}||(v,\mu)||_{\mathbf{H}}, \;\; \forall (u,\lambda),(v,\mu) \in \mathbf{H}. \tag{6}$$

Hence the finite element approximation of problem (3) has a unique solution $(u_h,\lambda_h) \in S_{h\times n}$.

We take any element $e \in T_h^j$, where $e = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e]$, and (x_e, y_e) is the centeral coordinate of e, $2h_e$ and $2k_e$ the two widths. Let A_i denote the four vertices of e and interpolation operator $I_h^{m_j} : C \longmapsto S_h(\Omega_j)$ satisfy

$$I_h^{m_j} u(A_i) = u(A_i), \quad i = 1,2,3,4,$$

$$\int_{l_k} (u - I_h^{m_j} u) v ds = 0, \quad \forall v \in P_{m_j-2}(l_k), \quad m_j \ge 2, \; k = 1,\cdots,4,$$

$$\int_e (u - I_h^{m_j} u) v dx dy = 0, \quad \forall v \in Q_{m_j-2}(e), \, m_j \ge 2,$$

where l_1, l_2 and l_3, l_4 denote the four edges of e parallel to x and y axis respectively. We know by [14] that $I_h^{m_j} u$ are only defined.

Lemma 1. *Let $u \in H^{m_j+2}(\Omega_j), (j = 1,\cdots,n_d)$. Then there exists a positive constant c, independent of h and n, such that*

$$|\int_{\Omega_j} \sum_{i=1}^{2} (u - I_h^{m_j} u)_{x_i} v_{x_i}| \le c h^{m_j+\frac{1}{2}} ||u||_{m_j+2,\Omega_j} ||v||_{1,\Omega_j}, \tag{7}$$

$$|\int_{\Omega_j} (u - I_h^{m_j} u) v| \le c h^{m_j+1} ||u||_{m_j+1,\Omega_j} ||v||_{0,\Omega_j}, \tag{8}$$

$$|\int_{\partial\Omega_j} (u - I_h^{m_j} u) \lambda ds| \le c h^{m_j+\frac{1}{2}} ||u||_{m_j+1,\partial\Omega_j} ||\lambda||_{-\frac{1}{2},\partial\Omega_j}, \tag{9}$$

for any $v \in S_h(\Omega_j)$ *and* $\lambda \in S_n(\partial\Omega_j)$, *where* v_{x_i} *denotes* $\frac{\partial v}{\partial x_i}$.

Let P represent the L^2- projection operator from $H(\Gamma)$ to $S_n(\Gamma)$. Then the following Lemma is easily proved (cf [2],[4],[13]).

Lemma 2. *If* $\mu \in H^l(\partial\Omega_j)$ *and* $\lambda \in S_n(\partial\Omega_j)$, *then we have*

$$||\mu - P\mu||_{k,\partial\Omega_j} \leq cn^{-(l-k)}d^{l-k}||\mu||_{l,\partial\Omega_j},\ -\frac{1}{2} \leq k \leq l, \tag{10}$$

and

$$||\lambda||_{0,\partial\Omega_j} \leq cnd^{-\frac{1}{2}}||\lambda||_{-\frac{1}{2},\partial\Omega_j}. \tag{11}$$

Theorem 1. *Let* $h < d, n, m_j \geq 1, l \geq 2m_j + \frac{1}{2}\,(j = 1,\cdots,n_d)$ *and let* $(u_h, \lambda_h) \in S_{h\times n}$ *satisfy (3). Then for* $\frac{n^2 h}{d}$ *sufficiently small we have*

$$||u_h - I_h^{m_j}u||_{1,\Omega} + ||\lambda_h - P\frac{\partial u}{\partial \vec{n}}||_{-\frac{1}{2},\Gamma} \leq c\sum_{j=1}^{n_d} h^{m_j+\frac{1}{2}}(||u||_{m_j+2,\Omega_j} + d^{\frac{l}{2}+\frac{1}{4}}||u||_{l+1,\partial\Omega_j}).$$

Proof. Let $\tilde{I}_h u \in S_h(\Omega)$ such that $\tilde{I}_h u|_{\Omega_j} = I_h^{m_j}(u|_{\Omega_j})$. Then $(u_h - \tilde{I}_h u, \lambda_h - P\frac{\partial u}{\partial \vec{n}}) \in S_{h\times n}$. Thus we have by (2)–(6)

$$\begin{aligned} c||(u_h - \tilde{I}_h u, \lambda_h - P\frac{\partial u}{\partial \vec{n}})||_{\mathbf{H}} &\leq \sup_{\substack{(v,\mu)\in S_{h\times n}\\ ||(v,\mu)||_{\mathbf{H}}\leq 1}} B(u_h - \tilde{I}_h u, \lambda_h - P\frac{\partial u}{\partial \vec{n}}; v, \mu) \\ &= \sup_{\substack{(v,\mu)\in S_{h\times n}\\ ||(v,\mu)||_{\mathbf{H}}\leq 1}} \{\sum_{j=1}^{n_d}\int_{\Omega_j}[\sum_{i=1}^{2}(u - I_h^{m_j}u)_{x_i}v_{x_i} + (u - I_h^{m_j}u)v] \\ &\quad -\sum_{j=1}^{n_d}\int_{\partial\Omega_j}[(\frac{\partial u}{\partial \vec{n}} - P\frac{\partial u}{\partial \vec{n}})v + (u - I_h^{m_j}u)\mu]\}, \end{aligned}$$

which, combining Lemma 1 with Lemma 2, implies our Theorem.

In order to obtain the error estimates between the solution of equation (1) and its finite element solution, we will use high order interpolation postprocessing. We assume that $T_h^j(j = 1,\cdots,n_d)$ has been obtained from T_{2h}^j of mesh size $2h_j$ and $2k_j$ by subdividing each element in T_{2h}^j into 4 congruent subrectangles of size h_j and k_j. Let $e_i(i = 1,2,3,4)$ denote the 4 congruent subrectangles of size h_j and k_j, $T = \{A_i, i = 1,\cdots,9\}$ the vertice sets of e_i and $S = \{l_i, i = 1,\cdots,12\}$ the edge sets of e_i.

For $u \in C(\Omega)$, we define high order interpolation operator $J_{2h}^{2m_j}$ satisfying

$$\begin{aligned} J_{2h}^{2m_j}u(A_i) &= u(A_i), \qquad (i = 1,\cdots.9), \\ \int_{l_j} J_{2h}^{2m_j}uv &= \int_{l_j} uv, \qquad \forall v \in P_{m_j-2}(l_j),\ (j = 1,\cdots,12), \\ \int_{e_i}(u - J_{2h}^{2m_j}u)v &= 0, \qquad \forall v \in Q_{m_j-2}(e_i), \quad (i = 1,\cdots,4). \end{aligned} \tag{12}$$

It is easy to check that

$$J_{2h}^{2m_j} I_h^{m_j} = J_{2h}^{2m_j}, \quad ||J_{2h}^{2m_j} v||_{s,p,\Omega_j} \le c||v||_{s,p,\Omega_j}, \quad \forall v \in S_h(\Omega_j),$$

$$||u - J_{2h}^{2m_j} u||_{s,p,\Omega_j} \quad \le ch^{2m_j+1-s}||u||_{2m_j+1,p,\Omega_j}, \tag{13}$$

$$s = 0, 1, \quad 1 \le p \le \infty.$$

Theorem 2. *Under the assumption of Theorem 1, there exists a positive constant c, independent of h and n, such that*

$$\begin{aligned} &||u - J_{2h}^{2m_j} u_h||_{1,\Omega} + ||\tfrac{\partial u}{\partial \vec{n}} - \lambda_h||_{-\frac{1}{2},\Gamma} \\ &\le c\sum_{j=1}^{n_d} h^{m_j+\frac{1}{2}}(||u||_{m_j+2,\Omega_j} + d^{\frac{l}{2}+\frac{1}{4}}||u||_{l+1,\partial\Omega_j}). \end{aligned} \tag{14}$$

Proof. We have by (13) and triangular inequality

$$\begin{aligned} ||u - J_{2h}^{2m_j} u_h||_{1,\Omega_j} \quad &\le \quad c(||u - J_{2h}^{2m_j} u||_{1,\Omega_j} + ||I_h^{m_j} u - u_h||_{1,\Omega_j}) \\ &\le \quad c(h^{m_j+1}||u||_{m_j+2,\Omega_j} + ||I_h^{m_j} u - u_h||_{1,\Omega_j}) \end{aligned}$$

and

$$||\frac{\partial u}{\partial \vec{n}} - \lambda_h||_{-\frac{1}{2},\partial\Omega_j} \le ||\frac{\partial u}{\partial \vec{n}} - P\frac{\partial u}{\partial \vec{n}}||_{-\frac{1}{2},\partial\Omega_j} + ||P\frac{\partial u}{\partial \vec{n}} - \lambda_h||_{-\frac{1}{2},\partial\Omega_j},$$

which, in combination with Theorem 1 and Lemma 2, shows inequality (14).

3. Correction Scheme and L^∞ Estimates

Theorem 3. *Let (u, λ) and (u_h, λ_h) satisfy (2) and (3) respectively. Then*

$$||u - u_h||_{0,\Omega} \le c\tilde{h}||(u - u_h, \frac{\partial u}{\partial \vec{n}} - \lambda_h)||_{\mathbf{H}},$$

where $\tilde{h} = h + n^{-1}$.

Proof. Let $w_j \in H_0^1(\Omega_j)$ be a weak solution of the following equation

$$\begin{cases} -\triangle w + w &= u - u_h \quad \text{in} \quad \Omega_j, \\ w &= 0 \quad \text{on}\, \partial\Omega_j, \end{cases}$$

Then we have

$$||w_j||_{2,\Omega_j} \le c||u - u_h||_{0,\Omega_j}. \tag{15}$$

Let $W \in H(\Omega)$ such that $W|_{\Omega_j} = w_j$ and $\frac{\partial W}{\partial \vec{n}}|_{\partial\Omega_j} = \frac{\partial w_j}{\partial \vec{n}} \quad (j = 1, \cdots, n_d)$. We get by Green's formula, for any $(v, \mu) \in \mathbf{H}$,

$$\begin{aligned} B(W, \tfrac{\partial W}{\partial \vec{n}}; v, \mu) = &\quad \sum_{j=1}^{n_d}\{\int_{\Omega_j}(\sum_{i=1}^{2}\frac{\partial W}{\partial x_i}\frac{\partial v}{\partial x_i} + Wv) - \int_{\partial\Omega_j}(\frac{\partial W}{\partial \vec{n}}v + W\mu)\} \\ = &\quad \sum_{j=1}^{n_d}\{\int_{\Omega_j}(-\triangle W + W)v + \int_{\partial\Omega_j}[\frac{\partial W}{\partial \vec{n}}v - (\frac{\partial W}{\partial \vec{n}}v + W\mu)]\} \\ = &\quad \sum_{j=1}^{n_d}\int_{\Omega_j}(u - u_h)v. \end{aligned} \tag{16}$$

Hence, $(W, \frac{\partial W}{\partial \vec{n}})$ is a weak solution of problem (2). Taking $v = u - u_h$ and $\mu = \frac{\partial u}{\partial \vec{n}} - \lambda_h$ in (16), we deduce that

$$B(W, \frac{\partial W}{\partial \vec{n}}; u - u_h, \frac{\partial u}{\partial \vec{n}} - \lambda_h) = \sum_{j=1}^{n_d} ||u - u_h||_{0,\Omega_j}^2. \tag{17}$$

By the symmetry of B we find that

$$B(v, \mu; u - u_h, \frac{\partial u}{\partial \vec{n}} - \lambda_h) = 0, \quad \forall (v, \mu) \in S_{h\times n}. \tag{18}$$

Combining (17),(18) with the continuity of B, we obtain, for any $(v, \mu) \in S_{h\times n}$,

$$\begin{aligned}\sum_{j=1}^{n_d} ||u - u_h||_{0,\Omega_j}^2 &= B(W - v, \frac{\partial W}{\partial \vec{n}} - \mu; u - u_h, \frac{\partial u}{\partial \vec{n}} - \lambda_h) \\ &\leq c||(W - v, \frac{\partial W}{\partial \vec{n}} - \mu)||_{\mathbf{H}}||(u - u_h, \frac{\partial u}{\partial \vec{n}} - \lambda_h)||_{\mathbf{H}}.\end{aligned}$$

It shows that

$$\sum_{j=1}^{n_d} ||u - u_h||_{0,\Omega_j}^2 \leq c \inf_{(v,\mu)\in S_{h\times n}} ||(W - v, \frac{\partial W}{\partial \vec{n}} - \mu)||_{\mathbf{H}}||(u - u_h, \frac{\partial u}{\partial \vec{n}} - \lambda_h)||_{\mathbf{H}}.$$

Further, we have by (15), Lemma 2, interpolation estimate and above inequality

$$\begin{aligned}\sum_{j=1}^{n_d} ||u - u_h||_{0,\Omega_j}^2 &\leq c(h||W||_{2,\Omega} + n^{-1}d||\frac{\partial W}{\partial \vec{n}}||_{\frac{1}{2},\Gamma})||(u - u_h, \frac{\partial u}{\partial \vec{n}} - \lambda_h)||_{\mathbf{H}} \\ &\leq c(h + n^{-1})||u - u_h||_{0,\Omega}||(u - u_h, \frac{\partial u}{\partial \vec{n}} - \lambda_h)||_{\mathbf{H}}.\end{aligned}$$

This proves our Theorem.

For any $(W, \frac{\partial W}{\partial \vec{n}}) \in \mathbf{H}$, let $(W_h, \tilde{\lambda}_h) \in S_{h\times n}$ be its finite element approximation solution satisfying

$$B(W - W_h, \frac{\partial W}{\partial \vec{n}} - \tilde{\lambda}_h; v, \mu) = 0, \quad \forall (v, \mu) \in S_{h\times n}.$$

Then we may define a projection operator R_h: $(W, \frac{\partial W}{\partial \vec{n}}) \longmapsto (W_h, \tilde{\lambda}_h)$ and a correction scheme:

$$\tilde{u}_h = J_{2h}^{2m_j} u_h + u_h - R_h J_{2h}^{2m_j} u_h.$$

Hence

$$u - \tilde{u}_h = (I - R_h)(I - J_{2h}^{2m_j} R_h)u,$$

where $u_h = R_h u$.

Theorem 4. *Under the assumption of Theorem 1, there exists a positive constant c, independent of h and n, such that*

$$||u - \tilde{u}_h||_{0,\Omega} \leq \quad c\tilde{h} \sum_{j=1}^{n_d} h^{m_j+\frac{1}{2}}(||u||_{m_j+2,\Omega_j} + d^{\frac{l}{2}+\frac{1}{4}}||u||_{l+1,\partial\Omega_j}). \tag{19}$$

Proof. Set $\bar{h} = h^2 + n^{-2}$ and $W = u - J_{2h}^{2m_j} u_h$. Then $(W, \frac{\partial W}{\partial \vec{n}}) \in \mathbf{H}$. We assume that $R_h(W, \frac{\partial W}{\partial \vec{n}}) \in S_{h\times n}$ is the finite element approximation solution of $(W, \frac{\partial W}{\partial \vec{n}})$. From (13), Theorem 2 and Theorem 3 imply that

$$
\begin{aligned}
||u-\tilde{u}_h||_{0,\Omega}^2 \leq\ & c\bar{h}\sum_{j=1}^{n_d}(||W-R_hW||_{1,\Omega_j}^2+||\frac{\partial W}{\partial \vec{n}}-R_h\frac{\partial W}{\partial \vec{n}}||_{-\frac{1}{2},\partial\Omega_j}^2) \\
\leq\ & c\bar{h}\sum_{j=1}^{n_d}(||W||_{1,\Omega_j}^2+||R_hW||_{1,\Omega_j}^2+n^{-2(l+\frac{1}{2})}d^{2(l+\frac{1}{2})}||u||_{l+1,\partial\Omega_j}^2) \\
\leq\ & c\bar{h}\sum_{j=1}^{n_d}(||W||_{1,\Omega_j}^2+n^{-2(l+\frac{1}{2})}d^{2(l+\frac{1}{2})}||u||_{l+1,\partial\Omega_j}^2) \\
\leq\ & c\bar{h}\sum_{j=1}^{n_d}h^{2m_j+1}(||u||_{m_j+2,\Omega_j}^2+d^{(l+\frac{1}{2})}||u||_{l+1,\partial\Omega_j}^2).
\end{aligned}
\tag{20}
$$

This implies inquality (19).

Theorem 5. *Let u and (u_h,λ_h) satisfy (2) and (3) respectively. Then under the assumption of Theorem 1, there exists a positive constant c, independent of h and n, such that*

$$
||\frac{\partial u}{\partial \vec{n}}-\lambda_h||_{0,2,\Gamma} \leq\ cnd^{-\frac{1}{2}}\sum_{j=1}^{n_d}h^{m_j+\frac{1}{2}}(||u||_{m_j+2,\Omega_j}+d^{\frac{l}{2}+\frac{1}{4}}||u||_{l+1,\partial\Omega_j}) \tag{21}
$$

and

$$
||u-u_h||_{0,\infty,\Omega} \leq\ c(\ln h)^{\frac{1}{2}}\sum_{j=1}^{n_d}h^{m_j+\frac{1}{2}}(||u||_{m_j+2,\Omega_j}+d^{\frac{l}{2}+\frac{1}{4}}||u||_{l+1,\partial\Omega_j}). \tag{22}
$$

Proof. From (10) and inverse estimate we see that

$$
\begin{aligned}
||\frac{\partial u}{\partial \vec{n}}-\lambda_h||_{0,2,\partial\Omega_j} \leq\ & ||\frac{\partial u}{\partial \vec{n}}-P\frac{\partial u}{\partial \vec{n}}||_{0,2,\partial\Omega_j}+||P\frac{\partial u}{\partial \vec{n}}-\lambda_h||_{0,2,\partial\Omega_j} \\
\leq\ & cn^{-l}d^l||u||_{l+1,\partial\Omega_j}+cnd^{-\frac{1}{2}}||P\frac{\partial u}{\partial \vec{n}}-\lambda_h||_{-\frac{1}{2},\partial\Omega_j} \\
\leq\ & cn^{-l}d^l||u||_{l+1,\partial\Omega_j}+cnd^{-\frac{1}{2}}||\frac{\partial u}{\partial \vec{n}}-\lambda_h||_{-\frac{1}{2},\partial\Omega_j}.
\end{aligned}
$$

This implies inequality (21) by Theorem 2.

We have by triangular inequality, inverse estimate and interpolation estimate

$$
\begin{aligned}
||u-u_h||_{0,\infty,\Omega_j} \leq\ & ||u-I_h^{m_j}u||_{0,\infty,\Omega_j}+||I_h^{m_j}u-u_h||_{0,\infty,\Omega_j} \\
\leq\ & ch^{m_j+\frac{1}{2}}||u||_{m_j+\frac{1}{2},\infty,\Omega_j}+(\ln h)^{\frac{1}{2}}||I_h^{m_j}u-u_h||_{1,2,\Omega_j},
\end{aligned}
$$

which in combination with Theorem 1 shows inequality (22).

References

[1] F. Arnold, F. Brezzi, Mixed and non-conforming finite element methods: implementation post-processing and error estimates, *Math.Modelling. Numer.Anal.,* **19** (1985), 7-35.

[2] I. Babuska, The finite element method with Lagrangian multipliers, *Numer.Math.,* **20** (1973), 179-192.

[3] I. Babuska, B.Q. Guo, The $h-p$ version of the finite element method for problems with nonhomogeous essential boundary condition, *Comp. Meth. App. Mech. Engin,* **74** (1989), 1-18.

[4] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods Springer, New York, 1991.

[5] C.M. Chen, Finite Element Method and Its Analysis in Improving Accuracy, Hunan Science Press, 1982.

[6] L.C. Cowsar, J. Manded, M.F. Whealer, Balancing domain decomposition for mixed finite elements, *Math. Comp.,* **64** (1995), 989-1015.

[7] R.E. Ewing, R. Lazarov, J. Wang, Superconvergence of the velocity along the Gauss lines in mixed finite element methods, *SIAM. J. Numer. Anal.,* **28** (1991), 1015-1029.

[8] M. Feistauer, On the finite element approximation of functions with noninteger derivatives, *Numer. Funct. Anal. and Optimiz.,* **10**:182 (1989), 91-110.

[9] W. Gui, I. Babuska, The h, p and $h-p$ version of the finite element method in one dimension, part 1, the error analysis of the p version, part 2,the error analysis of the h and $h-P$ versions part 3, the adaptive $h-p$ version, *Numer. Math.,* **49** (1986), 577-683.

[10] Howard Swann, On the use of Lagrange multipliers in domain decomposition for solving elliptic problem, *Math. Comp.,* **60** (1993), 49-78.

[11] M. Křížek, P.Neittaanmäki, On superconvergence techniques, *Acta. Appl. Math,* **9** (1987), 175-198.

[12] M. Křížek, Superconvergence phenomena in the finite element method, *Comput. Math., in Appl.Mech & Engrg,* **116** (1994), 157-163.

[13] Liang G.P., He J,, The nonconforming domain decomposition method for elliptic problems with Lagrangian multipliers, *Math. Num. Sin.,* **14,** 207-215.

[14] Q. Lin, Q. Zhu, The Preprocessing and Postprocessing for FEM, Shanghai Sci.Tech, Publishers, 1994.

[15] Q. Lin, J. Xu, Linear finite element with high accuracy, *J. Comp. Math.,* **3** (1985), 115-133.

[16] Q. Lin, P. Luo, High accuracy analysis for a nonconforming membrane element, Math. Study, **28**:3 (1995), 1-5.

[17] P. Luo, Q. Lin, High accuracy analysis of the wilson element, *Math. Comp.,* **17**:2 (1999), 113-124.

[18] P. Luo, The high accuracy analysis for finite element methods, Ph.D thesis, Institute of Systems Science Academia Sinica, 1996.

[19] P.A. Raviart, J.M. Thomas, Primal hybrid finite element methods for 2nd order elliptic equations, *Math. Comp.,* **31**:139 (1977), 391-413.

[20] Rannacher, R, Defect correction techniques in the finite element method, Metz Days on Numer. Anal., Univ. Metz, June, 1990.

[21] R. Scott, S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, *Math. Comp.,* **54** (1990), 483-493.

[22] B. Smith, A domain decomposition algorithm for elliptic problems in three dimension, *Numer. Math.,* **60** (1991), 219-234.

[23] P.Le. Tallec, T. Sassi, M. Vidrascu, Three-dimensional domain decomposition methods with nonmatching grids and unstructured coarse solvers, *Contemporary. Math,* **180** (1994), 61-74.

[24] H. Thomas, The primal mixed finite element method and the LBB condition, *Numer. Methods for P.D.E.,* **8** (1992), 357-379.

[25] L.B. Wahlbin, Superconvergence in Garlerkin finite element method, Lecture Notes in Math, 1605, Springer,1995.

[26] M.F. Wheeler, J.R. Whiteman, Superconvergent recovery of gradients on subdomains from piecewise linear finite element approximations, *Numer. Methods Partial Differential equations,* **3** (1987), 357-374.

[27] Q.D. Zhu, Q. Lin, Superconvergence Theory of the Finite Element Methods, Hunan Science Press, 1990.