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A TRUST REGION ALGORITHM FOR CONSTRAINED NONSMOOTH OPTIMIZATION PROBLEMS*1)

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Abstract

This paper presents a new inexact trust region algorithm for solving constrained nonsmooth optimization problems. Under certain conditions, we prove that the algorithm is globally convergent.

Key words: Trust region method, Nonsmooth function, Constrained optimization, Global convergence.

1. Introduction

Trust region methods are an important class of iterative methods for solving nonlinear optimization problems, and have been developed rapidly in recent twenty years (see [1]–[9], [15], [16] etc.). For nonsmooth optimization problems, as early as in 1984, Y. Yuan [2] [3] proposed a trust region method for the composite function $f(x) = h(g(x))$, where h is convex and $g \in C^1$; L. Qi and J. Sun [4] proposed an inexact trust region method for the general unconstrained nonsmooth optimization problems; A. Friedlander et al. [7] also proposed an inexact trust region method for the box constrained smooth optimization problems, and J.M. Martínez and A.C. Moretti [8] extended the method to the nonsmooth case with convex polyhedron constraints.

In this paper, we will consider the nonsmooth case with the general closed convex set constraint, i.e., we will consider the following constrained nonsmooth optimizaton problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \Omega. \end{array} \tag{1}$$

where Ω is a closed convex set in R^n and $f : \Omega \to R$ is a locally Lipschitzian function in Ω.

We propose a new inexact trust region algorithm for solving (1), where the subproblem is similar to that in [8], but the adjustment of the trust region radius is different. In contrast to the model in [8], our method is suitable to more general closed convex set constraint. Moreover, we discuss the convergence not only for the case $\{\|B_k\|\}$ uniformly bounded, but also for the case

$$\|B_k\| \le c_5 + c_6 \sum_{i=1}^{k} \Delta_i, \forall k \tag{2}$$

or

$$\|B_k\| \le c_7 + c_8 k, \forall k \tag{3}$$

where c_5, c_6, c_7, c_8 are all constants, Δ_i is the trust region radius in the i-th iteration.

The organization of the remainder of this paper is as follows. In section 2, we introduce the concept of the critical point, describe the algorithm and make some basic assumptions on the

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algorithm. In section 3, we establish the global convergence of the algorithm when $\{\|B_k\|\}$ is uniformly bounded or (3) holds.

2. Algorithm and Basic Assumptions

Throughout this paper, $\|\cdot\|$ denotes the 2-norm. In fact, according to the equivalence of the norms in R^n, the results in this paper hold for arbitrary norms.

Let $\varphi(x,s) : \Omega \times R^n \to R$ be a given iterative function satisfying the following assumptions:

A1: For all $x \in \Omega$, $\varphi(x,0) = 0$ and $\varphi(x,\cdot)$ is lower semicontinuous.

A2: For all $x \in \Omega$, if $x+s \in \Omega$, $t \in [0,1]$, then $\varphi(x,ts) \leq t\varphi(x,s)$.

The concept of iterative functions can be found in [13].

Definition 1. *For all $x \in \Omega, \Delta > 0$ and $M \geq 0$, let*

$$m(x,\Delta,M) = \min\{\varphi(x,s) + \frac{1}{2}M\|s\|^2 : x+s \in \Omega \text{ and } \|s\| \leq \Delta\} \tag{4}$$

we say that $x \in \Omega$ is a critical point of (1) if there exist $\Delta > 0, M \geq 0$ such that $m(x,\Delta,M) = 0$.

The above definition on the critical point is similar to that in [8], we also may use the definition on the critical point in [4]. In fact, it is easy to see that under Assumptions A1 and A2, the two definitions are equivalent.

Next, we introduce an useful lemma whose proof can also be found in [8].

Lemma 1. *Under Assumptions A1 and A2, $x \in \Omega$ is a critical point of (1) if and only if for all $\Delta > 0$ and $M \geq 0$,*

$$m(x,\Delta,M) = 0.$$

Now, we assume that $x_0 \in \Omega$ is the initial point, the level set $L_0 = \{x | f(x) \leq f(x_0), x \in \Omega\}$ is bounded, $D \subset R^n$ is a bounded open set containing L_0. Let Δ_0 be the diameter of D, and c_0, c_1, c_2, c_3, c_4 be constants satisfying $0 < c_0 \leq 1$, $0 < c_2 \leq c_1 < 1$, $c_3 < 1 < c_4$. A trust region algorithm for solving constrained nonsmooth optimization problem (1) is stated as follows:

Algorithm 1.

For all $k \geq 0$

Step 0. Choose the symmetric matrix $B_k \in R^{n\times n}$ and constant M_k satisfying $\|B_k\| \leq M_k$.

Step 1. Solve the subproblem

$$\begin{array}{ll} \text{minimize} & Q_k(s) = \varphi(x_k,s) + \frac{1}{2}M_k\|s\|^2 \\ \text{subject to} & x_k + s \in \Omega \text{ and } \|s\| \leq \Delta_k. \end{array} \tag{5}$$

Assume that the solution of (5) is s_k^Q.

If $Q_k(s_k^Q) = 0$, stop; otherwise

Step 2. Compute $s_k \in R^n$ such that

$$\Phi_k(s_k) \leq c_0 Q_k(s_k^Q) \text{ and } x_k + s_k \in \Omega, \|s_k\| \leq \Delta, \tag{6}$$

where, for all $s \in R^n$, $\Phi_k(s) = \varphi(x_k,s) + \frac{1}{2}s^T B_k s$.

Step 3. Let

$$r_k = \frac{f(x_k+s_k) - f(x_k)}{\Phi_k(s_k)}, \tag{7}$$

$$x_{k+1} = \begin{cases} x_k + s_k, & \text{if } r_k > c_2, \\ x_k, & \text{otherwise }, \end{cases} \tag{8}$$

$$\Delta_{k+1} = \begin{cases} c_3\Delta_k, & \text{if } r_k \le c_2, \\ \Delta_k, & \text{if } c_2 < r_k \le c_1, \\ \min\{c_4\Delta_k, \Delta_0\}, & \text{otherwise}. \end{cases} \tag{9}$$

To discuss the relation between critical points and the stationary points, we introduce several definitions, see [4] and [8] for details.

Definition 2. *Let $x \in \Omega$, $\quad s \in R^n$ and $x + s \in \Omega$. If*

$$\limsup_{t\downarrow 0} \frac{f(x+ts)-f(x)}{t}$$

exists, then we call it the upper Dini directional derivative of f at x in the direction s, and denote it by $f^+(x;s)$.

Definition 3. *A point $x \in \Omega$ is called a Dini stationary point of f if for all $s \in R^n$ and $x + s \in \Omega$,*

$$f^+(x;s) \ge 0.$$

A0. Assume that for all $x \in \Omega, s \in R^n$ such that $x + s \in \Omega$, $f^+(x;s)$ exists and that there exists $\theta > 0$ such that

$$\liminf_{t\downarrow 0} \frac{\varphi(x,ts)}{t} \le \theta f^+(x;s).$$

In smooth optimization, Dini stationary points coincide with the first-order stationary points. The following lemma shows that, under Assumption A0, critical points of (1) are Dini stationary points of f.

Lemma 2. *Under Assumptions A0 and A1, if $Q_k(s_k^Q) = 0$, then x_k is a Dini stationary point of f. Moreover, any critical point of (1) is also a Dini stationary point of f.*

Proof. If $Q_k(s_k^Q) = 0$, then $s = 0$ is the solution of the subproblem (5). Thus, for all $s \ne 0$ and $x_k + s \in \Omega, t \in (0,1)$, by Assumption A1, we have

$$0 = Q_k(0) \le Q_k(ts) = \varphi(x_k, ts) + \frac{t^2}{2} M_k \|s\|^2.$$

So,

$$\frac{\varphi(x_k,ts)}{t} + \frac{t}{2} M_k \|s\|^2 \ge 0,$$

that is,

$$\liminf_{t\downarrow 0} \frac{\varphi(x_k,ts)}{t} \ge 0. \tag{10}$$

By Assumption A0, $f^+(x_k;s) \ge 0$. This shows that x_k is a Dini stationary point of f.

Suppose that x_k is a critical point of (1), then (10) is also true. So, any critical point of (1) is also a Dini stationary point of f.

Lemma 3. *Under Assumptions A1 and A2, for all $x \in L_0$ and $M \ge 0$, the function $m(x,\cdot,M)$ is nonpositive, nonincreasing and for $\Delta > 0$ and any $t \in [0,1]$,*

$$m(x, t\Delta, M) \le t\, m(x, \Delta, M).$$

Proof. By the definition of $m(x,\Delta,M)$ and Assumption A1, we deduce that $m(x,\cdot,M)$ is nonpositive, nonincreasing and for all $t \in [0,1]$,

$$\begin{aligned} m(x, t\Delta, M) &= \min\{\varphi(x, ts) + \tfrac{1}{2}M\|ts\|^2 : x + ts \in \Omega \text{ and } \|s\| \le \Delta\} \\ &\le \min\{t(\varphi(x, s) + \tfrac{1}{2}M\|s\|^2) : x + ts \in \Omega \text{ and } \|s\| \le \Delta\} \\ &\le \min\{t(\varphi(x, s) + \tfrac{1}{2}M\|s\|^2) : x + s \in \Omega \text{ and } \|s\| \le \Delta\} \\ &= t\, m(x, \Delta, M). \end{aligned}$$

This completes the proof.

Lemma 4. *Under Assumptions A1 and A2, for all* $\Delta \ge \Delta_k$,

$$\Phi_k(s_k) \le \frac{c_0}{2\Delta} m(x_k, \Delta, 0) \min\{\Delta_k, -m(x_k, \Delta, 0)/(M_k\Delta)\} \tag{11}$$

Proof. Let $m(x_k, \Delta_k, 0) = \varphi(x_k, \bar{s}_k)$, where $x_k + \bar{s}_k \in \Omega$ and $\|\bar{s}_k\| \le \Delta_k$. If $\bar{s}_k = 0$, it is obvious that (11) holds. So, in the following analysis, we assume $\bar{s}_k \ne 0$. Then for all $t \in [0, 1]$, we have $x_k + t\bar{s}_k \in \Omega$ and $\|t\bar{s}_k\| \le \Delta_k$. Thus,

$$\begin{aligned} \Phi_k(s_k) &\le c_0 Q_k(s_k^Q) \le c_0 \min_{0 \le t \le 1} Q_k(t\bar{s}_k) \\ &\le c_0 \min_{0 \le t \le 1} \{t\varphi(x_k, \bar{s}_k) + \tfrac{1}{2}M_k t^2 \|\bar{s}_k\|^2\} \\ &\le \frac{c_0}{2} \max\{\varphi(x_k, \bar{s}_k), -[\varphi(x_k, \bar{s}_k)]^2/(M_k\|\bar{s}_k\|^2)\} \\ &\le \frac{c_0}{2} \max\{m(x_k, \Delta_k, 0), -[m(x_k, \Delta_k, 0)]^2/(M_k{\Delta_k}^2)\}. \end{aligned}$$

For all $\Delta \ge \Delta_k$, by Lemma 3, $m(x_k, \Delta_k, 0) \le \frac{\Delta_k}{\Delta}\, m(x_k, \Delta, 0)$. Therefore,

$$\Phi_k(s_k) \le \frac{c_0}{2\Delta}\, m(x_k, \Delta, 0) \min\{\Delta_k, -m(x_k, \Delta, 0)/(M_k\Delta)\}.$$

This completes the proof.

To prove the global convergence of Algorithm 1, we need additional assumptions on φ. Let $\mathcal{N} = \{0, 1, 2, \ldots\}$.

A3. For any convergent subsequence $\{x_k : k \in K \subseteq \mathcal{N}\}$, if $s_k \to 0$, then

$$f(x_k + s_k) - f(x_k) \le \varphi(x_k, s_k) + o(\|s_k\|).$$

A4. There exists $\bar{\Delta} > 0$ such that, for all $\|s\| \le \bar{\Delta}$, $-\varphi(\cdot, s)$ is lower semicontinuous.

Lemma 5. *Under Assumptions A1–A4, if $\bar{x}$ is an accumulation point of $\{x_k\}$ but not a critical point, then there exist $\varepsilon > 0$ and $\eta > 0$ such that for all k satisfying*

$$M_k \le M, \quad \|x_k - \bar{x}\| < \varepsilon, \quad 0 < \Delta_k < \eta,$$

we have $r_k > c_2$, where M is a positive constant.

Proof. Since $\bar{x}$ is not a critical point, by Lemma 1 and Lemma 3, we may assume that $m(\bar{x}, \bar{\Delta}, 0) = -2\beta < 0$. From Assumption A4, there exists $\varepsilon > 0$ such that when $\|x_k - \bar{x}\| < \varepsilon, m(x_k, \bar{\Delta}, 0) < -\beta$. So, as $\Delta_k \to 0$,

$$\begin{aligned} 1 - r_k &= \frac{-\Phi_k(s_k) + f(x_k + s_k) - f(x_k)}{-\Phi_k(s_k)} \\ &\le \frac{-\frac{1}{2}s_k^T B_k s_k + o(\|s_k\|)}{-\Phi_k(s_k)} \\ &\le \frac{o(\Delta_k)}{\frac{c_o}{2\bar{\Delta}}\beta \min\{\Delta_k, \beta/(M_k\bar{\Delta})\}} \to 0. \end{aligned}$$

Therefore, when η is sufficiently small and $0 < \Delta_k < \eta$, $1 - r_k < 1 - c_2$, i.e., $r_k > c_2$. This completes the proof.

3. Convergence Analysis

In this section, we assume that $\bar{\Delta} \le \Delta_0$. By the above lemmas, we first deduce the convergence of Algorithm 1 when $\{M_k\}$ is bounded.

Theorem 1. *Under Assumptions A1–A4, suppose that there exists a positive constant M such that $M_k \le M, \forall k$. Then any accumulation point of the sequence $\{x_k\}$ generated by Algorithm 1 is a critical point of f.*

Proof. Let us assume that an infinite subsequence $\{x_k : k \in K\}$ converges to $\bar{x}$ which is not a critical point of f.

Let $K_0 = \{k : r_k > c_2\}$ be the set of successful iterations. If K_0 is finite, then for all k sufficiently large, $r_k \le c_2$, thus

$$x_{k+1} = x_k = \bar{x}, \quad \Delta_{k+1} = c_3\Delta_k \to 0(\text{ as } k \to +\infty).$$

By Lemma 5, when k is sufficiently large, $r_k > c_2$. This is a contradiction. Hence K_0 is infinite. Without loss of generality, we assume $K \subseteq K_0$. From (11) and (7), we have

$$\begin{aligned} & \frac{c_0c_2}{2\Delta_0}\sum_{k\in K_0}[-m(x_k,\Delta_0,0)]\min\{\Delta_k, -m(x_k,\Delta_0,0)/(M_k\Delta_0)\} \\ \le\ & \sum_{k\in K_0}[f(x_k)-f(x_{k+1})] = \sum_{k=0}^{\infty}[f(x_k)-f(x_{k+1})] \\ \le\ & f(x_0)-f(\bar{x}). \end{aligned}$$

So

$$\sum_{k\in K_0}[-m(x_k,\Delta_0,0)]\min\{\Delta_k, -m(x_k,\Delta_0,0)/(M_k\Delta_0)\} < +\infty. \tag{12}$$

Let $m(\bar{x},\bar{\Delta},0) = -2\beta < 0$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0]$, as $\|x_k - \bar{x}\| < \varepsilon$,

$$m(x_k,\Delta_0,0) \le m(x_k,\bar{\Delta},0) < -\beta < 0,$$

thus $\sum_{k\in K_0}\min\{\Delta_k, \beta/(M\Delta_0)\} < +\infty$, that is,

$$\sum_{k\in K_0}\Delta_k < +\infty. \tag{13}$$

Therefore, when $k \in K_0$ and $k \to \infty, \Delta_k \to 0$. From (9), we have

$$\lim_{k\to\infty}\Delta_k = 0. \tag{14}$$

By (13) and $x_k \to \bar{x}$ as $k \in K$ and $k \to \infty$, we deduce that for any $\varepsilon \in (0,\varepsilon_0]$, there exists a positive integer $N = N(\varepsilon)$ such that as $k \in K$ and $k \ge N$,

$$\|x_k - \bar{x}\| < \frac{\varepsilon}{2} \text{ and } \sum_{\substack{k\,\in\,K_0\\ k\,\ge\,N}}\Delta_k < \frac{\varepsilon}{2}.$$

Now, we prove by induction that

$$\lim_{k\to+\infty} x_k = \bar{x}. \tag{15}$$

Choose $k_0 \in K \subseteq K_0$ such that $k_0 \ge N$, then

$$\|x_{k_0} - \bar{x}\| < \frac{\varepsilon}{2} < \varepsilon,$$

$$\|x_{k_0+1}-\bar{x}\| < \|x_{k_0+1}-x_{k_0}\| + \frac{\varepsilon}{2} \le \Delta_{k_0} + \frac{\varepsilon}{2} < \varepsilon.$$

Suppose that for some $m > 0$, $\|x_{k_0+m}-\bar{x}\| < \varepsilon$. Then, if $k_0+m \in K_0$, we have

$$\begin{aligned} \|x_{k_0+m+1}-\bar{x}\| &\le \|x_{k_0}-\bar{x}\| + \sum_{i=0}^{m} \|x_{k_0+i+1}-x_{k_0+i}\| \\ &< \frac{\varepsilon}{2} + \sum_{\substack{k \in K_0 \\ k \ge N}} \Delta_k < \varepsilon; \end{aligned}$$

if $k_0+m \notin K_0$, we also have

$$\|x_{k_0+m+1}-\bar{x}\| = \|x_{k_0+m}-\bar{x}\| < \varepsilon.$$

Hence, (15) holds.

(15), together with (14) and Lemma 5, implies that when k is sufficiently large, $r_k > c_2$. So, $\Delta_{k+1} \ge \Delta_k$, which contradicts (14). This completes the proof of the theorem.

For (2) and (3), Y. Yuan [2] has shown that if (2) is valid, then $\{\|B_k\|\}$ is uniformly bounded. Hence, it suffices to discuss the case when (3) holds. We have the following theorem.

Theorem 2. *Under Assumptions A1–A4, if (3) holds, then*

$$\limsup_{k\to\infty} m(x_k, \bar{\Delta}, 0) = 0,$$

that is, at least one accumulation point of the sequence $\{x_k\}$ is a critical point of f.

For the proof of the above theorem, we quote Lemma 3.4 of Y. Yuan [2].

Lemma 6. *Let $\{\Delta_k\}$ and $\{M_k\}$ be two sequences such that $\Delta_k \ge c_{10}/M_k$ for all k, where c_{10} is a positive constant. Let l be a subset of $\mathcal{N}$. Assume*

$$\begin{aligned} &\Delta_{k+1} \le c_4\Delta_k, \quad k \in l, \\ &\Delta_{k+1} \le c_3\Delta_k, \quad k \notin l, \\ &M_{k+1} \ge M_k \textit{ for all } k, \textit{ and } \sum_{k\in l} \min\{\Delta_k, 1/M_k\} < \infty, \end{aligned}$$

where $c_4 > 1, c_3 < 1$ are positive constants. Then the sum

$$\sum_{k=1}^{\infty} 1/M_k < \infty.$$

Lemma 7. *Under Assumptions A1–A3, if $\limsup_{k\to\infty} m(x_k, \bar{\Delta}, 0) \ne 0$, then $\Delta_k \ge c_{10}/M_k$ for all $k \ge 1$, where $M_k = \max_{i\le k}\{\|B_i\|\} + 1, c_{10}$ is a positive constant.*

Proof. Since $\limsup_{k\to\infty} m(x_k, \bar{\Delta}, 0) \ne 0$, there exist $\delta > 0$ and a positive integer k_0 such that $\forall k \ge k_0$, $m(x_k, \bar{\Delta}, 0) < -\delta$. Thus,

$$m(x_k, \Delta_0, 0) \le m(x_k, \bar{\Delta}, 0) < -\delta, \quad \forall k \ge k_0. \tag{16}$$

From (11), we have

$$\begin{aligned} \Phi_k(s_k) &\le -\frac{c_0\delta}{2\Delta_0} \min\{\Delta_k, \delta/(M_k\Delta_0)\} \\ &\le -c_9 \min\{\Delta_k, 1/M_k\}, \quad \forall k \ge k_0 \end{aligned} \tag{17}$$

where c_9 is a positive constant.

By Assumption A3, we may deduce that there exists $\eta > 0$ such that when $\|s_k\| \le \eta$,

$$f(x_k+s_k)-f(x_k)\le\varphi(x_k,s_k)+\frac{1}{2}c_9(1-c_2)\|s_k\|.$$

Let $c_{10}=\min\{\Delta_1 M_1,\Delta_2 M_2,\cdots,\Delta_{k_0}M_{k_0},c_3\eta M_1,c_3,c_3c_9(1-c_2)\}$. Then, $\forall k\le k_0$, $\Delta_k\ge c_{10}/M_k$ holds.

Suppose that $\Delta_k\ge c_{10}/M_k$ holds for some $k\ge k_0$, by induction, it suffices to prove that it also holds for $k+1$.

If $\|s_k\|\ge\eta$, then $\Delta_{k+1}\ge c_3\Delta_k\ge c_3\eta\ge c_{10}/M_1\ge c_{10}/M_{k+1}$.

If $\|s_k\|<\eta$, then when $r_k>c_2$, $\Delta_{k+1}\ge\Delta_k\ge c_{10}/M_k\ge c_{10}/M_{k+1}$. On the other hand, when $r_k\le c_2$,

$$c_2\Phi_k(s_k)\le f(x_k+s_k)-f(x_k)\le\varphi(x_k,s_k)+\frac{1}{2}c_9(1-c_2)\|s_k\|. \tag{18}$$

Thus, we have

$$(1-c_2)\Phi_k(s_k)\ge-\frac{1}{2}c_9(1-c_2)\|s_k\|+\frac{1}{2}s_k^TB_ks_k,$$

that is,

$$\begin{aligned}-\tfrac{1}{2}s_k^TB_ks_k+\tfrac{1}{2}c_9(1-c_2)\|s_k\| &\ge(1-c_2)(-\Phi_k(s_k))\\ &\ge c_9(1-c_2)\min\{\Delta_k,1/M_k\}.\end{aligned}$$

From the above inequality, we deduce that

$$\|B_k\|\|s_k\|^2+c_9(1-c_2)\|s_k\|\ge c_9(1-c_2)\min\{2\|s_k\|,2/M_k\},$$

So,

$$M_k\|s_k\|^2\ge c_9(1-c_2)\min\{\|s_k\|,2/M_k-\|s_k\|\}.$$

If $\|s_k\|\ge 2/M_k-\|s_k\|$, then $\|s_k\|\ge 1/M_k$; otherwise, $M_k\|s_k\|^2\ge c_9(1-c_2)\|s_k\|$. Hence, $\|s_k\|\ge\min\{1,c_9(1-c_2)\}/M_k$. Consequently, $\Delta_{k+1}\ge c_3\|s_k\|\ge c_{10}/M_k\ge c_{10}/M_{k+1}$. This establishes the lemma.

Proof of Theorem 2. Let $M_k=\max\limits_{i\le k}\{\|B_i\|\}+1$, choose $l=\{k:r_k>c_2\}$. If $\limsup\limits_{k\to\infty}m(x_k,\bar\Delta,0)\ne0$, then by (12) and (16), we deduce

$$\sum_{k\in l}\min\{\Delta_k,\delta/(M_k\Delta_0)\}<\infty,$$

that is,

$$\sum_{k\in l}\min\{\Delta_k,1/M_k\}<\infty.$$

From Lemma 6 and Lemma 7, we have

$$\sum_{k=1}^{\infty}\frac{1}{M_k}<\infty.$$

This contradicts (3). Hence

$$\limsup_{k\to\infty}m(x_k,\bar\Delta,0)=0.$$

By Assumption A4, it follows that at least one accumulation point of the sequence $\{x_k\}$ is a critical point of f. This completes the proof.

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