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ON ENTROPY CONDITIONS OF HIGH RESOLUTION SCHEMES FOR SCALAR CONSERVATION LAWS*1)

Ning Zhao
(*Department of Aerodynamics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China*)

Hua-mu Wu
(*LSEC, ICMSEC, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper a kind of quadratic cell entropy inequalities of second order resolution SOR-TVD schemes is obtained for scalar hyperbolic conservation laws with strictly convex (concave) fluxes, which in turn implies the convergence of the schemes to the physically relevent solution of the problem. The theoretical results obtained in this paper improve the main results of Osher and Tadmor [6].

Key words: Entropy condition, High resolution schemes, Conservation laws.

1. Introduction

Let us consider the Cauchy problems for nonlinear hyperbolic scalar conservation laws:

$$\begin{aligned} \frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x} &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1.1}$$

where the function $f(u) \in C^2(R)$ and the initial data $u_0 \in BV(R)$. As is well known, this problem in general does not admit smooth solution, so that weak solution in the sense of distributions must be considered. Moreover, an entropy condition must be added in order to ensure the uniqueness of the weak solutions.

The research of numerical methods for solving the equation (1.1) has been developed rapidly in this decade. Since appearance of the concept of TVD (total variation diminishing) schemes, various high resolution schemes (TVD, TVB (total variation bounded), ENO (essentially non-oscillatory)) have been applied successfully to computational fluid dynamics. See [3, 4, 7]. The convergence of numerical methods for hyperbolic conservation laws depends on the discrete entropy condition and total variation stability of difference schemes, which has been investigated by many authors (see [1, 2, 5, 6, 8, 9, 10]). Especially, Osher and Tadmor in [6] discussed the convergence and cell entropy inequalities of a class of second order resolution TVD (SOR-TVD) schemes. They introduced a kind of modified flux functions for estimating entropy production, and made use of the Godunov numerical flux to obtain an entropy estimate of the general TVD schemes for the new conservation laws with the modified flux. Moreover, they constructed a class of SOR-TVD schemes and analysed the cell entropy conditions of the SOR-TVD schemes with upwind building block, which results in the convergence of the numerical solutions to the unique physical relevant solution. However, there are two points of their arguments should be noticed.

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Firstly, the modified flux functions introduced in [6] are piecewise linear functions. However the smoothness of the modified flux functions is essentially used in the entropy estimatation ([6], section 6). Secondly, in [6] the authors used the first order accurate upwind scheme as the building block for constructing SOR-TVD schemes and as is well known that the upwind scheme may admit entropy voilating solutions, so, some kind of artificial viscousity terms should be added to the upwind scheme for entropy satisfaction.

In this paper, we present a more direct method of proof, which not only avoids using the nonsmooth modified flux function but also simplifies the method of proving and improves the estimates of entropy production for general TVD schemes. In order to avoid adding artificial viscousity, instead of using upwind building block, we choose the Godunov scheme as the building block and obtained the condition for the SOR-TVD properties of the schemes with this Godunov building block, which implies the convergence of the numerical solution to the unique physical relevant solution of the conservation laws (1.1). Our approach seems more natural comparing with the Osher-Tadmor's arguments.

The outline of the paper is as follows. Section 2 reviews the main elements of the general theory of TVD schemes [8,6]. Section 3 is the heart of the paper, we give the discrete entropy estimates of the general TVD schemes by using a more directly method of proving. In Section 4, based on the Godunov building block, we obtain the entropy inequality of a class of SOR-TVD schemes for strict convex (or concave) conservation laws, which results in the convergence of the SOR-TVD schemes to the unique physical relevant solution.

2. TVD Schemes

Consider the following conservative finite difference schemes of (1.1)

$$u_j^{n+1} = u_j - \lambda(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) \tag{2.1}$$

where $\lambda = \Delta t/\Delta x$ is the mesh ratio, and Δt and Δx the variable mesh size in time and space directions, respectively. $h_{j+\frac{1}{2}}$ denotes the Lipschitz continuous numerical flux

$$h_{j+\frac{1}{2}} = h(u_{j-s+1}, ..., u_{j+s}) \tag{2.2}$$

consistent with the differential one,

$$h(w, ..., w) = f(w) \quad . \tag{2.3}$$

We also assume that the scheme (2.1) can be written in an incremental form

$$u_j^{n+1} = u_j + C^+_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} - C^-_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}} \tag{2.4}$$

where $\Delta u_{j+\frac{1}{2}} = u_{j+1} - u_j$.

From (2.1) and (2.4), we have

$$h_{j+\frac{1}{2}} + \frac{1}{\lambda} C^+_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} = h_{j-\frac{1}{2}} + \frac{1}{\lambda} C^-_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}} \ . \tag{2.5}$$

Let the modified flux

$$\begin{aligned} g_j &= h_{j\pm\frac{1}{2}} + \tfrac{1}{\lambda} C^{\pm}_{j\pm\frac{1}{2}} \Delta u_{j\pm\frac{1}{2}} \\ &= \tfrac{1}{2}\left[h_{j-\frac{1}{2}} + h_{j+\frac{1}{2}} + \tfrac{1}{\lambda} C^-_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}} + \tfrac{1}{\lambda} C^+_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} \right] . \end{aligned} \tag{2.6}$$

From (2.5) and (2.6) one can get

$$C^-_{j+\frac{1}{2}} - C^+_{j+\frac{1}{2}} = \lambda \frac{\Delta g_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}} \ . \tag{2.7}$$

Thus, we can finally recast the scheme (2.1) or (2.4) into its viscous form

$$u_j^{n+1} = u_j - \frac{\lambda}{2}\left(g_{j+1} - g_{j-1}\right) + \frac{1}{2}\left[Q_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} - Q_{j-\frac{1}{2}}\Delta u_{j-\frac{1}{2}}\right] \tag{2.8a}$$

with the numerical viscousity coefficient

$$Q_{j+\frac{1}{2}} = C^+_{j+\frac{1}{2}} + C^-_{j+\frac{1}{2}} = \lambda\frac{g_{j+1} + g_j - 2h_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}}\ . \tag{2.8b}$$

As is well known[6], the scheme (2.4) is TVD (total variation diminishing), provided its numerical viscousity coefficient $Q_{j+\frac{1}{2}}$ satisfies

$$\lambda\left|\frac{\Delta g_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}}\right| \le Q_{j+\frac{1}{2}} \le 1\ . \tag{2.9}$$

Indeed, the above inequalities will lead to the following sufficient TVD conditions:

$$C^{\pm}_{j+\frac{1}{2}} \ge 0, \quad 1 - C^+_{j+\frac{1}{2}} - C^-_{j+\frac{1}{2}} \ge 0. \tag{2.10}$$

3. The Entropy Estimate of the TVD Schemes

As is well known, the weak solution of (1.1) is not unique. So, let the function $U(u) \in C^2(R)$ be any convex function, so-called the entropy function, and corresponded to the function $F(u)$ (entropy flux) satisfies $F'(u) = U'(u)f'(u)$. (U, F) is called an entropy pair. If the weak solution u of (1.1) satisfies the inequality:

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \le 0 \tag{3.1}$$

in the sense of distribution to every entropy pair (U, F), the the weak solution is the unique physical solution. The inequality (3.1) is called the entropy inequality (or the entropy condition).

Corresponding to the conservative scheme (3.1), the discrete entropy inequality is defined as

$$U(u_j^{n+1}) - U(u_j) + \lambda(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}) \le 0 \tag{3.2}$$

where the discrete entropy flux

$$F_{j+\frac{1}{2}} = H(u_{j-s+1}, ..., u_{j+s}) \tag{3.3}$$

consistent with the differential one,

$$H(w, ..., w) = F(w). \tag{3.4}$$

To begin we distinguish the cases between critical points and noncritical points by using

$$|s_j| = \begin{cases} 0 & \text{if } \Delta u_{j+\frac{1}{2}}\Delta u_{j-\frac{1}{2}} < 0 \\ 1 & \text{otherwise} \end{cases} \tag{3.5}$$

To obtain the entropy estimate of the scheme (2.1),(2.4), (2.8a) or (2.8b), as in [6], we define the intermediate values

$$\begin{aligned} u^+_{j+\frac{1}{2}} &= u_j + \frac{2\lambda}{1+|s_j|}(g_j - h_{j+\frac{1}{2}}) \\ u^-_{j+\frac{1}{2}} &= u_{j+1} - \frac{2\lambda}{1+|s_{j+1}|}(g_{j+1} - h_{j+\frac{1}{2}}). \end{aligned} \tag{3.6}$$

For these intermediate values, we have

Lemma 3.1. *Assume the TVD condition*

$$\lambda \left| \frac{\Delta g_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}} \right| \leq Q_{j+\frac{1}{2}} \leq \frac{1}{2} \min \{1+|s_j|, 1+|s_{j+1}|\} \quad . \tag{3.7}$$

holds. Then

(i)

$$u_j \leq u^+_{j+\frac{1}{2}} \leq u^-_{j+\frac{1}{2}} \leq u_{j+1}, \quad \textit{if} \ \ u_j \leq u_{j+1} \tag{3.8}$$

(ii)

$$u_j \geq u^+_{j+\frac{1}{2}} \geq u^-_{j+\frac{1}{2}} \geq u_{j+1}, \quad \textit{if} \ \ u_j \geq u_{j+1} \tag{3.9}$$

Proof. We only consider the case when $u_j \leq u_{j+1}$, the case when $u_j \geq u_{j+1}$ is similar.

It is well known that the TVD condition (3.7) means

$$\begin{array}{l} C^-_{j+\frac{1}{2}} \geq 0, \quad C^+_{j+\frac{1}{2}} \geq 0 \\ Q_{j+\frac{1}{2}} = C^+_{j+\frac{1}{2}} + C^-_{j+\frac{1}{2}} \leq \frac{1}{2} \min \{1+|s_j|, 1+|s_{j+1}|\} \end{array} \tag{3.10}$$

When $u_j \leq u_{j+1}$, by (3.6) and (3.8), we have

$$\begin{array}{rcl} u^+_{j+\frac{1}{2}} - u_j & = & \dfrac{2\lambda}{1+|s_j|}(g_j - h_{j+\frac{1}{2}}) = \dfrac{2}{1+|s_j|} C^+_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} \geq 0 \\ u_{j+1} - u^-_{j+\frac{1}{2}} & = & \dfrac{2\lambda}{1+|s_{j+1}|}(g_{j+1} - h_{j+\frac{1}{2}}) \\ & = & \dfrac{2}{1+|s_{j+1}|} C^-_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} \geq 0 \\ u^-_{j+\frac{1}{2}} - u^+_{j+\frac{1}{2}} & = & \left[1 - \dfrac{2}{1+|s_j|} C^+_{j+\frac{1}{2}} - \dfrac{2}{1+|s_{j+1}|} C^-_{j+\frac{1}{2}} \right] \Delta u_{j+\frac{1}{2}} \\ & \geq & \Delta u_{j+\frac{1}{2}} \left[1 - \max \left\{ \dfrac{2}{1+|s_j|}, \dfrac{2}{1+|s_{j+1}|} \right\} Q_{j+\frac{1}{2}} \right] \geq 0 \ . \end{array} \tag{3.11}$$

For the scheme (2.8a),(2.8b) and any entropy pair (U, F). we define the numerical entropy flux

$$F_{j+\frac{1}{2}} = F(u_{j+1}) + U'(u_{j+1})(h_{j+\frac{1}{2}} - f(u_{j+1})). \tag{3.12}$$

We have

Lemma 3.2. *For the numerical entropy flux (3.12),*

$$\begin{aligned} U(u_j^{n+1}) &= U(u_j) - \lambda(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}) + \lambda \int_{u_j}^{u_{j+1}} U''(v)(h_{j+\frac{1}{2}} - f(v))dv \\ &+ \int_{u_j}^{u_j^{n+1}} U''(v)(u_j^{n+1} - v)dv \ . \end{aligned} \tag{3.13}$$

Proof.

$$U(u_j^{n+1}) = U(u_j) - \lambda U'(u_j)(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) + \int_{u_j}^{u_j^{n+1}} U''(v)(u_j^{n+1} - v)dv \ . \tag{3.14}$$

$$F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} = F(u_{j+1}) - F(u_j) + U'(u_{j+1})(h_{j+\frac{1}{2}} - f(u_{j+1})) - U'(u_j)(h_{j-\frac{1}{2}} - f(u_j)) \ , \tag{3.15}$$

$$\begin{aligned} F(u_{j+1})-F(u_j) &= \int_{u_j}^{u_{j+1}} U'(v)f'(v)dv \\ &= U'(u_{j+1})f(u_{j+1})-U'(u_j)f(u_j)-\int_{u_j}^{u_{j+1}} U''(v)f(v)dv. \end{aligned} \tag{3.16}$$

Thus,

$$F_{j+\frac{1}{2}}-F_{j-\frac{1}{2}}=U'(u_{j+1})h_{j+\frac{1}{2}}-U'(u_j)h_{j-\frac{1}{2}}-\int_{u_j}^{u_{j+1}} U''(v)f(v)dv. \tag{3.17}$$

Using the above expression one may find from (3.10) that

$$\begin{aligned} U(u_j^{n+1}) &= U(u_j)-\lambda(F_{j+\frac{1}{2}}-F_{j-\frac{1}{2}})+\lambda h_{j+\frac{1}{2}}(U'(u_{j+1})-U'(u_j)) \\ &- \lambda\int_{u_j}^{u_{j+1}} U''(v)f(v)dv+\int_{u_j}^{u_j^{n+1}} U''(v)(u_j^{n+1}-v)dv \\ &= U(u_j)-\lambda(F_{j+\frac{1}{2}}-F_{j-\frac{1}{2}})+\lambda\int_{u_j}^{u_{j+1}} U''(v)(h_{j+\frac{1}{2}}-f(v))dv \\ &+ \int_{u_j}^{u_j^{n+1}} U''(v)(u_j^{n+1}-v)dv. \end{aligned} \tag{3.18}$$

That is the desired result (3.13).

As in [6], we modify the numerical entropy flux (3.12) in

$$\mathbf{F}_{j+\frac{1}{2}}=F_{j+\frac{1}{2}}+\widetilde{F}_{j+\frac{1}{2}} \tag{3.19a}$$

where

$$\widetilde{F}_{j+\frac{1}{2}}=\frac{1+|s_{j+1}|}{2\lambda}\left[U(u^-_{j+\frac{1}{2}})-U(u_{j+1})-U'(u_{j+1})(u^-_{j+\frac{1}{2}}-u_{j+1})\right]. \tag{3.19b}$$

It is obvious that the numerical entropy flux defined in (3.19) is consistent with the differential one, and can be found easily that it is identical with that defined in [6] (6.22b).

Next theorem is in the heart of the matter.

Theorem 3.3. *Consider the scheme (2.8) satisfying the TVD condition*

$$\lambda\left|\frac{\Delta g_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}}\right|\le Q_{j+\frac{1}{2}}\le\frac{1}{2}\min\{1+|s_j|,1+|s_{j+1}|\}\ . \tag{3.20}$$

Then for all entropy pairs (U,F) we have

$$\begin{aligned} U(u_j^{n+1}) &\le U(u_j)-\lambda(\mathbf{F}_{j+\frac{1}{2}}-\mathbf{F}_{j-\frac{1}{2}})+\lambda\int_{u_j}^{u_{j+1}} U''(v)(h_{j+\frac{1}{2}}-f(v))dv \\ &+\frac{1+|s_{j+1}|}{2}\int_{u^-_{j+\frac{1}{2}}}^{u_{j+1}} U''(v)(v-u^-_{j+\frac{1}{2}})dv+\frac{1+|s_j|}{2}\int_{u_j}^{u^+_{j+\frac{1}{2}}} U''(v)(u^+_{j+\frac{1}{2}}-v)dv\ . \end{aligned} \tag{3.21}$$

Proof. Comparing (3.21) with (3.13), we can find what we need to prove is that

$$\begin{aligned} &\lambda(\widetilde{F}_{j+\frac{1}{2}}-\widetilde{F}_{j-\frac{1}{2}})+\int_{u_j}^{u_j^{n+1}} U''(v)(u_j^{n+1}-v)dv \\ \le\ &\frac{1+|s_{j+1}|}{2}\int_{u^-_{j+\frac{1}{2}}}^{u_{j+1}} U''(v)(v-u^-_{j+\frac{1}{2}})dv+\frac{1+|s_j|}{2}\int_{u_j}^{u^+_{j+\frac{1}{2}}} U''(v)(u^+_{j+\frac{1}{2}}-v)dv\ . \end{aligned} \tag{3.22}$$

By (3.19b),

$$\begin{aligned}\lambda\widetilde{F}_{j+\frac{1}{2}} &= \frac{1+|s_{j+1}|}{2}\Big[U(u^-_{j+\frac{1}{2}})-U(u_{j+1})-U'(u_{j+1})(u^-_{j+\frac{1}{2}}-u_{j+1})\Big]\\ &= \frac{1+|s_{j+1}|}{2}\int_{u^-_{j+\frac{1}{2}}}^{u_{j+1}} U''(v)(v-u^-_{j+\frac{1}{2}})dv\ .\end{aligned} \tag{3.23}$$

Hence, the desired inequality (3.22) becomes

$$\begin{aligned}\text{LHS} &= -\frac{1+|s_j|}{2}\Big[U(u^-_{j-\frac{1}{2}})-U(u_j)-U'(u_j)(u^-_{j-\frac{1}{2}}-u_j)\Big]\\ &\quad+\int_{u_j}^{u_j^{n+1}} U''(v)(u_j^{n+1}-v)dv\ .\\ &\le \frac{1+|s_j|}{2}\int_{u_j}^{u^+_{j+\frac{1}{2}}} U''(v)(u^+_{j+\frac{1}{2}}-v)dv\ .\end{aligned} \tag{3.24}$$

By (3.6),

$$\begin{aligned}u_j^{n+1} &= u_j-\lambda(h_{j+\frac{1}{2}}-h_{j-\frac{1}{2}}) = u_j-\lambda(h_{j+\frac{1}{2}}-g_j)-\lambda(g_j-h_{j-\frac{1}{2}})\\ &= u_j+\frac{1+|s_j|}{2}(u^+_{j+\frac{1}{2}}-u_j)+\frac{1+|s_j|}{2}(u^-_{j-\frac{1}{2}}-u_j)\ .\end{aligned} \tag{3.25}$$

So,

$$\int_{u_j}^{u_j^{n+1}} U''(v)(u_j^{n+1}-v)dv = -U'(u_j)(u_j^{n+1}-u_j)+U(u_j^{n+1})-U(u_j)$$

$$= U(u_j+\frac{1+|s_j|}{2}(u^+_{j+\frac{1}{2}}+u^-_{j-\frac{1}{2}}-2u_j))-U(u_j)-\frac{1+|s_j|}{2}U'(u_j)(u^+_{j+\frac{1}{2}}+u^-_{j-\frac{1}{2}}-2u_j) \tag{3.26}$$

Thus,

$$\begin{aligned}\text{LHS} &= U(u_j+\frac{1+|s_j|}{2}(u^+_{j+\frac{1}{2}}+u^-_{j-\frac{1}{2}}-2u_j))-U(u_j)\\ &\quad-\frac{1+|s_j|}{2}U'(u_j)(u^+_{j+\frac{1}{2}}-u_j)-\frac{1+|s_j|}{2}[U(u^-_{j-\frac{1}{2}})-U(u_j)].\end{aligned} \tag{3.27}$$

(i) $|s_j|=0$,

$$\begin{aligned}\text{LHS} = U(\frac{1}{2}(u^+_{j+\frac{1}{2}}+u^-_{j-\frac{1}{2}}))-U(u_j)-\frac{1}{2}(U(u^-_{j-\frac{1}{2}})-U(u_j))-\frac{1}{2}U'(u_j)(u^+_{j+\frac{1}{2}}-u_j)\\ \le \tfrac{1}{2}U(u^+_{j+\frac{1}{2}})-\tfrac{1}{2}U(u_j)-\tfrac{1}{2}U'(u_j)(u^+_{j+\frac{1}{2}}-u_j)\\ = \tfrac{1}{2}\int_{u_j}^{u^+_{j+\frac{1}{2}}} U''(v)(u^+_{j+\frac{1}{2}}-v)dv\ .\end{aligned} \tag{3.28}$$

(ii) $|s_j|=1$,

$$\begin{aligned}&\text{LHS} = U(u^-_{j-\frac{1}{2}}+u^+_{j+\frac{1}{2}}-u_j)-U(u^-_{j-\frac{1}{2}})-U'(u_j)(u^+_{j+\frac{1}{2}}-u_j)\\ &= \int_{u_j}^{u^+_{j+\frac{1}{2}}} U''(v)(u^+_{j+\frac{1}{2}}-v)dv-U(u^-_{j-\frac{1}{2}})-U(u^+_{j+\frac{1}{2}})+U(u^-_{j-\frac{1}{2}}+u^+_{j+\frac{1}{2}}-u_j)+U(u_j)\\ &\le \int_{u_j}^{u^+_{j+\frac{1}{2}}} U''(v)(u^+_{j+\frac{1}{2}}-v)dv.\end{aligned} \tag{3.29}$$

Indeed, by $|s_j| = 1$ and the convexity of $U(v)$,

$$-U(u^-_{j-\frac{1}{2}}) - U(u^+_{j+\frac{1}{2}}) + U(u^-_{j-\frac{1}{2}} + u^+_{j+\frac{1}{2}} - u_j) + U(u_j) \leq 0 \ . \tag{3.30}$$

Combining these two cases when $|s_j| = 1$ and $|s_j| = 0$, we have

$$\text{LHS} \leq \frac{1+|s_j|}{2} \int_{u_j}^{u^+_{j+\frac{1}{2}}} U''(v)(u^+_{j+\frac{1}{2}} - v)dv \quad , \tag{3.31}$$

and the desired result (3.22) follows.

Remark 3.4. The first integrate of the right hand side of (3.13) refers to the E-condition [5]. By the convexity of the entropy function U, the second and third integrates of RHS of (3.13) are all nonnegative. So, in general, the E-type numerical flux is necessary for the scheme (2.7) to satisfy the entropy condition (3.2) for all entropy pairs (U, F).

As is well known, E-schemes are at most first order accurate. Next, we shall consider the special entropy pair $U(v) = v^2/2$, $F(v) = \int^v wf'(w)dw$. In the genuinely nonlinear case where f is, say, strictly convex (or concave), the special entropy condition is enough for the convergence to the unique solution. For this special quadratic entropy pair we have

Corollary 3.5. *Consider the scheme (2.7) satisfying the TVD condition (3.6)*

$$\lambda \left| \frac{\Delta g_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}} \right| \leq Q_{j+\frac{1}{2}} \leq \frac{1}{2} \min\{1+|s_j|, 1+|s_{j+1}|\} \ . \tag{3.32}$$

Then the cell entropy inequality

$$\frac{1}{2}(u_j^{n+1})^2 - \frac{1}{2}(u_j)^2 + \lambda(\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}}) \leq 0 \tag{3.33}$$

holds provided

$$h_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} - \int_{u_j}^{u_{j+1}} f(v)dv + \frac{1}{2\lambda}\left[\frac{2}{1+|s_{j+1}|}(C^-_{j+\frac{1}{2}})^2 + \frac{2}{1+|s_j|}(C^+_{j+\frac{1}{2}})^2\right](\Delta u_{j+\frac{1}{2}})^2 \leq 0 \ , \tag{3.34}$$

where the numerical flux $\mathbf{F}_{j+\frac{1}{2}}$ is defined in (3.11).

Proof. For our special quadratic entropy, (3.13) reads

$$\begin{aligned} &\tfrac{1}{2}(u_j^{n+1})^2 - \tfrac{1}{2}(u_j)^2 + \lambda(\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}}) \\ &\leq \lambda \int_{u_j}^{u_{j+1}} (h_{j+\frac{1}{2}} - f(v))dv + \tfrac{1+|s_{j+1}|}{4}(u_{j+1} - u^-_{j+\frac{1}{2}})^2 + \tfrac{1+|s_j|}{4}(u^+_{j+\frac{1}{2}} - u_j)^2 \ . \end{aligned} \tag{3.35}$$

Thus (3.33) holds if

$$h_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} - \int_{u_j}^{u_{j+1}} f(v)dv + \frac{1+|s_{j+1}|}{4\lambda}(u_{j+1} - u^-_{j+\frac{1}{2}})^2 + \frac{1+|s_j|}{4\lambda}(u^+_{j+\frac{1}{2}} - u_j)^2 \leq 0 \tag{3.36}$$

But,

$$\begin{aligned} u_{j+1} - u^-_{j+\frac{1}{2}} &= \frac{2}{1+|s_{j+1}|} C^-_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}}, \\ u^+_{j+\frac{1}{2}} - u_j &= \frac{2}{1+|s_j|} C^+_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} \ . \end{aligned} \tag{3.37}$$

Therefore, the left hand side of (3.36)

$$\begin{aligned} \text{LHS} &= h_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} - \int_{u_j}^{u_{j+1}} f(v)dv \\ &+ \tfrac{1}{2\lambda}\left[\frac{2}{1+|s_{j+1}|}(C^-_{j+\frac{1}{2}})^2 + \frac{2}{1+|s_j|}(C^+_{j+\frac{1}{2}})^2\right](\Delta u_{j+\frac{1}{2}})^2 \end{aligned} \tag{3.38}$$

and the inequality (3.34) results.

Remark 3.6. The result of Corollary 3.5 is identical with (8.5a) in [6].

From the proof procedure of Theorem 3.3, we can also obtain the following more precise entropy estimate of the scheme (2.7) in the case of quadratic entropy.

Theorem 3.7. *Consider the scheme (2.7) satisfying the TVD condition*

$$\lambda\left|\frac{\Delta g_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}}\right| \le Q_{j+\frac{1}{2}} \le \frac{1}{2}\min\{1+|s_j|, 1+|s_{j+1}|\} \ . \tag{3.39}$$

Then the cell entropy inequality (3.33) holds if and only if

$$\begin{aligned} &h_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} - \int_{u_j}^{u_{j+1}} f(v)dv + \frac{1}{2\lambda}\left[\frac{2}{1+|s_{j+1}|}(C^-_{j+\frac{1}{2}})^2 + \frac{2}{1+|s_j|}(C^+_{j+\frac{1}{2}})^2\right](\Delta u_{j+\frac{1}{2}})^2 \\ &\le \frac{1-|s_j|}{2\lambda}(C^+_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} + C^-_{j-\frac{1}{2}}\Delta u_{j-\frac{1}{2}})^2 + \frac{|s_j|}{\lambda}C^+_{j+\frac{1}{2}}C^-_{j-\frac{1}{2}}\Delta u_{j+\frac{1}{2}}\Delta u_{j-\frac{1}{2}} \end{aligned} \tag{3.40}$$

where the numerical flux $\mathbf{F}_{j+\frac{1}{2}}$ *is defined in (3.33).*

Proof. The result can be obtained directly by using that

$$\begin{aligned} &U(\tfrac{1}{2}(u^+_{j+\frac{1}{2}} + u^-_{j-\frac{1}{2}})) - \tfrac{1}{2}U(u^+_{j+\frac{1}{2}}) - \tfrac{1}{2}U(u^-_{j-\frac{1}{2}}) \\ &= -\tfrac{1}{2}\int_{u^-_{j-\frac{1}{2}}}^{(u^+_{j+\frac{1}{2}}+u^-_{j-\frac{1}{2}})/2} U''(v)(v-u^-_{j-\frac{1}{2}})dv - \tfrac{1}{2}\int_{(u^+_{j+\frac{1}{2}}+u^-_{j-\frac{1}{2}})/2}^{u^+_{j+\frac{1}{2}}} U''(v)(u^+_{j+\frac{1}{2}}-v)dv \le 0 \end{aligned} \tag{3.41}$$

when $|s_j| = 0$, and

$$-U(u^-_{j-\frac{1}{2}}) - U(u^+_{j+\frac{1}{2}}) + U(u^-_{j-\frac{1}{2}} + u^+_{j+\frac{1}{2}} - u_j) + U(u_j) \le 0 \tag{3.42}$$

when $|s_j| = 1$.

4. Entropy Estimate of the SOR Godunov Scheme

Let us give the estimate of entropy dissipation of first order Godunov scheme.

Theorem 4.1. *(Entropy dissipation of the Godunov scheme)*
Consider the Godunov scheme under the CFL condition

$$\lambda\max_u|f'(u)| \le \frac{1}{2} \ , \tag{4.1}$$

Then we have

$$U(u^{n+1}_j) - U(u_j) + \lambda(\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}}) \le -C_0|u_{j+1}-u_j|^3 \tag{4.2}$$

where C_0 *is a positive constant,* $U(v) = v^2/2$*, and the discrete entropy flux* $\mathbf{F}_{j+\frac{1}{2}}$ *is defined in (3.11).*

Proof. In fact, a more careful estimate of the inequality (4.2) can be found in **Appendix.**

To obtain the SOR version of the Godunov scheme, as in [6, section 4], we modify the numercal flux f_j into

$$g_j = f_j + \frac{1}{\lambda}\widetilde{g}_j \tag{4.3}$$

where

$$\widetilde{g}_j = \frac{s_j}{2}\ min\left[[Q^G_{j\pm\frac{1}{2}}(f) - \lambda^2(a^2)_{j\pm\frac{1}{2}}]|\Delta u_{j\pm\frac{1}{2}}|\right] \tag{4.4}$$

and $s_j = (s_{j+\frac{1}{2}} + s_{j-\frac{1}{2}})/2$, $s_{j+\frac{1}{2}} = sgn(\Delta u_{j+\frac{1}{2}})$.

As is well known (cf. [6]), if the term $\lambda^2(a^2)_{j+\frac{1}{2}}$ is chosen so that

$$\lambda^2(a^2)_{j+\frac{1}{2}} = (\lambda a_{j+\frac{1}{2}})^2 + O(|\Delta u_{j+\frac{1}{2}}|) \quad \text{and} \quad \lambda^2(a^2)_{j+\frac{1}{2}} \le Q^G_{j+\frac{1}{2}}(f) \tag{4.5}$$

the difference scheme

$$u_j^{n+1} = u_j - \frac{\lambda}{2}(g_{j+1} - g_{j-1}) + \frac{1}{2}\left[Q^{SOR}_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} - Q^{SOR}_{j-\frac{1}{2}}\Delta u_{j-\frac{1}{2}}\right] \tag{4.6}$$

is SOR-TVD under the CFL restriction

$$\lambda|a_{j+\frac{1}{2}}| \le Q^G_{j+\frac{1}{2}} \le \frac{1}{3}\ , \tag{4.7}$$

where

$$Q^{SOR}_{j+\frac{1}{2}}(f) = Q^G_{j+\frac{1}{2}}(f) + \left|\frac{\Delta\tilde{g}_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}}\right| \tag{4.8}$$

$$a_{j+\frac{1}{2}} = \begin{cases} \dfrac{\Delta f_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}} & \text{if } u_j \neq u_{j+1} \\ f'(u_j) & \text{if } u_j = u_{j+1} \end{cases} \tag{4.9}$$

here $Q^G_{j+\frac{1}{2}}(f)$ is the numerical viscosity of the Godunov scheme, and can be described as follows:

$$Q^G_{j+\frac{1}{2}} = \begin{cases} \lambda\dfrac{f_j + f_{j+1} - 2f(u^*)}{\Delta u_{j+\frac{1}{2}}} & \text{if } u_{j+1} > u_j \text{ and } f'(u_j)f'(u_{j+1}) < 0 \\ \lambda|a_{j+\frac{1}{2}}| & \text{otherwise} \end{cases} \tag{4.10}$$

where u^* is the sonic point such that $f'(u^*) = 0$.

Now, let us estimate the entropy production of the SOR-TVD scheme (4.6). From the definition of the scheme, we have

$$h_{j+\frac{1}{2}} = \frac{f_j + f_{j+1}}{2} + \frac{1}{2\lambda}(\widetilde{g}_j + \widetilde{g}_{j+1}) - \frac{1}{2\lambda}Q^{SOR}_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}}\ . \tag{4.11}$$

From the inequality (3.21), the entropy production of the SOR-TVD scheme (4.6)

$$\begin{aligned} &\tfrac{1}{2}(u_j^{n+1})^2 - \tfrac{1}{2}(u_j)^2 + \lambda(\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}}) \\ &\le \lambda\left[\Delta u_{j+\frac{1}{2}}\frac{f_j + f_{j+1}}{2} - \int_{u_j}^{u_{j+1}} f(v)dv + \frac{(\Delta u_{j+\frac{1}{2}})^2}{2\lambda}\left[\frac{\widetilde{g}_j + \widetilde{g}_{j+1}}{\Delta u_{j+\frac{1}{2}}}\right.\right. \\ &\left.\left. -Q^{SOR}_{j+\frac{1}{2}} + \frac{2}{1+|s_j|}(C^+_{j+\frac{1}{2}})^2 + \frac{2}{1+|s_{j+1}|}(C^-_{j+\frac{1}{2}})^2\right]\right] \end{aligned} \tag{4.12}$$

where the entropy flux $\mathbf{F}$ is defined as in (3.19a). In [6], Osher and Tadmor analysed the entropy inequality of the SOR scheme with the upwind building block

$$Q_{j+\frac{1}{2}}(f) = \lambda|a_{j+\frac{1}{2}}|\ . \tag{4.13}$$

For the completeness of description, we review their discussion in [6, section 8]. Let

$$ET_{j+\frac{1}{2}} = \Delta u_{j+\frac{1}{2}} \left[\frac{f_j + f_{j+1}}{2} \right] - \int_{u_j}^{u_{j+1}} f(v)dv \quad , \tag{4.14}$$

then

$$ET_{j+\frac{1}{2}} = \frac{(\Delta u_{j+\frac{1}{2}})^3}{12} f''(\xi) \; , \qquad \xi \in I_{j+\frac{1}{2}} \tag{4.15}$$

where $I_{j+\frac{1}{2}} = (min(u_j, u_{j+1}), max(u_j, u_{j+1}))$. By the convexity of the flux function $f(u)$ we have $ET_{j+\frac{1}{2}} \leq 0$ if $\Delta u_{j+\frac{1}{2}} < 0$ (shock), otherwise (rarefaction), $ET_{j+\frac{1}{2}} \geq 0$. For the other terms of the right hand side of (4.12), we have

Lemma 4.2. *The following estimate holds:*

$$\begin{aligned} &\frac{\widetilde{g}_j + \widetilde{g}_{j+1}}{\Delta u_{j+\frac{1}{2}}} - Q^{SOR}_{j+\frac{1}{2}} + \frac{2}{1+|s_j|}(C^+_{j+\frac{1}{2}})^2 + \frac{2}{1+|s_{j+1}|}(C^-_{j+\frac{1}{2}})^2 \\ &\qquad \leq \frac{|s_j| + |s_{j+1}|}{2} \left[Q^2_{j+\frac{1}{2}}(f) - \lambda^2 (a^2)_{j+\frac{1}{2}} \right] \; . \end{aligned} \tag{4.16}$$

In the case of shock $(u_j > u_{j+1})$, they take $(a^2)_{j+\frac{1}{2}} = |a_{j+\frac{1}{2}}|^2$, which is enough for making the entropy inequality hold.

For the case of rarefaction, they choose

$$\lambda^2 (a^2)_{j+\frac{1}{2}} = \lambda^2 |a_{j+\frac{1}{2}}|^2 + \frac{\lambda\delta}{3} |\Delta u_{j+\frac{1}{2}}| \tag{4.17}$$

which may eliminate the positive term $ET_{j+\frac{1}{2}}$ under the restriction

$$(a_{j+\frac{1}{2}}) \geq \frac{\delta}{2} |\Delta u_{j+\frac{1}{2}}| \tag{4.18}$$

and by using the modified flux

$$\widetilde{g}_j = \frac{s_j}{2} \left[\, min \; [Q_{j\pm\frac{1}{2}} - \lambda^2 (a^2)_{j\pm\frac{1}{2}}]^+ |\Delta u_{j\pm\frac{1}{2}}| \right] \tag{4.19}$$

where f^+ means the positive part of the function f,i.e.,$f^+ = f$, if$f > 0$,otherwise,$f = 0$.

Obviously, it will be successful to do so except the case of $|s_j| = |s_{j+1}| = 0$. In the case of $|s_j| = |s_{j+1}| = 0$, the SOR-TVD scheme becomes the three point upwind scheme. As is well known, if no artificial viscousity is added to the upwind building block, the upwind scheme will not satisfy the entropy inequality. It can also be seen from (4.15) and (4.16) even if $Q_{j+\frac{1}{2}}(f)$ is modified.

In order to overcome the above difficulty we adopt the $Q^G_{j+\frac{1}{2}}(f)$ of Godunov scheme as the building block instead of the upwind one. Now, in the critical case when $|s_j| = |s_{j+1}| = 0$, the SOR-TVD scheme changes into the first order Godunov scheme. From Theorem 4.1, we have that the entropy inequality holds. As is well known, the viscosity of the Godunov scheme is the same as the upwind one except the case of $u_j < u_{j+1}$ and $\sigma^L_{j+\frac{1}{2}} < 0 < \sigma^R_{j+\frac{1}{2}}$, that is $s^R_{j+\frac{1}{2}} - s^L_{j+\frac{1}{2}} = 2$, where $\sigma^L_{j+\frac{1}{2}}$, $\sigma^R_{j+\frac{1}{2}}$, $s^L_{j+\frac{1}{2}}$ and $s^R_{j+\frac{1}{2}}$ are defined as follows:

$$\sigma^L_{j+\frac{1}{2}} = f'(u_j) \; , \quad \sigma_{j+\frac{1}{2}} = \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j} \; , \quad \sigma^R_{j+\frac{1}{2}} = f'(u_{j+1}) \; , \tag{4.20}$$

and

$$s^\alpha_{j+\frac{1}{2}} = \begin{cases} 1, & \text{if } \sigma^\alpha_{j+\frac{1}{2}} > 0 \\ 0, & \text{if } \sigma^\alpha_{j+\frac{1}{2}} \leq 0 \end{cases} \; , \qquad (\alpha = L \; or \; R). \tag{4.21}$$

At that time

$$Q^G_{j+\frac{1}{2}}(f) = \lambda \frac{f_j + f_{j+1} - 2f(u^*)}{\Delta u_{j+\frac{1}{2}}} \tag{4.22}$$

where u^* is the sonic point ($f'(u^*) = 0$).

Because

$$Q^G_{j+\frac{1}{2}}(f) = \lambda \frac{f_j + f_{j+1} - 2f(u^*)}{\Delta u_{j+\frac{1}{2}}} \geq \lambda |a_{j+\frac{1}{2}}| \quad , \tag{4.23}$$

the proof of Lemma 4.2 still keeps valid. Moreover, by $\sigma^L_{j+\frac{1}{2}} < 0 < \sigma^R_{j+\frac{1}{2}}$, $|s_j| + |s_{j+1}| \leq 1$. If $|s_j| + |s_{j+1}| = 0$, the SOR scheme changes into the Godunov scheme, otherwise, the entropy inequality holds provided

$$\left[\left(Q^G_{j+\frac{1}{2}}(f) \right)^2 - \lambda^2 (a^2)_{j+\frac{1}{2}} \right] + \frac{\Delta u_{j+\frac{1}{2}}}{3} \delta \leq 0 \ . \tag{4.24}$$

Taking

$$\lambda^2 (a^2)_{j+\frac{1}{2}} = \lambda^2 |a_{j+\frac{1}{2}}|^2 + \frac{\lambda \delta}{2} \Delta u_{j+\frac{1}{2}} \tag{4.25}$$

and noting

$$\begin{aligned} &Q^G_{j+\frac{1}{2}}(f) = \lambda \frac{f_j + f_{j+1} - 2f(u^*)}{\Delta u_{j+\frac{1}{2}}} \leq \frac{1}{3} \\ &\lambda \frac{f_j + f_{j+1} - 2f(u^*)}{\Delta u_{j+\frac{1}{2}}} \leq \frac{\lambda \delta}{2} \Delta u_{j+\frac{1}{2}}, \end{aligned} \tag{4.26}$$

we can find that the inequality (4.24) holds.

Unifying the critical and noncritical cases, we arrive at

$$\lambda^2 (a^2)_{j+\frac{1}{2}} = \lambda^2 |a_{j+\frac{1}{2}}|^2 + \frac{\lambda \delta}{6} \left(2 + \left(s^R_{j+\frac{1}{2}} - s^L_{j+\frac{1}{2}} \right) / 2 \right) \left(\Delta u_{j+\frac{1}{2}} \right)^+ \tag{4.27}$$

So, $\tilde{g}_j$ and $\tilde{g}_{j+1}$ do not vanish provided that

$$\lambda |a_{j+\frac{1}{2}}| - \lambda^2 |a_{j+\frac{1}{2}}|^2 - \frac{\lambda \delta}{6} \left(2 + \left(s^R_{j+\frac{1}{2}} - s^L_{j+\frac{1}{2}} \right) / 2 \right) \left(\Delta u_{j+\frac{1}{2}} \right)^+ \geq Q^G_{j+\frac{1}{2}} - \lambda^2 (a^2)_{j+\frac{1}{2}} \tag{4.28}$$

$$\lambda |a_{j+\frac{1}{2}}| - \lambda^2 |a_{j+\frac{1}{2}}|^2 - \frac{\lambda \delta}{6} \left(2 + \left(s^R_{j+\frac{1}{2}} - s^L_{j+\frac{1}{2}} \right) / 2 \right) \left(\Delta u_{j+\frac{1}{2}} \right)^+ \geq 0 \tag{4.29}$$

which requires

$$|a_{j+\frac{1}{2}}| \geq \frac{\delta}{4} \left(2 + \left(s^R_{j+\frac{1}{2}} - s^L_{j+\frac{1}{2}} \right) / 2 \right) \left(\Delta u_{j+\frac{1}{2}} \right)^+ \ . \tag{4.30}$$

Thus, we have

Theorem 4.3. *Assume $f(u)$ is strictly convex, $Q^G_{j+\frac{1}{2}}$ is the Godunov viscosity and CFL condition $Q^G_{j+\frac{1}{2}} \leq 1/3$ holds. The SOR scheme (4.6) satisfies*

(i) *total variation diminishing;*

(ii) *second order resolution, where*

$$|a_{j+\frac{1}{2}}| \geq \frac{\delta}{4} \left(2 + \left(s^R_{j+\frac{1}{2}} - s^L_{j+\frac{1}{2}} \right) / 2 \right) \left(\Delta u_{j+\frac{1}{2}} \right)^+ \ ; \tag{4.31}$$

(iii) *a consistent quadratic cell entropy inequality;*
and as a consequence of (i) and (iii),

(iv) *convergence to the unique physically relevant solution.*

5. Conclusion

The establishment of entropy conditions of high rsolution schemes for hyperbolic conservation laws is one of the subjects of intensive research in the past decades [2,5,6,8,9,10,.etc]. Especially Osher and Tadmor[6] investigated a kind of cell entropy conditions of the SOR-TVD schemes of second order accuracy. However the modified flux function they considered is not smooth enough which is essential for their entropy estimation. Besides, their building block for the SOR-TVD schemes is the first order accurate upwind scheme which may produce nonphysical entropy voilated solutions ,hence some kind of artificial viscosity terms have to be added to the upwind scheme to avoid the entropy voilation.

In this paper a more direct method of proof is given, which simplifies the method of proving and improves the estimates of entropy production for general TVD schemes. As the building block for the SOR-TVD scheme we choose the Godunov scheme. Our approach is more natural comparing with the one in [6]. The discussion present here not only simplifies the method of proving in the paper [6] by Osher and Tadmor, but also improves their theoretical results.

6. Appendix: The Sharp Entropy Estimates of the Godunov Scheme

For the conservation law (1.1), we assume that the given flux function f to be C^2 class and uniformly convex. The convex modulus given by

$$\delta = \inf_u f''(u) \ , \tag{A.1}$$

the infinimum being taken on all u under consideration. The sonic point u^* is defined by

$$f'(u^*) = 0 \ . \tag{A.2}$$

In this section the entropy pair (U, F) of (1.1) is defined as

$$U(u) = \frac{u^2}{2}, \qquad F(u) = \int_0^u vf'(v)dv \ . \tag{A.3}$$

Assume that the CFL stability restriction on λ

$$\lambda \sup_u |f'(u)| \le \frac{1}{2} \tag{A.4}$$

is satisfied. The discrete entropy flux, $\mathbf{F}_{j+\frac{1}{2}}$ defined as in (3.11a). Let $w = \mathcal{W}(0; u_j, u_{j+1})$ denote the Riemann solution at $t = \Delta t$ and $x = 0$ of the problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$$u(0,x) = \begin{cases} u_j & \text{if } x < 0 \\ u_{j+1} & \text{if } x > 0. \end{cases} \tag{A.5}$$

For the Godunov scheme, we have

$$C^+_{j+\frac{1}{2}} = \lambda \frac{f(u_j) - f(w)}{u_{j+1} - u_j}, \qquad C^-_{j+\frac{1}{2}} = \lambda \frac{f(u_{j+1}) - f(w)}{u_{j+1} - u_j} \tag{A.6}$$

Then

$$\begin{aligned} \text{RHS}/\lambda \ &= f(w)(u_{j+1} - u_j) - \textstyle\int_{u_j}^{u_{j+1}} f(v)dv \\ &+\lambda \left[\frac{1}{1+|s_{j+1}|}(f(u_{j+1}) - f(w))^2 + \frac{1}{1+|s_j|}(f(u_j) - f(w))^2 \right] \ . \end{aligned} \tag{A.7}$$

Next, we shall distinguish between two cases, depending if $\mathcal{W}(0; u_j, u_{j+1})$ is a shock wave or a rarefaction wave.

Case 1. Shock Wave $(u_j > u_{j+1})$.

(a)

$$\sigma_{j+\frac{1}{2}} = \frac{f(u_{j+1} - f(u_j)}{u_{j+1} - u_j} > 0 \qquad \text{then} \qquad w = u_j \ . \tag{A.8}$$

(b)

$$\sigma_{j+\frac{1}{2}} = \frac{f(u_{j+1} - f(u_j)}{u_{j+1} - u_j} < 0 \qquad \text{then} \qquad w = u_{j+1} \ . \tag{A.9}$$

Through a rather complicated algebraic manipulation, we can get

Theorem A.1. *(Entropy dissipation of a shock wave in the Godunov scheme)*

Consider the Godunov scheme under the CFL condition (A.4). Suppose that, for some $n \in N$ *and* $j \in Z$*, the Riemann solution* $\mathcal{W}(.; u_j, u_{j+1})$ *consists of a shock wave (i.e.* $u_j > u_{j+1}$*) whose the speed is denoted by* $\sigma_{j+\frac{1}{2}}$*. Then we have*

$$\begin{aligned} &U(u_j^{n+1}) - U(u_j) + \lambda(\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}}) \\ &\leq -\frac{\lambda\delta}{12}|u_{j+1} - u_j|^3 - \frac{\lambda|\sigma_{j+\frac{1}{2}}|}{2}\left[1 - \frac{2}{1+|s^*|}\lambda|\sigma_{j+\frac{1}{2}}|\right]|u_{j+1} - u_j|^2 \leq 0 \ , \end{aligned} \tag{A.10}$$

where

$$\sigma_{j+\frac{1}{2}} = \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j}, \qquad |s^*| = \begin{cases} |s_{j+1}|, & \textit{if } \sigma_{j+\frac{1}{2}} > 0 \\ |s_j|, & \textit{if } \sigma_{j+\frac{1}{2}} < 0 \ . \end{cases} \tag{A.11}$$

Case 2. Rarefaction Wave $(u_{j+1} \geq u_j)$.

Denotes

$$\sigma^L_{j+\frac{1}{2}} = f'(u_j), \qquad \sigma^R_{j+\frac{1}{2}} = f'(u_{j+1}) \ . \tag{A.12}$$

Then,

$$\sigma^L_{j+\frac{1}{2}} \leq \sigma_{j+\frac{1}{2}} \leq \sigma^R_{j+\frac{1}{2}} \ . \tag{A.13}$$

(a) $\sigma^L_{j+\frac{1}{2}} > 0$, then $w = u_j$ and $f'(v) > 0$ for all $v \in (u_j, u_{j+1})$.

(b) $\sigma^R_{j+\frac{1}{2}} < 0$, then $w = u_{j+1}$ and $f'(v) < 0$ for all $v \in (u_j, u_{j+1})$.

(c) $\sigma^L_{j+\frac{1}{2}} < 0 < \sigma^R_{j+\frac{1}{2}}$, then $w = u^*$ where $f'(u^*) = 0$.

After a complicated algebraic calculation, we can get

Theorem A.2. *(Entropy dissipation of a rarefaction wave in the Godunov scheme) Consider the Godunov scheme under the CFL condition (A.4). Suppose that, for some* $n \in N$ *and* $j \in Z$*, the Riemann solution* $\mathcal{W}(.; u_j, u_{j+1})$ *consists of a rarefaction wave (i.e.* $u_j < u_{j+1}$*) .*

Then we have

$$
\begin{aligned}
&U(u_j^{n+1}) - U(u_j) + \lambda(\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}}) \\
&\le -\frac{\lambda}{2}\Bigg[s^L_{j+\frac{1}{2}}s^R_{j+\frac{1}{2}}\left(\frac{2\lambda}{(}\sigma_{j+\frac{1}{2}})^2 1 + |s_{j+1}|(1 - \frac{2\lambda\sigma^R_{j+\frac{1}{2}}}{1+|s_{j+1}|}) + \sigma^L_{j+\frac{1}{2}}(1 - \frac{2\lambda\sigma_{j+\frac{1}{2}}}{1+|s_{j+1}|})^2\right) \\
&+(1-s^L_{j+\frac{1}{2}})(1-s^R_{j+\frac{1}{2}})\left(\frac{2\lambda(\sigma_{j+\frac{1}{2}})^2}{1+|s_j|}(1+\frac{2\lambda\sigma^L_{j+\frac{1}{2}}}{1+|s_j|}) - \sigma^R_{j+\frac{1}{2}}(1+\frac{2\lambda\sigma_{j+\frac{1}{2}}}{1+|s_j|})^2\right) \\
&+(1-\sigma^L_{j+\frac{1}{2}})\sigma^R_{j+\frac{1}{2}}2\lambda(\sigma_{j+\frac{1}{2}})^2\Bigg(\frac{s_{j+\frac{1}{2}}}{1+|s_{j+1}|}(1-\frac{2\lambda}{1+|s_{j+1}|}\sigma^R_{j+\frac{1}{2}}) \\
&+\ \frac{1-s_{j+\frac{1}{2}}}{1+|s_j|}(1+\frac{2\lambda}{1+|s_j|}\sigma^L_{j+\frac{1}{2}})\Bigg)\Bigg](u_{j+1}-u_j)^2 \\
&-\frac{\lambda\delta}{6}\Bigg[s^L_{j+\frac{1}{2}}s^R_{j+\frac{1}{2}}(1-\frac{2\lambda}{1+|s_{j+1}|}\sigma_{j+\frac{1}{2}})^3 \\
&+(1-s^L_{j+\frac{1}{2}})(1-s^R_{j+\frac{1}{2}})(1+\frac{2\lambda}{1+|s_j|}\sigma_{j+\frac{1}{2}})^3\Bigg]|u_{j+1}-u_j|^3 \\
&-\frac{\lambda\delta}{24}(1-s^L_{j+\frac{1}{2}})s^R_{j+\frac{1}{2}}\Bigg[\left(1-\frac{2\lambda\sigma^R_{j+\frac{1}{2}}}{1+|s_{j+1}|}\right)^3|u_{j+1}-u^*|^3 \\
&+\left(1+\frac{2\lambda\sigma^L_{j+\frac{1}{2}}}{1+|s_j|}\right)|u^*-u_j|^3\Bigg]\ ,
\end{aligned}
\tag{A.14}
$$

where

$$
\sigma^L_{j+\frac{1}{2}} = f'(u_j)\ , \qquad \sigma_{j+\frac{1}{2}} = \frac{f(u_{j+1})-f(u_j)}{u_{j+1}-u_j}, \qquad \sigma^R_{j+\frac{1}{2}} = f'(u_{j+1})\ , \tag{A.15}
$$

and

$$
s^\alpha_{j+\frac{1}{2}} = \begin{cases} 1, & \text{if } \sigma^\alpha_{j+\frac{1}{2}} > 0 \\ 0, & \text{if } \sigma^\alpha_{j+\frac{1}{2}} \le 0 \end{cases}, \qquad (\alpha = L \text{ or } R). \tag{A.16}
$$

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