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IMPROVED ERROR ESTIMATES FOR MIXED FINITE ELEMENT FOR NONLINEAR HYPERBOLIC EQUATIONS: THE CONTINUOUS-TIME CASE*1)

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Abstract

Improved L^2-error estimates are computed for mixed finite element methods for second order nonlinear hyperbolic equations. Results are given for the continuous-time case. The convergence of the values for both the scalar function and the flux is demonstrated. The technique used here covers the lowest-order Raviart-Thomas spaces, as well as the higher-order spaces. A second paper will present the analysis of a fully discrete scheme (Numer. Math. J. Chinese Univ. vol.9, no.2, 2000, 181-192).

Key words: Nonlinear hyperbolic equations, Mixed finite element methods, Error estimates, Superconvergence.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^2$ with Lipschitz boundary $\partial\Omega$, and unit outward normal ν. For fixed $0 < T < \infty$, $J = (0, T]$, we discuss mixed finite element approximations of second order nonlinear hyperbolic equation

$$c(x,u)u_{tt} - \nabla \cdot (a(x,u)\nabla u) = f(x,u,t), \qquad x \in \Omega, \quad t \in J, \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{1.2}$$

and Dirichlet boundary condition

$$u(x,t) = -g(x,t), \qquad (x,t) \in \partial\Omega \times J. \tag{1.3}$$

We shall assume that the functions $c(x,u)$, $a(x,u)$, $f(x,u,t)$, $g(x,t)$ and solution $u(x,t)$ have sufficient regularity. Additionally, we assume that there exist constants c_*, c^*, a_*, and a^* such that

$$0 < c_* \le c(x,u) \le c^*, \qquad 0 < a_* \le a(x,u) \le a^*, \tag{1.4}$$

Optimal rates of convergence for Galerkin approximations to a class of second order nonlinear hyperbolic equations have been previously derived by Yuan yi-rang and Wang hong [9, 12-13]. The study of superconvergence for the gradient of the solution of second order hyperbolic equation was provided in [1, 6, 10-11]. Recently, several works have been devoted to the analysis of the mixed finite element methods (see [2-5, 8]). Cowsar, Dupont, Wheeler [2] have considered the convergence of the mixed finite element methods for second order linear hyperbolic equation.

In this paper, we formulate a mixed finite elment scheme for the approximation of (1.1)-(1.3) and establish the superconvergence L^2-estimate between the finite element solution and its elliptic projection. The method here gives a direct approximation of the flux, rather than

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one that requires differentiation and multiplication by a possibly rapidly varying coefficient; so, the direct evaluation of the flux can be expected to give improved accuracy for the same computational effort.

2. Mixed Finite Element Formulation for Nonlinear Hyperbolic Problem

Let $V = H(\mathrm{div};\Omega)$, $W = L^2(\Omega)$. Introduce the flux variable z:

$$z = -a(x,u)\nabla u, \tag{2.1}$$

and let $\alpha(u) = \alpha(x,u) = 1/a(x,u)$, $c(u) = c(x,u)$, $f(u) = f(x,u,t)$. Before defining a mixed finite element procedure we rewrite (1.1)-(1.3) in the following weak formulation

$$(u(0), w) = (u_0, w), \qquad w \in W, \tag{2.2}$$

$$(u_t(0), w) = (u_1, w), \qquad w \in W, \tag{2.3}$$

$$(c(u)u_{tt}, w) + (\nabla \cdot z, w) = (f(u), w), \qquad w \in W, \tag{2.4}$$

$$(\alpha(u)z, v) - (\nabla \cdot v, u) =< g, v \cdot \nu >, \qquad v \in V, \tag{2.5}$$

where $(\cdot,\cdot)$ is $L^2(\Omega)$ inner product, $< \cdot,\cdot >$ is the $L^2(\partial\Omega)$ inner product.

For h a small positive parameter we take $W_h \times V_h \subset W \times V$ to be the Raviart-Thomas space [8] of index k, where k is fixed nonnegative integer, associated with $\mathcal{T}_h$.

The continuous-time mixed finite element approximation to (2.2)-(2.5) is defined as a map from $[0,T]$ into $W_h \times V_h$ given by the pair $(U(\cdot,t), Z(\cdot,t))$ satisfying

$$(c(U)U_{tt}, w) + (\nabla \cdot Z, w) = (f(U), w), \qquad w \in W_h, \tag{2.6}$$

$$(\alpha(U)Z, v) - (\nabla \cdot v, U) =< g, v \cdot \nu >, \qquad v \in V_h, \tag{2.7}$$

with initial conditions

$$U(0) = \widetilde{U}(0), \qquad U_t(0) = \widetilde{U}_t(0), \qquad Z(0) = \widetilde{Z}(0), \tag{2.8}$$

where $(\widetilde{U}(\cdot,t), \widetilde{Z}(\cdot,t))$ is the elliptic mixed method projection to be defined later.

3. Mixed Method Projection

For the solution $u(x,t)$, let $\alpha_1 = \alpha(u)$, $\gamma(u) = \alpha_u(u)z$, $\alpha_u(u) = \frac{\partial\alpha(u)}{\partial u}$, and $\gamma_1 = \gamma(u)$. Define a linear mixed elliptic projection of $W \times V$ onto $W_h \times V_h$ by the $(u,z) \to (\widetilde{U},\widetilde{Z})$ determined by the relations

$$\begin{aligned} &(\nabla \cdot (z - \widetilde{Z}), w) + \lambda(u - \widetilde{U}, w) = 0, && w \in W_h, \\ &(\alpha_1(z - \widetilde{Z}), v) - (\nabla \cdot v, u - \widetilde{U}) + (\gamma_1(u - \widetilde{U}), v) = 0, && v \in V_h, \end{aligned} \tag{3.1}$$

for each $t \in J$. The positive constant λ will be assumed to be a sufficiently large constant such that

$$(\alpha_1\zeta, \zeta) + \lambda(\xi,\xi) + (\gamma_1\xi, \zeta) \geq \lambda_0(||\zeta||_0^2 + ||\xi||_0^2), \quad \text{for } \zeta \in V \text{ and } \xi \in W, \tag{3.2}$$

where $\lambda_0 > 0$ is independent of $t \in J$. Let

$$\begin{aligned} \eta &= u - \widetilde{U}, & \rho &= z - \widetilde{Z}, \\ \xi &= \widetilde{U} - U, & \zeta &= \widetilde{Z} - Z. \end{aligned} \tag{3.3}$$

As shown in [3, 4, 8], there exist the Raviart-Thomas projection $\Pi_h : \ V \to V_h$ and L^2-projection $P_h : \ W \to W_h$ such that for $0 < q \leq \infty$,

$$\mathrm{div} \circ \Pi_h = P_h \circ \mathrm{div}, \tag{3.4}$$

$$||v - \Pi_h v||_{0,q} \le C||v||_{r,q} h^r, \qquad 1/q < r \le k+1, \tag{3.5}$$

$$||w - P_h w||_{-s,q} \le C||w||_{r,q} h^{r+s}, \qquad 0 < r+s \le k+1, \tag{3.6}$$

$$||\nabla \cdot (v - \Pi_h v)||_{0,q} \le C||\nabla \cdot v||_{r,q} h^r, \qquad 1 \le r \le k+1. \tag{3.7}$$

We point out $(\Pi_h \psi)_t = \Pi_h \psi_t$, $(P_h \phi)_t = P_h \phi_t$, $(P_h \phi)_{tt} = P_h \phi_{tt}$. Let

$$\begin{array}{ll} \theta = u - P_h u, & \sigma = z - \Pi_h z, \\ \tau = P_h u - \widetilde{U}, & \delta = \Pi_h z - \widetilde{Z}, \end{array} \tag{3.8}$$

then $\eta = \theta + \tau$, $\rho = \sigma + \delta$. Recall a duality lemma from [3-4], which is restated here as Lemma 1. Assume that $a_1 = a(u) \in L^\infty(J; H^{s+1}(\Omega))$ and Ω is $s+2$ regular for $0 \le s \le k$ in this paper. The domain Ω is said to be $s+2$ regular if the Dirichlet problem

$$\begin{array}{ll} L^*_{t,\lambda}\phi = -\nabla \cdot (a_1 \nabla \phi) + b_1 \nabla \phi + \lambda \phi = \psi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{array} \tag{3.9}$$

is uniquely solvable for $\psi \in L^2(\Omega)$ and if $||\phi||_{s+2} \le C||\psi||_s$ for all $\psi \in H^s(\Omega)$.

Lemma 1. *Let $\zeta \in V$, $f \in V'$, and $g \in L^2(\Omega)$. Let $\tau \in W_h$ satisfy the relation*

$$\begin{array}{ll} (\nabla \cdot \zeta, w) + \lambda(\tau, w) = g(w), & w \in W_h, \\ (\alpha\zeta, v) - (\nabla \cdot v, \tau) + (\beta\tau, v) = f(v), & v \in V_h. \end{array} \tag{3.10}$$

If $k \ge 1$, $0 \le s \le k-1$ and Ω is $s+2$ regular, then for h sufficiently small,

$$\begin{aligned} ||\tau||_{-s} \le C(& h^{s+1}||\zeta||_0 + h^{s+2}||\nabla \cdot \zeta||_0 + ||f||_{-s-1} + ||g||_{-s-2} \\ & + h^{s+1}||f||_0 + h^{s+2}||g||_0). \end{aligned} \tag{3.11}$$

If $k \ge 0$ and Ω is $k+2$ regular, then for h sufficiently small,

$$||\tau||_{-k} \le Ch^{k+1}(||\zeta||_0 + ||\nabla \cdot \zeta||_0 + ||f||_0 + ||g||_0) + ||f||_{-k-1} + ||g||_{-k-2}. \tag{3.12}$$

Estimates for η, ρ, $\nabla \cdot \rho$, η_t, ρ_t, and $\nabla \cdot \rho_t$ are given in [3-5] and are presented in Lemma 2, 3, and 4 without proof.

Lemma 2. *For $t \in J$ and for h sufficiently small,*

$$||u - \widetilde{U}||_{-s} \le Ch^{r+s} \left\{ \begin{array}{llll} ||u||_r & \text{for} \ \ 0 \le s \le k-1 & \text{and} \ \ 1 \le r \le k+1 \\ ||u||_{r+1} & \text{for} \ \ s = k & \text{and} \ \ 1 \le r \le k+1 \\ ||u||_{r+2} & \text{for} \ \ s = k+1 & \text{and} \ \ 0 \le r \le k+1, \end{array} \right. \tag{3.13}$$

$$||z - \widetilde{Z}||_{-s} \le Ch^{r+s} \left\{ \begin{array}{lll} ||u||_{r+1} & \text{for} \ \ 0 \le s \le k-1 & \text{and} \ \ 1 \le r \le k+1 \\ ||u||_{r+2} & \text{for} \ \ s = k+1 & \text{and} \ \ 0 \le r \le k+1, \end{array} \right. \tag{3.14}$$

$$||\nabla \cdot (z - \widetilde{Z})||_{-s} \le Ch^{r+s}||u||_{r+2} \quad \text{for} \ \ 0 \le s \le k+1 \ \ \text{and} \ \ 0 \le r \le k+1. \tag{3.15}$$

Lemma 3. *For $t \in J$ and for h sufficiently small,*

$$||u_t - \widetilde{U}_t||_0 \le C \left\{ \begin{array}{ll} (||u_t||_r + ||u||_r)h^r & \text{for} \ \ 2 \le r \le k+1, \ \ k \ge 1 \\ (||u_t||_3 + ||u||_3)h & \text{for} \ \ k = 0, \end{array} \right. \tag{3.16}$$

$$||z_t - \widetilde{Z}_t||_0 \le C(||u_t||_{r+1} + ||u||_{r+1})h^r \quad \text{for} \ \ 1 \le r \le k+1, \ \ k \ge 0, \tag{3.17}$$

$$||\nabla \cdot (z_t - \widetilde{Z}_t)||_0 \le C(||u_t||_{r+2} + ||u||_{r+2})h^r \quad \text{for} \ \ 0 \le r \le k+1 \tag{3.18}$$

Lemma 4. *For $t \in J$ and for h sufficiently small,*

$$||u - \widetilde{U}||_{0,\infty} \le C(||u||_{r,\infty} + ||u||_{r+1+\delta_{k0}})h^r \qquad \text{for} \ \ 1 \le r \le k+1, \tag{3.19}$$

$$||z - \widetilde{Z}||_{0,\infty} \le Ch^{r-1/2}|\ln h| \cdot ||u||_{r+1+\delta_{k0},\infty} \qquad \text{for} \ \ 1 \le r \le k+1, \tag{3.20}$$

$$||u_t - \widetilde{U}_t||_{0,\infty} \le C(||u_t||_{r,\infty} + ||u||_{r+1} + ||u_t||_{r+1})h^r \ 1 \le r \le k+1, \ k > 0, \tag{3.21a}$$

$$||u_t - \widetilde{U}_t||_{0,\infty} \le C(||u_t||_{1,\infty} + ||u||_3 + ||u_t||_3)h \qquad \text{for} \quad k = 0. \tag{3.21b}$$

It also follows from [3-4] that

$$||\tau||_0 \le C||u||_{r+1+\delta_{k0}} h^{r+1}, \qquad 1 \le r \le k+1, \tag{3.22}$$

$$||\tau_t||_0 \le C(||u_t||_{r+1+\delta_{k0}} + ||u||_{r+1+\delta_{k0}})h^{r+1}, \quad 1 \le r \le k+1, \tag{3.23}$$

The following inverse hypothesis will be useful in the argument below

$$||v||_{0,\infty} \le Ch^{-1}||v||_0, \quad ||w||_{0,\infty} \le Ch^{-1}||w||_0, \quad v \in V_h, \ \ w \in W_h. \tag{3.24}$$

In order to estimate τ_{tt}, differentiate (3.1) with respect to t twice:

$$\begin{aligned} &(\nabla \cdot \rho_t, w) + \lambda(\tau_t, w) = 0, && w \in W_h, \\ &(\alpha_1 \rho_t, v) - (\nabla \cdot v, \tau_t) + (\gamma_1 \tau_t, v) \\ &\quad = -(\gamma_1 \theta_t, v) - (\tfrac{\partial \alpha_1}{\partial u} u_t \rho, v) - (\tfrac{\partial \gamma_1}{\partial u} u_t \eta, v), && v \in V_h, \end{aligned} \tag{3.25}$$

$$\begin{aligned} &(\nabla \cdot \rho_{tt}, w) + \lambda(\tau_{tt}, w) = 0, && w \in W_h, \\ &(\alpha_1 \rho_{tt}, v) - (\nabla \cdot v, \tau_{tt}) + (\gamma_1 \tau_{tt}, v) = -(\gamma_1 \theta_{tt}, v) - 2(\tfrac{\partial \alpha_1}{\partial u} u_t \rho_t, v) \\ &\quad - 2(\tfrac{\partial \gamma_1}{\partial u} u_t \eta_t, v) - ((\tfrac{\partial \alpha_1}{\partial u} u_t)_t \rho, v) - ((\tfrac{\partial \gamma_1}{\partial u} u_t)_t \eta, v) \equiv F(v), && v \in V_h. \end{aligned} \tag{3.26}$$

It follows from Lemma 1 that for h sufficiently small,

$$\begin{aligned} ||\tau_{tt}||_0 \le \ & Ch(||\rho_{tt}||_0 + ||\nabla \cdot \rho_{tt}||_0 + ||\rho_t||_0 + ||\eta_t||_0 + ||\rho||_0 + ||\eta||_0 \\ & + ||\theta_{tt}||_0) + ||\rho_t||_{-1} + ||\eta_t||_{-1} + ||\rho||_{-1} + ||\eta||_{-1} + ||\theta_{tt}||_{-1}. \end{aligned} \tag{3.27}$$

Note that $||\eta_t||_0 \le C(||\tau_t||_0 + ||\theta_t||_{-1})$. It is necessary to estimate $||\rho_{tt}||_0$, $||\nabla \cdot \rho_{tt}||_0$, and $||\rho_t||_{-1}$. By (3.4), (3.26) can be rewritten as

$$\begin{aligned} &(\nabla \cdot \delta_{tt}, w) + \lambda(\tau_{tt}, w) = 0, && w \in W_h, \\ &(\alpha_1 \delta_{tt}, v) - (\nabla \cdot v, \tau_{tt}) + (\gamma_1 \tau_{tt}, v) = -(\alpha_1 \sigma_{tt}, v) + F(v), && v \in V_h. \end{aligned} \tag{3.28}$$

Take $v = \delta_{tt}$ and $w = \tau_{tt}$:

$$(\alpha_1 \delta_{tt}, \delta_{tt}) + \lambda||\tau_{tt}||^2 + (\gamma_1 \tau_{tt}, \delta_{tt}) = -(\alpha_1 \sigma_{tt}, \delta_{tt}) + F(\delta_{tt}). \tag{3.29}$$

Using (3.2), it implies

$$||\delta_{tt}||_0 \le C(||\sigma_{tt}||_0 + ||\theta_{tt}||_0 + ||\rho_t||_0 + ||\eta_t||_0 + ||\rho||_0 + ||\eta||_0). \tag{3.30}$$

It also follows from (3.28) that

$$||\nabla \cdot \delta_{tt}||_0 \le \lambda||\tau_{tt}||_0. \tag{3.31}$$

Note that $||\rho_{tt}||_0 \le ||\sigma_{tt}||_0 + ||\delta_{tt}||_0$ and $||\nabla \cdot \rho_{tt}||_0 \le ||\nabla \cdot \sigma_{tt}||_0 + ||\nabla \cdot \delta_{tt}||_0$. Substitute inequalities (3.30) and (3.31) into (3.27), then

$$\begin{aligned} ||\tau_{tt}||_0 \le \ & Ch(||\sigma_{tt}||_0 + ||\nabla \cdot \sigma_{tt}||_0 + ||\theta_{tt}||_0 + ||\tau_{tt}||_0 + ||\rho_t||_0 + ||\eta_t||_0 \\ & + ||\rho||_0 + ||\eta||_0) + ||\rho_t||_{-1} + ||\eta_t||_{-1} + ||\rho||_{-1} + ||\eta||_{-1} + ||\theta_{tt}||_{-1}. \end{aligned} \tag{3.32}$$

For h sufficiently small, the $h||\tau_{tt}||_0$-term on the right-hand side can be absorbed into the left-hand side. Now, we bound $||\rho_t||_{-1}$. Let $\psi \in H^1(\Omega)^2$, and $\phi \in H^2(\Omega)$ satisfying that $-\nabla \cdot (a_1 \nabla \phi) = \nabla \cdot \psi$ in Ω and $\phi = 0$ on $\partial\Omega$. Assume that $||\phi||_2 \le C||\nabla \psi||_0 \le C||\psi||_1$. Furthermore, $\psi = -a_1 \nabla \phi + \mu$, where $\nabla \cdot \mu = 0$ and $||\mu||_1 \le C||\psi||_1$. Then

$$(\alpha_1 \rho_t, \psi) = -(\alpha_1 \rho_t, a_1 \nabla \phi) + (\alpha_1 \rho_t, \mu) = (\nabla \cdot \rho_t, \phi) + (\alpha_1 \rho_t, \mu). \tag{3.33}$$

By (3.25),

$$\begin{aligned} (\nabla \cdot \rho_t, \phi) \ & = (\nabla \cdot \rho_t, P_h \phi) + (\nabla \cdot \rho_t, \phi - P_h \phi) \\ & = -\lambda(\tau_t, \phi) + (\lambda \tau_t + \nabla \cdot \rho_t, \phi - P_h \phi) \\ & \le C\{||\tau_t||_0 ||\phi||_0 + h^2(||\tau_t||_0 + ||\nabla \cdot \rho_t||_0)||\phi||_2\} \\ & \le C(||\tau_t||_0 + h||\nabla \cdot \rho_t||_0)||\psi||_1. \end{aligned} \tag{3.34}$$

Then, since $\nabla\cdot\mu=0$ and by (3.25)

$$\begin{aligned}(\alpha_1\rho_t,\mu) &= (\alpha_1\rho_t,\Pi_h\mu)+(\alpha_1\rho_t,\mu-\Pi_h\mu)\\ &= (\nabla\cdot(\Pi_h\mu-\mu),\tau_t)-(\gamma_1\tau_t+\gamma_1\theta_t+\tfrac{\partial\alpha_1}{\partial u}u_t\rho+\tfrac{\partial\gamma_1}{\partial u}u_t\eta,\mu)\\ &\quad+(\alpha_1\rho_t+\gamma_1\eta_t+\tfrac{\partial\alpha_1}{\partial u}u_t\rho+\tfrac{\partial\gamma_1}{\partial u}u_t\eta,\mu-\Pi_h\mu)\\ &\le C\left\{h(||\rho_|||_0+||\eta_t||_0+||\rho||_0+||\eta||_0)+||\tau_t||_0\right.\\ &\quad\left.+||\theta_t||_{-1}+||\rho||_{-1}+||\eta||_{-1}\right\}\cdot||\psi||_1.\end{aligned}\tag{3.35}$$

The inequalities (3.34) and (3.35) can be combined with (3.33) to imply that

$$\begin{aligned}||\rho_t||_{-1}\le\ & C\{h(||\rho_t||_0+||\nabla\cdot\rho_t||_0+||\eta_t||_0+||\rho||_0+||\eta||_0)\\ &+||\tau_t||_0+||\theta_t||_{-1}+||\rho||_{-1}+||\eta||_{-1}\}.\end{aligned}\tag{3.36}$$

Thus, combining (3.36) with (3.32) and using Lemma 2-3, (3.5)-(3.7), and (3.23), we derive that

$$||\tau_{tt}||_0\le C(||u_{tt}||_{r+1}+||u_t||_{r+2}+||u||_{r+2})h^{r+1},\qquad 1\le r\le k+1.\tag{3.37}$$

Hence, by $||\eta_{tt}||_0\le||\theta_{tt}||_0+||\tau_{tt}||_0$, we have

$$||\eta_{tt}||_0\le C(||u_{tt}||_{r+1}+||u_t||_{r+2}+||u||_{r+2})h^{r},\qquad 1\le r\le k+1.\tag{3.38}$$

From (3.6), (3.24) and (3.37), we also have

$$||\eta_{tt}||_{0,\infty}\le C(||u_{tt}||_{r,\infty}+||u_{tt}||_{r+1}+||u_t||_{r+2}+||u||_{r+2})h^{r},\quad 1\le r\le k+1.\tag{3.39}$$

4. Improved Error Estimates

In this section, we derive the following superconvergence result

$$||(\widetilde{U}-U)_t||_{L^\infty(L^2)}+||\widetilde{U}-U||_{L^\infty(L^2)}+||\widetilde{Z}-Z||_{L^\infty(L^2)}=O(h^{k+2}),$$

where $L^\infty(L^2)=L^\infty(J;L^2(\Omega))$. This superconvergence result is useful to prove the following theorem.

Theorem 1. *There is a constant $C>0$, independent of h, such that*

$$||(u-U)_t||_{L^\infty(L^2)}+||u-U||_{L^\infty(L^2)}+||z-Z||_{L^\infty(L^2)}\le C(u)h^r,\tag{4.1}$$

for $1\le r\le k+1$, $k\ge0$, where $C(u)$ is given in (4.23).

The following result from [4] will often be used in the argument below.

Lemma 5. *If $\bar g$ is the average value of $g(u)$ on each element of the $\mathcal{T}_h$, and $||\nabla g||_{0,\infty}\le K$, then*

$$|(g(u)\theta,\psi)-(\bar g\theta,\psi)|\le CKh||\theta||_0||\psi||_0.$$

First, we prove the following lemma in order to prove Theorem 1.

Lemma 6. *If $\xi=\widetilde{U}-U$ and $\zeta=\widetilde{Z}-Z$, then there is a constant C independent of h such that*

$$||\xi_t||_{L^\infty(L^2)}+||\xi||_{L^\infty(L^2)}+||\zeta||_{L^\infty(L^2)}\le C(u)h^{r+1},\tag{4.2}$$

for $1\le r\le k+1$, $k\ge0$, where $C(u)$ is given in (4.23).

Proof. Using Eqs. (2.2)-(2.7) and (3.1), we obtain the error equations:

$$\begin{aligned}(c(U)\xi_{tt},w)+(\nabla\cdot\zeta,w)=\ & -([c(u)-c(U)]u_{tt},w)+(f(u)-f(U),w)\\ &-(c(U)\eta_{tt},w)+\lambda(\tau,w),\qquad w\in W_h,\end{aligned}\tag{4.3}$$

$$\begin{aligned}(\alpha(U)\zeta,v)-(\nabla\cdot v,\xi)=\ & ([\alpha(u)-\alpha(U)]\rho,v)-([\alpha(u)-\alpha(U)]z,v)\\ &+(\gamma_1\eta,v),\qquad v\in V_h.\end{aligned}\tag{4.4}$$

Differentiating (4.4) with respect to t and setting $v=\zeta$, $w=\xi_t$, add the two relations (4.3)-(4.4):

$$(c(U)\xi_{tt},\xi_t)+(\alpha(U)\zeta_t,\zeta)=-([c(u)-c(U)]u_{tt},\xi_t)+(f(u)-f(U),\xi_t)$$

$$-(c(U)\eta_{tt}-\lambda\tau,\xi_t)-(\alpha_u(U)U_t\zeta+\alpha_u(u)z\xi_t,\zeta)+([\alpha_u(u)-\alpha_u(U)]U_t\rho,\zeta)$$
$$+([\alpha(u)-\alpha(U)]\rho_t+\alpha_u(u)(u_t-U_t)\rho,\zeta)+([\alpha_u(u)-\alpha_u(U)](u_t-U_t)z,\zeta)$$
$$+(\alpha_{uu}u_tz\eta-[\alpha_u(u)-\alpha_u(U)]u_tz,\zeta)+(\alpha_uz_t\eta-[\alpha(u)-\alpha(U)]z_t,\zeta)\equiv\sum_{i=1}^{8}I_i. \tag{4.5}$$

First, integrating (4.5) in time, let us estimate the left-hand side of (4.5) using (1.4) and (2.8),

$$\int_0^t[(c(U)\xi_{tt},\xi_t)+(\alpha(U)\zeta_t,\zeta)]\,ds=\frac{1}{2}\int_0^t\left[\frac{d}{dt}(c(U)\xi_t,\xi_t)-(c_u(U)U_t\xi_t,\xi_t)\right.$$
$$\left.+\frac{d}{dt}(\alpha(U)\zeta,\zeta)-(\alpha_u(U)U_t\zeta,\zeta)\right]ds\geq\frac{c_0}{2}||\xi_t||_0^2+\frac{1}{2a^*}||\zeta||_0^2$$
$$-C\left(||\xi_t||_{L^\infty(L^\infty)}+1\right)\int_0^t(||\xi_t||_0^2+||\zeta||_0^2)ds, \tag{4.6}$$

Then, we bound each of the terms on the right-hand side of (4.5).

$$|I_1|=|([c(u)-c(U)]u_{tt},\xi_t)|=\Big|\Big([c(u)-c(P_hu)+c(P_hu)-c(\widetilde{U})+c(\widetilde{U})$$
$$-c(U)]u_{tt},\xi_t\Big)\Big|\leq\left|(c_u(u)(u-P_hu)u_{tt},\xi_t)-\left(\frac{\tilde{c}_{uu}}{2}(u-P_hu)^2u_{tt},\xi_t\right)\right|$$
$$+C||\tau||_0||\xi_t||_0+C||\xi||_0||\xi_t||_0, \tag{4.7}$$

where $\tilde{c}_{uu}$ means c_{uu} evaluated at a point $u(x^*,t^*)$ between $u(x,t)$ and $P_hu(x,t)$. Using Lemma 5 with $g(u)=c_u(u)u_{tt}$, we obtain

$$|I_1|\leq C\{h||\theta||_0+||\theta||_{0,\infty}||\theta||_0+||\tau||_0+||\xi||_0\}||\xi_t||_0. \tag{4.8}$$

Similarly

$$|I_2|=|(f(u)-f(U),\xi_t)|=\Big|\Big(f(u)-f(P_hu)+f(P_hu)-f(\widetilde{U})+f(\widetilde{U})$$
$$-f(U),\xi_t\Big)\Big|\leq C\{h||\theta||_0+||\theta||_{0,\infty}||\theta||_0+||\tau||_0+||\xi||_0\}||\xi_t||_0, \tag{4.9}$$

Next

$$|I_3|=|(c(U)\eta_{tt}-\lambda\tau,\xi_t)+(\alpha_u(U)U_t\zeta+\alpha_u(u)z\xi_t,\zeta)|\leq\Big|(c(u)\theta_{tt},\xi_t)+(c(u)\tau_{tt},\xi_t)$$
$$\Big([c(u)-c(\widetilde{U})]\eta_{tt},\xi_t\Big)+\Big([c(\widetilde{U})-c(U)\eta_{tt},\xi_t\Big)\Big|+C\Big\{(||\tau||_0+||\zeta||_0)||\xi_t||_0$$
$$+\left(||\xi_t||_{L^\infty(L^\infty)}+1\right)||\zeta||_0^2\Big\}\leq C\left\{h||\theta_{tt}||_0+||\tau_{tt}||_0+||\eta||_{0,\infty}||\eta_{tt}||_0\right.$$
$$\left.+||\eta_{tt}||_{0,\infty}||\xi||_0+||\tau||_0+||\zeta||_0\right\}\cdot||\xi_t||_0+C\left(||\xi_t||_{L^\infty(L^\infty)}+1\right)||\zeta||_0^2, \tag{4.10}$$

$$|I_4|=|([\alpha_u(u)-\alpha_u(U)]U_t\rho,\zeta)|=\Big|\Big([\alpha_u(u)-\alpha_u(\widetilde{U})+\alpha_u(\widetilde{U})-\alpha_u(U)]$$
$$U_t\rho,\zeta\Big)\Big|\leq C\left\{||\eta||_{0,\infty}||\rho||_0+||\rho||_{0,\infty}||\xi||_0\right\}\left(||\xi_t||_{L^\infty(L^\infty)}+1\right)||\zeta||_0, \tag{4.11}$$

$$|I_5|=|([\alpha(u)-\alpha(U)]\rho_t+\alpha_u(u)(u_t-U_t)\rho,\zeta)|$$
$$=\Big|\Big([\alpha(u)-\alpha(\widetilde{U})+\alpha(\widetilde{U})-\alpha(U)]\rho_t+\alpha_u(u)(u_t-\widetilde{U}_t+\widetilde{U}_t-U_t)\rho,\zeta\Big)\Big|$$
$$\leq C\left\{(||\eta||_{0,\infty}+h^{-1}||\xi||_0)||\rho_t||_0+(||\eta||_{0,\infty}+h^{-1}||\xi_t||_0)||\rho||_0\right\}||\zeta||_0. \tag{4.12}$$

Let us make an induction hypothesis

$$\sup_{0\le t\le T}(||\xi(t)||_0+||\xi_t(t)||_0)h^{-1}=O(h), \qquad \text{as } \ h\to 0, \tag{4.13}$$

which is consistent with our goal of showing that $||\xi||_0+||\xi_t||_0=O(h^{k+2})$. As of (4.13), (4.12) becomes

$$|I_5|\le C\left\{||\eta||_{0,\infty}||\rho_t||_0+h||\rho_t||_0+||\eta_t||_{0,\infty}||\rho||_0+h||\rho||_0\right\}||\zeta||_0, \tag{4.14}$$

Again in the next term,

$$\begin{aligned}|I_6| &= |([\alpha_u(u)-\alpha_u(U)](u_t-U_t)z,\zeta)|\\ &= \left|\left([\alpha_u(u)-\alpha_u(\widetilde{U})+\alpha_u(\widetilde{U})-\alpha_u(U)](u_t-\widetilde{U}_t+\widetilde{U}_t-U_t)z,\zeta\right)\right|\\ &\le C\left\{||\eta||_{0,\infty}||\eta_t||_0+||\eta||_{0,\infty}||\xi_t||_0+||\eta_t||_{0,\infty}||\xi||_0+h||\xi_t||_0\right\}||\zeta||_0,\end{aligned} \tag{4.15}$$

where the hypothesis (4.13) was again used.

$$\begin{aligned}|I_7| &= |(\alpha_{uu}u_tz\eta-[\alpha_u(u)-\alpha_u(U)]u_tz,\zeta)|\\ &= \left|\left(\tfrac{1}{2}\tilde{\alpha}_{uuu}u_tz\eta^2+[\alpha_u(\widetilde{U})-\alpha_u(U)]u_tz,\zeta\right)\right|\\ &\le C\left\{||\eta||_{0,\infty}||\eta||_0+||\xi||_0\right\}||\zeta||_0,\end{aligned} \tag{4.16}$$

$$|I_8|=\left|\left(\frac{1}{2}\tilde{\alpha}_{uu}z_t\eta^2+[\alpha(\widetilde{U})-\alpha(U)]z_t,\zeta\right)\right|\le C\left\{||\eta||_{0,\infty}||\eta||_0+||\xi||_0\right\}||\zeta||_0, \tag{4.17}$$

Note that (4.13) implies

$$||\xi_t||_{L^\infty(L^\infty)}\le 1, \qquad \text{as } \ h\to 0. \tag{4.18}$$

Now, we can find an estimate for (4.5) using the estimates above and (3.20), (3.39), and (4.18) to obtain the evolution inequality

$$||\xi_t||_0^2+||\zeta||_0^2\le C\int_0^t[||\xi_t||_0^2+||\xi||_0^2+||\zeta||_0^2]ds+\mathcal{R}, \tag{4.19}$$

where $\mathcal{R}$ was used to simplify

$$\begin{aligned}\mathcal{R}= \ &\textstyle\int_0^t\{h^2||\theta||_0^2+||\theta||_{0,\infty}^2||\theta||_0^2+||\tau||_0^2+h^2||\theta_{tt}||_0^2+||\tau_{tt}||_0^2\\ &+||\eta||_{0,\infty}^2(||\eta||_0^2+||\eta_t||_0^2+||\eta_{tt}||_0^2+||\rho||_0^2+||\rho_t||_0^2)\\ &+h^2||\rho_t||_0^2+||\eta_t||_{0,\infty}^2||\rho||_0^2+h^2||\rho||_0^2\}\,ds.\end{aligned} \tag{4.20}$$

Using (3.6), (3.22), (3.37)-(3.38), and Lemma 2-4, it is easy to see that for $1\le r\le k+1$, $k\ge 0$,

$$\mathcal{R}\le C\int_0^t\left\{||u_{tt}||_{r+1}^2+||u_t||_{r+2}^2+||u_t||_{r,\infty}^2+||u||_{r+2}^2+||u||_{r,\infty}^2\right\}ds\cdot h^{2r+2}. \tag{4.21}$$

Finally, adding a term $||\xi||_0^2$ at the two sides of (4.19) and using that $||\xi||_0^2\le T\int_0^t||\xi_t||_0^2ds$ since $\xi(0)=0$ and Gronwall$'$ lemma, we obtain the following estimate:

$$||\xi_t||_{L^\infty(L^2)}+||\xi||_{L^\infty(L^2)}+||\zeta||_{L^\infty(L^2)}\le C(u)h^{r+1}, \tag{4.22}$$

for $1\le r\le k+1$, $k\ge 0$, where

$$C(u)=C\left(||u_{tt}||_{L^2(H^{r+1})}+||u_t||_{L^2(H^{r+2}\cap W^{r,\infty})}+||u||_{L^2(H^{r+2}\cap W^{r,\infty})}\right). \tag{4.23}$$

Thus, Lemma 6 follows if we verify the hypothsis (4.13). Note that for $k\ge 0$, (4.22) implies

$$\sup_{0\le t\le T}(||\xi(t)||_0+||\xi_t(t)||_0)h^{-1}=O(h^{k+2})\cdot h^{-1}=O(h), \qquad \text{as } \ h\to 0,$$

So (4.13) is demostrated, and Lemma 6 is established.

Now, the proof of Theorem 1 follows directly from Lemma 6 and the triangle inequality.

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