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EXTRAPOLATION AND A-POSTERIORI ERROR ESTIMATORS OF PETROV-GALERKIN METHODS FOR NON-LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*[1)]

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Abstract

In this paper we will show that the Richardson extrapolation can be used to enhance the numerical solution generated by a Petrov-Galerkin finite element method for the initial-value problem for a nonlinear Volterra integro-differential equation. As by-products, we will also show that these enhanced approximations can be used to form a class of a-posteriori estimators for this Petrov-Galerkin finite element method. Numerical examples are supplied to illustrate the theoretical results.

Key words: Volterra integro-differential equations, Petrov-Galerkin finite element methods, Asymptotic expansions, Interpolation post-processing, A-posteriori error estimators.

1. Introduction

The purpose of this paper is to show that the Richardson extrapolation can be used to enhance the numerical solutions generated by a class of Petrov-Galerkin finite element methods for the nonlinear Volterra integro-differential equation (VIDE):

$$y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s))ds, \quad t \in I := [0, T], \quad y(0) = 0, \tag{1.1}$$

where $f = f(t, y) : I \times R \to R$ and $k = k(t, s, y) : D \times R \to R$ (with $D := \{(t, s) : 0 \le s \le t \le T\}$) denote given functions.

Throughout this paper, it will always be assumed that the problem (1.1) possesses a unique solution $y \in C^1(I)$, namely, the given functions $f(t, y)$ and $k(t, s, y)$, which are, respectively, continuous for $t \in I$ and $(t, s) \in D$, will be subject to the following (uniform) Lipschitz conditions [3]:

$$\begin{array}{ll} (V1) & |f(t, y_1) - f(t, y_2)| \le L_1|y_1 - y_2|, \\ (V2) & |k(t, s, y_1) - k(t, s, y_2)| \le L_2|y_1 - y_2|, \end{array}$$

for all $t \in I$, $(t, s) \in D$, and $|y_i| < \infty$ $(i = 1, 2)$.

The Volterra integro-differential equation (1.1) plays an important role in the mathematical modeling of various physical and biological phenomena, and the study of various numerical

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methods has received considerable attention in the past, see, for example, the survey paper [2] by Brunner and the monograph [3] by Brunner et al., as well as references cited therein. Also, see [7] and [16] for some related recent publications.

It is well known that the extrapolation method is a quite effective numerical method in producing more accurate approximations. This technique has been well demonstrated in its applications to the numerical solutions for the elliptic partial differential equations in [1, 5, 10, 11], the parabolic partial differential and integro-differential equations in [6, 8, 12]. This method has also been discussed for boundary element approximations [17, 18]. In [19] multi-parameter parallel algorithms have been introduced into the extrapolation for accelerating the computational speeds.

Here, we shall investigate the extrapolation of the numerical solutions generated by a class of Petrov-Galerkin finite element (PGFE) methods [9] for the initial value problem of a nonlinear VIDE (1.1). The Petrov-Galerkin finite element method has an advantage over the standard Galerkin finite element method, a special case of the Petrov-Galerkin finite element method, in that it allows the trial and test function spaces to be different. This feature provides us with a freedom in choosing a pair of trial and test function spaces for a better computational efficiency than the standard Galerkin finite element method. Moreover, our analysis reveals that the PGFE solutions have a rich collection of approximation properties, and a variety of post-processing techniques have been developed to take advantage of them [15]. In particular, we will show that the Richardson extrapolation can be used to efficiently generate better approximations from the PGFE solutions for both the solution and its derivative to the initial value problem of the nonlinear VIDE.

This paper is organized in the following way. In Section 2, we introduce the Petrov-Galerkin finite element scheme for the problem (1.1) and recall some basic error estimates and asymptotic expansions obtained in our previous work. Section 3 is devoted to the study of the asymptotic expansions for the PGFE solutions and the iterated PGFE derivatives. In Section 4, we will investigate the Richardson extrapolation for the PGFE solutions and the iterated PGFE derivatives, and present some a-posteriori error estimators that use the superconvergent approximations obtained by post-processing the PGFE solutions with Richardson extrapolation. In Section 5 we will present some numerical examples to illustrate the theoretical results.

2. Petrov-Galerkin Finite Element Methods

In this section we will introduce the Petrov-Galerkin finite element methods and recall the global convergence results and asymptotic expansions for the L^2-projection operator obtained in [15] and [4], respectively. For this purpose, we first define a nonlinear integral operator $G : C(I) \to C(I)$ by

$$(G\varphi)(t) := f(t, \varphi(t)) + \int_0^t k(t, s, \varphi(s))ds.$$

Then, the problem (1.1) reads: Find $y = y(t)$ such that

$$y'(t) = (Gy)(t), \quad t \in I, \tag{2.1}$$

and its Petrov-Galerkin weak form consists in finding $y \in H_0^1(I)$ (and then $y' \in L^2(I)$) such that

$$(y', v) = (Gy, v), \quad \forall v \in L^2(I), \tag{2.2}$$

where $(\cdot, \cdot)$ denotes the usual inner product in $L^2(I)$ and $H_0^1(I) := \{v \in H^1(I) : v(0) = 0\}$ is the Sobolev space.

Let $T_h : 0 = t_0 < t_1 < \cdots < t_N = T$ be a given mesh for the interval I, and denote the finite element trial and test function spaces, respectively, by

$$S_m^{(0)}(T_h) := \{v \in H_0^1(I) : v|_{\sigma_k} \in P_m, 0 \le k \le N-1\}$$

and

$$S_{m-1}^{(-1)}(T_h) := \{v \in L^2(I) : v|_{\sigma_k} \in P_{m-1}, 0 \le k \le N-1\} \quad \text{with} \quad m \ge 1,$$

where P_r denotes the space of (real) polynomials of degree not exceeding r, $\sigma_k := [t_k, t_{k+1}]$ $(0 \le k \le N-1)$, $h_k := t_{k+1} - t_k$ and $h := \max_{(k)}\{h_k\}$. Clearly, the dimensions of $S_m^{(0)}(T_h)$ and $S_{m-1}^{(-1)}(T_h)$ are equal to Nm. Note that the superscript (-1) in $S_{m-1}^{(-1)}(T_h)$ emphasizes the fact that functions in the space $S_{m-1}^{(-1)}(T_h)$ are allowed to have discontinuity.

Our Petrov-Galerkin finite element (PGFE) method of (2.2) is now defined as: Find $u \in S_m^{(0)}(T_h)$ (and then $u' \in S_{m-1}^{(-1)}(T_h)$) such that

$$(u', v) = (Gu, v), \quad \forall v \in S_{m-1}^{(-1)}(T_h). \tag{2.3}$$

Let $P_h : L^2(I) \to S_{m-1}^{(-1)}(T_h)$ be the L^2-projection operator defined by

$$(\varphi, v) = (P_h\varphi, v), \quad \forall v \in S_{m-1}^{(-1)}(T_h).$$

Then, the problem (2.3) can be equivalently written as: Find $u \in S_m^{(0)}(T_h)$ (and then $u' \in S_{m-1}^{(-1)}(T_h)$) such that

$$u'(t) = (P_h Gu)(t), \quad t \in I. \tag{2.4}$$

Since $S_{m-1}^{(-1)}(T_h)$ is a discontinuous piecewise-polynomial space, and P_h possesses localization, we have

$$\int_{\sigma_k} vP_h\varphi dt = \int_{\sigma_k} v\varphi dt, \quad \forall v \in P_{m-1}, \tag{2.5}$$

with

$$\|P_h\varphi - \varphi\|_{0,\infty} \le Ch^m \|\varphi\|_{m,\infty},$$

where $\|v\|_{r,\infty} := \max_{0 \le k \le r}\{\|v^{(k)}\|_\infty\}$ for any nonnegative integer r and C is a generic constant independent of h whose meanings will become clear by the context in which it arises. In this case, P_h is defined on each element, and it can be regarded as an interpolation operator of degree $m-1$ (it is a kind of interpolation in average meaning which is different from the standard Lagrange interpolation) associated with the mesh T_h. Moreover, for an arbitrary mesh point t_n $(1 \le n \le N)$ of T_h we have the following identity by (2.5)

$$\int_0^{t_n} v(P_h - I)\varphi dt = \sum_{k=0}^{n-1} \int_{\sigma_k} (I - P_h)v(P_h - I)\varphi dt, \tag{2.6}$$

which will frequently be employed in our analysis later. In [15] we have shown that under the conditions $(V1)$ and $(V2)$ the problem (2.3) (or (2.4)) is uniquely solvable whenever the mesh size h is sufficiently small. We have also obtained the following global convergence results in [15].

Lemma 2.1. *Assume that $f \in C^m(I \times R)$ and $k \in C^m(D \times R)$. Then the Petrov-Galerkin finite element error $e := u - y$ satisfies*

$$\|e\|_{0,\infty} := \max\{|e(t)| : t \in I\} \le Ch^{m+1}\|y\|_{m+1,\infty},$$
$$\|e'\|_{0,\infty} := sup\{|e'(t)| : t \in \sigma_k, 0 \le k \le N-1\} \le Ch^m\|y\|_{m+1,\infty}.$$

In addition, we also recall from [4] the following asymptotic expansion with respect to the L^2-projection operator P_h.

Lemma 2.2. *If $\varphi \in C^{m+2}(I)$ and $v \in C^{m+2}(I)$, then there exists a constant $\tilde{C} = \tilde{C}(m)$, independently of the mesh T_h, such that for $0 \leq k \leq N-1$*

$$\int_{\sigma_k} v(t)(\varphi - P_h\varphi)(t)dt = \tilde{C}h_k^{2m} \int_{\sigma_k} \varphi^{(m)}(t)v^{(m)}(t)dt + O(h_k^{2m+3}). \tag{2.7}$$

3. Asymptotic Expansions

In this section, we will derive the asymptotic expansions for the solution of the initial value problem (1.1) and its derivative, which are the key to the investigation of the global extrapolation approximations.

3.1. Asymptotic Expansions for PGFE Solutions

First, we discuss the asymptotic expansion of the PGFE solutions of the problem (1.1). We start by recalling the following lemma [3].

Lemma 3.1. *Let the functions g and K^* characterizing the integral equation,*

$$y(t) = g(t) + \int_0^t K^*(t,s)y(s)ds,$$

be continuous on I and D, respectively. Then this equation has a unique solution $y \in C(I)$ given by

$$y(t) = g(t) + \int_0^t R^*(t,s)g(s)ds, \quad t \in I,$$

where $R^* \in C(D)$ is the resolvent kernel associated with the given kernel K^* and defined by $R^*(t,s) := \sum_{m=1}^{\infty} K_m^*(t,s)$, $(t,s) \in D$ with $K_1^*(t,s) := K^*(t,s)$ and $K_n^*(t,s) := \int_s^t K_1^*(t,\tau)K_{n-1}^*(\tau,s)d\tau$, $(t,s) \in D$ $(n \geq 2)$. Moreover, the resolvent kernel satisfies the identities (usually called the Fredholm identities)

$$R^*(t,s) = K^*(t,s) + \int_s^t K^*(t,\tau)R^*(\tau,s)d\tau, \quad (t,s) \in D,$$

and

$$R^*(t,s) = K^*(t,s) + \int_s^t R^*(t,\tau)K^*(\tau,s)d\tau, \quad (t,s) \in D.$$

Now, let δ be the residual (or: defect) function,

$$\delta(t) := u'(t) - (Gu)(t), \quad t \in I.$$

It is easy to see from (2.4) that

$$\delta = P_hGu - Gu = (P_h - I)Gu. \tag{3.1}$$

Subtracting (2.1) from (2.4), we have by (3.1) that

$$e' = P_hGu - Gy = \delta + (Gu - Gy), \quad t \in I, \tag{3.2}$$

with $e(0) = 0$. Furthermore, let $G' : C(I) \to C(I)$ be the linear Volterra integral operator defined by

$$(G'\varphi)(t) := f_y(t,y(t))\varphi(t) + \int_0^t k_y(t,s,y(s))\varphi(s)ds, \tag{3.3}$$

where $y(t)$ is the exact solution of the problem (1.1). And then, from (3.2), (3.3), Lemma 2.1 and Taylor's formula we know that there are functions ξ and η, whose values $\xi(t)$ and $\eta(t)$ at t are between $y(t)$ and $u(t)$, such that

$$\begin{aligned} e'(t) &= \delta(t) + (Gu - Gy)(t) \\ &= \delta(t) + (G'e)(t) + O(h^{2m+2}) \\ &:= \delta(t) + p(t)e(t) + \int_0^t K(t,s)e(s)ds + O(h^{2m+2}) \end{aligned} \tag{3.4}$$

under the conditions that $f_{yy}(t,y)$ and $k_{yy}(t,s,y)$ are bounded uniformly on their respective domains $I \times R$ and $D \times R$, where $p(t) := f_y(t, y(t))$ and $K(t,s) := k_y(t,s,y(s))$ are independent of the mesh size h.

By setting $\widetilde{e}(t) = e(t)\exp\left(-\int_0^t p(s)ds\right)$, it is easy to see from a simple calculation that (3.4) becomes

$$\widetilde{e}'(t) = \widetilde{\delta}(t) + (\widetilde{K}\widetilde{e})(t), \quad \widetilde{e}(0) = 0, \tag{3.5}$$

where $\widetilde{\delta}(t) := \delta(t)\exp\left(-\int_0^t p(s)ds\right)$ and $\widetilde{K}$ is the linear Volterra integral operator defined by

$$(\widetilde{K}\varphi)(t) := \int_0^t \widetilde{K}(t,s)\varphi(s)ds, \quad \widetilde{K}(t,s) := K(t,s)\exp\left(-\int_s^t p(\tau)d\tau\right), \quad t \in I.$$

To simplify the notation, we will use e, K and δ instead of using $\widetilde{e}, \widetilde{K}$ and $\widetilde{\delta}$ in the discussion from now on. By exchanging the order of integration with respect to s and τ, we obtain

$$\begin{aligned} e' &= \delta + \int_0^t K(t,s)\left(\int_0^s e'(\tau)d\tau\right)ds + O(h^{2m+2}) \\ &= \delta + \int_0^t K_1(t,s)e'(s)ds + O(h^{2m+2}), \end{aligned}$$

where the kernel function $K_1(t,s)$ is defined by

$$K_1(t,s) := \int_s^t K(t,\tau)d\tau.$$

Therefore, $K_1(t,s)$ is independent of the mesh size h. And then, setting $F := \delta + O(h^{2m+2})$ and following from Lemma 3.1 we have

$$e' = F + \int_0^t R_1(t,s)F(s)ds, \tag{3.6}$$

where $R_1(t,s)$ is the resolvent kernel associated with $K_1(t,s)$ defined by

$$R_1(t,s) = K_1(t,s) + \int_s^t K_1(t,\tau)R_1(\tau,s)d\tau, \quad (t,s) \in D.$$

Note that $R_1(t,s)$ inherits the same smoothness of $K_1(t,s)$, and $R_1(t,s)$ (therefore $R(t,s)$) is independent of the mesh size h. Thus, by integrating from 0 to t on both sides of (3.6) and exchanging the order of integration, we have

$$e = \int_0^t R(t,s)F(s)ds, \tag{3.7}$$

where $R(t,s) := 1 + \int_s^t R_1(\tau,s)d\tau$. Then, we know from (3.7) that

$$e = \int_0^t R(t,s)\delta(s)ds + O(h^{2m+2}) \tag{3.8}$$

under the condition that $k_y(t,s,y)$ is bounded uniformly on $D \times R$.

Let

$$(R_h\varphi)(t) := \int_0^t R(t,s)(P_h - I)\varphi(s)ds, \tag{3.9}$$

then, it follows from (2.1), (3.1) and (3.3) that

$$\begin{aligned} e &= R_hGu + O(h^{2m+2}) = R_hGy + R_h(Gu - Gy) + O(h^{2m+2}) \\ &= R_hy' + R_hG'e + O(h^{2m+2}), \end{aligned} \tag{3.10}$$

which leads to a recurrence formula:

$$e = R_hy' + R_hG'R_hy' + (R_hG')^2e + O(h^{2m+2}). \tag{3.11}$$

Therefore, we have

Theorem 3.1. *Assume that $f \in C^{m+2}(I \times R)$ and $k \in C^{m+2}(D \times R)$ for $m \geq 1$. Then, the PGFE error $e = u - y$ has the following asymptotic expansion at the mesh points of T_h:*

$$e(t_n) = \alpha(t_n)h^{2m} + O(h^{2m+1}), \quad 1 \leq n \leq N, \tag{3.12}$$

where $\alpha \in C^1(I)$ is invariable when the mesh is refined uniformly.

Proof. Let

$$\alpha(t) := C\sum_{k=0}^{n-1}\left(\frac{h_k}{h}\right)^{2m}\int_{\sigma_k}\frac{\partial^m R}{\partial s^m}(t,s)y^{(m+1)}(s)ds,$$

where $\alpha(t)$ is invariable when the mesh is refined uniformly. Then, it follows from (2.7), (3.9) and $\sum_{k=0}^{n-1} h_k \leq T$ that

$$\begin{aligned} (R_hy')(t_n) &= \sum_{k=0}^{n-1}\int_{\sigma_k} R(t_n,s)(P_hy' - y')(s)ds \\ &= h^{2m}\sum_{k=0}^{n-1} C\left(\frac{h_k}{h}\right)^{2m}\int_{\sigma_k}\frac{\partial^m R}{\partial s^m}(t_n,s)y^{(m+1)}(s)ds + O(h^{2m+2}) \\ &= \alpha(t_n)h^{2m} + O(h^{2m+2}), \end{aligned} \tag{3.13}$$

where $C = C(m)$ is a constant. In addition, since $k \in C^{m+2}(D \times R)$, then $R \in C^{m+1}(D)$ such that $\alpha \in C^1(I)$.

Since the operators G' and P_h are bounded uniformly, from (2.6), (3.9) and Lemma 2.1 we know

$$\begin{aligned} |(R_hG'e)(t_n)| &= \left|\sum_{k=0}^{n-1}\int_{\sigma_k}(I - P_h)R(t_n,s)(P_h - I)(G'e)(s)ds\right| \\ &\leq \sum_{k=0}^{n-1} Ch^m\int_{\sigma_k}\left|(P_h - I)(G'e)(s)\right|ds \\ &\leq Ch^m\|G'e\|_{0,\infty} \leq Ch^m\|e\|_{0,\infty} \leq Ch^{2m+1}\|y\|_{m+1,\infty}. \end{aligned} \tag{3.14}$$

Now, by combining (3.13) and (3.14) with (3.10), we can obtain the result predicted in this theorem. *Q.E.D.*

For $m = 1$, which is the practically important case, we can show that the residual error on the right hand side of (3.12) is in fact $O(h^4)$ instead of $O(h^3)$.

Theorem 3.2. *Assume that $f \in C^3(I \times R)$ and $k \in C^3(D \times R)$. Then the linear PGFE error e has the following asymptotic expansion at the mesh points of T_h:*

$$e(t_n) = \alpha(t_n)h^2 + O(h^4), \quad 1 \le n \le N, \tag{3.15}$$

where $\alpha \in C^1(I)$ is invariable when the mesh is refined uniformly.

Proof. For any $t \in \sigma_k$ $(0 \le k \le N-1)$, by virtue of (2.6) and Schwartz's inequality we have that, for an arbitrary $\varphi \in C(I)$,

$$\begin{aligned} |(R_h\varphi)(t)| &\le \left|\sum_{j=0}^{k-1}\int_{\sigma_j}(I-P_h)R(t,s)(I-P_h)\varphi(s)ds\right| \\ &+\left|\int_{t_k}^{t}R(t,s)(I-P_h)\varphi(s)ds\right| \\ &\le Ch\|\varphi\|_{0,\infty} + C\|\varphi\|_{0,\infty}(t-t_k) \le Ch\|\varphi\|_{0,\infty} \end{aligned}$$

which, together with $\|G'\|_{C(I)\to C(I)} \le C$ and Lemma 2.1, yields that

$$\|(R_hG')^2e\|_{0,\infty} \le Ch\|(R_hG')e\|_{0,\infty} \le Ch^2\|e\|_{0,\infty} \le Ch^4\|y\|_{2,\infty}. \tag{3.16}$$

Next, it follows from exchanging the order of integration with respect to σ and τ and the definitions of the operators G' and R_h that

$$\begin{aligned} (G'R_hy')(s) &= p(s)(R_hy')(s) + \int_0^s K(s,\tau)(R_hy')(\tau)d\tau \\ &= \int_0^s A(s,\sigma)(P_h-I)y'(\sigma)d\sigma, \end{aligned}$$

where the kernel function $A(s,\sigma) := p(s)\cdot R(s,\sigma) + \int_\sigma^s K(s,\tau)R(\tau,\sigma)d\tau$ is independent of the mesh size h.

Since

$$(G'R_hy')'(s) = A(s,s)(P_h-I)y'(s) + \int_0^s A'_s(s,\sigma)(P_h-I)y'(\sigma)d\sigma,$$

we know from Lemma 2.2 and (2.6) that

$$\begin{aligned} (R_hG'R_hy')(t_j) &= \sum_{k=0}^{j-1}\int_{\sigma_k}R(t_j,s)(P_h-I)(G'R_hy')(s)ds \\ &= \sum_{k=0}^{j-1}Ch_k^2\int_{\sigma_k}R'_s(t_j,s)A(s,s)(P_h-I)y'(s)ds \\ &+\sum_{k=0}^{j-1}Ch_k^2\int_{\sigma_k}R'_s(t_j,s)\left(\int_0^s A'_s(s,\sigma)(P_h-I)y'(\sigma)d\sigma\right)ds \\ &+O(h^4) := I_0 + O(h^4), \end{aligned} \tag{3.17}$$

where C is a constant and $I_0 := \sum_{k=0}^{j-1}Ch_k^2\int_{\sigma_k}R'_s(t_j,s)\left(\int_0^s A'_s(s,\sigma)(P_h-I)y'(\sigma)d\sigma\right)ds$. Again, it follows from (2.6) and exchanging the order of integration with respect to s and σ that

$$\begin{aligned} |I_0| &= \left|\sum_{k=0}^{j-1}Ch_k^2\int_0^{t_k}(P_h-I)y'(\sigma)\left(\int_{\sigma_k}R'_s(t_j,s)A'_s(s,\sigma)ds\right)d\sigma\right. \\ &\left.+\sum_{k=0}^{j-1}Ch_k^2\int_{\sigma_k}(I-P_h)y'(\sigma)\left(\int_\sigma^{t_{k+1}}R'_s(t_j,s)A'_s(s,\sigma)ds\right)d\sigma\right| \\ &\le Ch^4\|y\|_{2,\infty} \end{aligned} \tag{3.18}$$

which, together with (3.17), implies

$$\max_{1\le j\le N} |(R_h G' R_h y')(t_j)| \le Ch^4 \|y\|_{2,\infty}. \tag{3.19}$$

Combining (3.13), (3.16) and (3.19) with (3.11) leads to Theorem 3.2. *Q.E.D.*

3.2. Asymptotic Expansions for Iterated PGFE Derivatives

This subsection is devoted to the discussion of obtaining asymptotic expansions for the derivative of the exact solution to the problem (1.1). For this purpose, we first introduce the iterated PGFE derivative u'_{it}:

$$u'_{it}(t) := f(t,u(t)) + \int_0^t k(t,s,u(s))ds, \quad t\in I, \tag{3.20}$$

where $u(t)\in S_m^{(0)}(T_h)$ is the PGFE solution of the problem (1.1). Obviously, we have $P_h u'_{it} = u'$. According to [15], we know that u'_{it} is a better approximation to y' than u' itself. Therefore, we consequently want to generate an even better approximation by applying the Richardson extrapolation to u'_{it}.

Letting $e'_{it} := u'_{it} - y'$ be the error corresponding to the iterated PGFE derivative, we know from (2.1), (3.4) and Lemma 2.1 that

$$e'_{it}(t) = (Gu - Gy)(t) = p(t)e(t) + (Ke)(t) + O(h^{2m+2}),$$

which, together with (3.10), leads to

$$\begin{aligned} e'_{it}(t) &= p(t)(R_h y')(t) + (KR_h y')(t) \\ &+ p(t)(R_h G'e)(t) + (KR_h G'e)(t) + O(h^{2m+2}), \end{aligned} \tag{3.21}$$

where the linear Volterra integral operator K is defined by

$$(K\varphi)(t) := \int_0^t K(t,s)\varphi(s)ds.$$

Furthermore, it is easy to see from (2.6) and exchanging the order of integration with respect to s and τ that

$$\begin{aligned} |(KR_h\varphi)(t_n)| &= \left|\int_0^{t_n}\left(\int_\tau^{t_n} K(t_n,s)R(s,\tau)ds\right)(P_h - I)\varphi(\tau)d\tau\right| \\ &\le Ch^m\|\varphi\|_{0,\infty} \end{aligned} \tag{3.22}$$

This, together with Lemma 2.1, implies

$$|(KR_h G'e)(t_n)| \le Ch^m\|e\|_{0,\infty} \le Ch^{2m+1}\|y\|_{m+1,\infty}. \tag{3.23}$$

And then, in light of (3.14) and (3.23) we have at the mesh points of T_h that

$$e'_{it}(t_n) = p(t_n)(R_h y')(t_n) + (KR_h y')(t_n) + O(h^{2m+1}), \quad 1\le n\le N. \tag{3.24}$$

By (3.22) there holds

$$(KR_h y')(t_n) = \int_0^{t_n}\left(\int_\tau^{t_n} K(t_n,s)R(s,\tau)ds\right)(P_h - I)y'(\tau)d\tau, \tag{3.25}$$

which, together with (2.7), (3.13) and (3.24), leads to

Theorem 3.3. *Assume that the conditions of Theorem 3.1 hold. Then the iterated PGFE derivative error $e'_{it}(t)$ has the following asymptotic expansion at the mesh points of T_h:*

$$e'_{it}(t_n) = \beta(t_n)h^{2m} + O(h^{2m+1}), \quad 1\le n\le N, \tag{3.26}$$

where $\beta\in C^1(I)$ is invariable when the mesh is refined uniformly.

Again, we can show that the residual error of the asymptotic expansion for the iterated PGFE derivative produced by $u \in S_1^{(0)}(T_h)$ is $O(h^4)$ instead of $O(h^3)$. In fact, by means of (3.16), substituting (3.10) into (3.21) gives

$$\begin{aligned} e'_{it}(t) &= p(t)(R_h y')(t) + (KR_h y')(t) + p(t)(R_h G' R_h y')(t) \\ &\quad + (KR_h G' R_h y')(t) + p(t)((R_h G')^2 e)(t) + (K(R_h G')^2 e)(t) + O(h^4) \\ &= p(t)(R_h y')(t) + (KR_h y')(t) + p(t)(R_h G' R_h y')(t) + (KR_h G' R_h y')(t) + O(h^4). \end{aligned} \tag{3.27}$$

And then, we further see by (3.19) that there holds at the mesh points of T_h:

$$e'_{it}(t_n) = p(t_n)(R_h y')(t_n) + (KR_h y')(t_n) + (KR_h G' R_h y')(t_n) + O(h^4), \quad 1 \le n \le N. \tag{3.28}$$

Again, it follows from exchanging the order of integration with respect to s and τ that

$$\begin{aligned} (KR_h G' R_h y')(t_n) &= \int_0^{t_n} K(t_n, s) \left(\int_0^s R(s,\tau)(P_h - I)(G' R_h y')(\tau) d\tau \right) ds \\ &= \int_0^{t_n} A_1(t_n, s)(P_h - I)(G' R_h y')(s) ds, \end{aligned} \tag{3.29}$$

where

$$A_1(t,s) := \int_s^t K(t,\tau) R(\tau, s) d\tau.$$

Since (3.29) is of the form similar to (3.17), we obtain according to the same arguments as those for getting (3.19) that

$$\max_{1 \le n \le N} |(KR_h G' R_h y')(t_n)| \le Ch^4 \|y\|_{2,\infty}, \tag{3.30}$$

which, together with (3.25), (3.28) and (3.13) for the case of $m = 1$, leads to

Theorem 3.4. *Assume that the conditions of Theorem 3.2 hold. Then the iterated PGFE derivative error e'_{it} corresponding to the linear PGFE solution $u \in S_1^{(0)}(T_h)$ has the following asymptotic expansion at the mesh points of T_h:*

$$e'_{it}(t_n) = \beta(t_n) h^2 + O(h^4), \quad 1 \le n \le N, \tag{3.31}$$

where $\beta \in C^1(I)$ is invariable when the mesh is refined uniformly.

4. Richardson Extrapolation and A-posteriori Estimates

In this section, on the basis of the asymptotic expansions obtained in Section 3, we will show that much better approximations to the solution of the initial value problem (1.1) and its derivative can be generated efficiently by applying the Richardson extrapolation to the PGFE solutions. Also, in this section we will present some a-posteriori error estimators based on the Richardson extrapolation for both the PGFE solutions and the iterated PGFE derivatives, since it is very important for a finite element method to have a computable a-posteriori error bound by which we can access the actual numerical error of this method. This is in fact a natural application of the superconvergent approximations generated by the Richardson extrapolation.

4.1. Richardson Extrapolation

We now give a brief description of the Richardson extrapolation of the PGFE solutions and the iterated PGFE derivatives of the problem (1.1). In addition to a PGFE solution $u \in S_m^{(0)}(T_h)$ and an iterated PGFE derivative u'_{it}, we generate another PGFE solution $u^{1/2} \in S_m^{(0)}(T_{h/2})$ and another iterated PGFE derivative $\left(u_{it}^{1/2}\right)'$, where the partition

$$T_{h/2}: \quad 0 = t_0 < t_{1/2} < t_1 < t_{3/2} < \cdots < t_{N-\frac{1}{2}} < t_N = T$$

is formed such that $t_{k+1/2} := (t_k + t_{k+1})/2$ is the midpoint of σ_k for $0 \le k \le N-1$. Then it follows from Theorems 3.1-3.4 that

$$\begin{aligned}
&u^{1/2}(t_n) - y(t_n) = \alpha(t_n)\left(\frac{h}{2}\right)^{2m} + O(h^{2m+1}) \ (m \ge 2),\\
&u^{1/2}(t_n) - y(t_n) = \alpha(t_n)\left(\frac{h}{2}\right)^{2} + O(h^{4}),\\
&\left(u_{it}^{1/2}\right)'(t_n) - y'(t_n) = \beta(t_n)\left(\frac{h}{2}\right)^{2m} + O(h^{2m+1}) \ (m \ge 2),\\
&\left(u_{it}^{1/2}\right)'(t_n) - y'(t_n) = \beta(t_n)\left(\frac{h}{2}\right)^{2} + O(h^{4}).
\end{aligned}$$

Hence, applying the Richardson extrapolation once yields the following new approximations of higher accuracy:

$$\begin{aligned}
&u_m^{(e)}(t_n) := \frac{2^{2m}u^{1/2}(t_n) - u(t_n)}{2^{2m}-1} = y(t_n) + O(h^{2m+1}) \ (m \ge 2),\\
&u_1^{(e)}(t_n) := \frac{4u^{1/2}(t_n) - u(t_n)}{3} = y(t_n) + O(h^{4}),\\
&\left(u_{it,m}^{(e)}\right)'(t_n) := \frac{2^{2m}\left(u_{it}^{1/2}\right)'(t_n) - u_{it}'(t_n)}{2^{2m}-1} = y'(t_n) + O(h^{2m+1}) \ (m \ge 2),\\
&\left(u_{it,1}^{(e)}\right)'(t_n) := \frac{4\left(u_{it}^{1/2}\right)'(t_n) - u_{it}'(t_n)}{3} = y'(t_n) + O(h^{4}).
\end{aligned} \tag{4.1}$$

Note that these higher order approximations based on extrapolation are only applicable at the mesh points of T_h. However, we can show that an interpolation post-processing method can extend it globally to any point in the whole solution domain with only an almost negligible extra computational cost.

For ease of exposition, we demonstrate our idea mainly for the case of $m = 1$. To this end, we assume that T_h is obtained from T_{3h} with mesh size $3h$ by subdividing each element of T_{3h} into three elements (i.e., each element of T_{3h} is obtained by a combination of each 3-element in T_h), so that the number N of elements for T_h is a multiple of 3. Then, for any $\varphi \in C(I)$, we define a Lagrange interpolation $I_{3h}^3\varphi$ of degree 3 associated with the mesh T_{3h} according to the following conditions:

$$I_{3h}^3\varphi|_{\sigma_{k-1}\cup\sigma_k\cup\sigma_{k+1}} \in P_3, \quad k = 3l+1, \quad l = 0, 1, \cdots, \frac{N}{3} - 1$$

and

$$I_{3h}^3\varphi(t_i) = \varphi(t_i), \quad i = k-1, k, k+1, k+2 \quad (1 \le k \le N-2).$$

Similarly, for any $\varphi \in C(I)$ we can also define an interpolation $I_{(2m)h}^{2m}\varphi$ of degree $2m$ $(m \ge 2)$ associated with the mesh $T_{(2m)h}$. Then, following the procedure of Theorem 3.2 in [4] we have from (4.1)

Theorem 4.1. *Assume that the conditions of Theorem 3.2 hold. Then, the extrapolation approximation $u_1^{(e)}(t)$ satisfies*

$$\|I_{3h}^3 u_1^{(e)} - y\|_{0,\infty} \le Ch^4\|y\|_{4,\infty}.$$

Furthermore, if $f \in C^{2m}(I \times R)$ and $K \in C^{2m}(D \times R)$, then the extrapolation approximation $u_m^{(e)}(t)$, $m \ge 2$, satisfies

$$\|I_{(2m)h}^{2m} u_m^{(e)} - y\|_{0,\infty} \le Ch^{2m+1}\|y\|_{2m+1,\infty}.$$

Theorem 4.2. *Assume that $f \in C^4(I \times R)$ and $k \in C^4(D \times R)$, then the extrapolation approximation $\left(u_{it,1}^{(e)}\right)'$ satisfies*

$$\|I_{3h}^3 \left(u_{it,1}^{(e)}\right)' - y'\|_{0,\infty} \leq Ch^4 \|y\|_{5,\infty}.$$

Furthermore, if $f \in C^{2m+1}(I \times R)$ and $k \in C^{2m+1}(D \times R)$. Then, the extrapolation approximation $\left(u_{it,m}^{(e)}\right)'$, $m \geq 2$, satisfies

$$\|I_{(2m)h}^{2m} \left(u_{it,m}^{(e)}\right)' - y'\|_{0,\infty} \leq Ch^{2m+1} \|y\|_{2m+2}.$$

4.2. A-posteriori Error Estimates

Theorem 4.3. *Under the conditions of Theorem 3.2 we have*

$$\max_{1\leq k\leq N} \left|y(t_k) - u^{1/2}(t_k)\right| = \max_{1\leq k\leq N} \left|\frac{u(t_k) - u^{1/2}(t_k)}{3}\right| + O(h^4), \tag{4.2}$$

where $u \in S_1^{(0)}(T_h)$ and $u^{1/2} \in S_1^{(0)}(T_{h/2})$ are the linear PGFE solutions of (1.1).

Proof. It follows from (4.1) that

$$y(t_k) - u^{1/2}(t_k) = \frac{u^{1/2}(t_k) - u(t_k)}{3} + O(h^4),$$

which leads to

$$\max_{1\leq k\leq N} \left|y(t_k) - u^{1/2}(t_k)\right| = \max_{1\leq k\leq N} \left|\frac{u(t_k) - u^{1/2}(t_k)}{3}\right| + O(h^4).$$

Q.E.D.

We know from (4.2) that the computable estimate bound $\max_{1\leq k\leq N} \left|\frac{u(t_k) - u^{1/2}(t_k)}{3}\right|$ is the principal part of the PGFE error $\max_{1\leq k\leq N} \left|y(t_k) - u^{1/2}(t_k)\right|$, and can be used as an a-posteriori error indicator to obtain the bound of the actual error of a PGFE solution.

Analogously, from (4.1) we can also obtain:

$$\max_{1\leq k\leq N} \left|y(t_k) - u^{1/2}(t_k)\right| = \max_{1\leq k\leq N} \left|\frac{u(t_k) - u^{1/2}(t_k)}{2^{2m} - 1}\right| + O(h^{2m+1}), \quad m \geq 2,$$

under the conditions of Theorem 3.1;

$$\max_{1\leq k\leq N} \left|y'(t_k) - \left(u_{it}^{1/2}\right)'(t_k)\right| = \max_{1\leq k\leq N} \left|\frac{u'_{it}(t_k) - \left(u_{it}^{1/2}\right)'(t_k)}{3}\right| + O(h^4)$$

under the conditions of Theorem 3.2; and

$$\max_{1\leq k\leq N} \left|y'(t_k) - \left(u_{it}^{1/2}\right)'(t_k)\right| = \max_{1\leq k\leq N} \left|\frac{u'_{it}(t_k) - \left(u_{it}^{1/2}\right)'(t_k)}{2^{2m} - 1}\right| + O(h^{2m+1}), \quad m \geq 2,$$

under the conditions of Theorem 3.1.

5. Some Numerical Examples

In this section we present some numerical results which illustrate features of the Richardson extrpolation applied to the PGFE solutions. All the numerical solutions given here are generated by the PGFE methods for the nonlinear Volterra equation (1.1) in which

$$k(t,s,y) = sin(t) + 2s + cos(s)e^{y},$$
$$f(t,y) = 1 - e^{sin(t)} - t^2 + cos(t) + cos(t+2y) - cos(t+2sin(t)) - tsin(t)$$

so that $y(t) = sin(t)$ is the exact solution. In all of our computations, Newton's method is used to solve the nonlinear algebraic equations produced by the PGFE methods, and we have observed quadratic convergence in Newton iterations provided that initial guess and the exact solutions are close enough.

Example 1. The purpose of this example is to demonstrate the convergence rate of the numerical solutions generated by using Richardson extrapolation to post-process the PGFE solutions. As suggested by the analysis in Section 4, the Richardson extrapolation of the linear PGFE solutions at the mesh points is given by

$$u_1^{(e)}(t_j) = \frac{4u^{1/2}(t_j) - u(t_j)}{3},$$

and this approximation can be extended to the whole solution domain as $I_{3h}^3 u_1^{(e)}(t)$. Table 1 contains errors of these approximations. For the data here, we have

$$\left\| u_1^{(e)} - y \right\|_{\infty,h} := \max_{1\le j\le N} \left| u_1^{(e)}(t_j) - y(t_j) \right| \approx 0.0048\ h^{4.1070}$$

and

$$\left\| I_{3h}^3 u_1^{(e)} - y \right\|_{\infty} \approx 0.0525\ h^{4.0847},$$

which agree with the estimates given by (4.1) and Theorem 4.1 very well. As a comparison, we also have listed the errors of the linear PGFE solutions for various mesh size h in the second column of Table 1. Similarly, the numerical approximation generated by post-processing the PGFE solution in $S_2^{(0)}(T_h)$ with the Richardson extrapolation is given by

$$u_2^{(e)}(t_j) = \frac{16u^{1/2}(t_j) - u(t_j)}{15},$$

and its extension to the solution domain is given by $I_{4h}^4 u_2^{(e)}(t)$. Table 2 lists the errors in $I_{4h}^4 u_2^{(e)}(t)$. The data here obey

$$\left\| I_{4h}^4 u_2^{(e)} - y \right\|_{\infty} \approx 0.0584\ h^{5.1292},$$

which is within the prediction of Theorem 4.1.

The Richardson extrapolation can also be used to treat the iterated PGFE derivatives. Again, the approximation after the Richardson extrapolation is given similarly by

$$\left(u_{it,1}^{(e)}\right)'(t_j) = \frac{4\left(u_{it}^{1/2}\right)'(t_j) - u_{it}'(t_j)}{3}, \text{ for } u \in S_1^{(0)}(T_h).$$

The errors of this approximation are listed in Table 3 for a series of values of the mesh size h. The data here obey

$$\left\| \left(u_{it,1}^{(e)}\right)' - y' \right\|_{\infty,h} := \max_{1\le j\le N} \left| \left(u_{it,1}^{(e)}\right)'(t_j) - y'(t_j) \right| \approx 0.0101\ h^{4.1064}$$

and

$$\left\| I_{3h}^3 \left(u_{it,1}^{(e)}\right)' - y' \right\|_{\infty} \approx 0.0706\ h^{4.1035},$$

which are within the prediction of Theorem 4.2. Similar results are also observed for the iterated PGFE derivatives produced by the PGFE solutions in $S_2^{(0)}(T_h)$.

h	$\|u-y\|_{\infty,h}$	$\left\|u_1^{(e)}-y\right\|_{\infty,h}$	$\left\|I_{3h}^3 u_1^{(e)}-y\right\|_{\infty}$
1/12	$0.19102152544459 \times 10^{-3}$	$0.00013622033168\times 10^{-3}$	$0.00154012645492\times 10^{-3}$
1/24	$0.04772018024823\times 10^{-3}$	$0.00000848566617\times 10^{-3}$	$0.00009949256363\times 10^{-3}$
1/48	$0.01192785441850\times 10^{-3}$	$0.00000053068927\times 10^{-3}$	$0.00000631979069\times 10^{-3}$
1/96	$0.00298182683645\times 10^{-3}$	$0.00000003316936\times 10^{-3}$	$0.00000039818504\times 10^{-3}$
1/192	$0.00074544816342\times 10^{-3}$	$0.00000000207329 \times 10^{-3}$	$0.00000002498690\times 10^{-3}$
1/384	$0.00018636150678\times 10^{-3}$	N/A	N/A

Table 1. Errors of the PGFE solutions in $S_1^{(0)}(T_h)$ and those post-processed by the Richardson extrapolation.

h	$\|u-y\|_{\infty,h}$	$\left\|I_{4h}^4 u_2^{(e)}-y\right\|_{\infty}$
1/12	$0.46279185013376\times 10^{-5}$	$0.01202347537470\times 10^{-5}$
1/24	$0.05796509425938\times 10^{-5}$	$0.00037896221838\times 10^{-5}$
1/48	$0.00724929977644\times 10^{-5}$	$0.00001186785384\times 10^{-5}$
1/96	$0.00090627725719\times 10^{-5}$	$0.00000037106832\times 10^{-5}$
1/192	$0.00011328824732\times 10^{-5}$	$0.00000001159730\times 10^{-5}$
1/384	$0.00001416114312\times 10^{-5}$	N/A

Table 2. Errors of the PGFE solutions in $S_2^{(0)}(T_h)$ and those post-processed by the Richardson extrapolation.

h	$\|u'-y'\|_{\infty,h}$	$\left\|\left(u_{it,1}^{(e)}\right)'-y'\right\|_{\infty,h}$	$\left\|I_{3h}^3 \left(u_{it,1}^{(e)}\right)'-y'\right\|_{\infty}$
1/12	$0.44074885954326\times 10^{-3}$	$0.00028513859007\times 10^{-3}$	$0.00199203903806\times 10^{-3}$
1/24	$0.11040458526201\times 10^{-3}$	$0.00001773577163\times 10^{-3}$	$0.00012527323845\times 10^{-3}$
1/48	$0.02761270534613\times 10^{-3}$	$0.00000111215681\times 10^{-3}$	$0.00000784309895\times 10^{-3}$
1/96	$0.00690387766256\times 10^{-3}$	$0.00000006948664\times 10^{-3}$	$0.00000049043869\times 10^{-3}$
1/192	$0.00172602407744\times 10^{-3}$	$0.00000000434286\times 10^{-3}$	$0.00000003065725\times 10^{-3}$
1/384	$0.00043150782014 \times 10^{-3}$	N/A	N/A

Table 3. Errors of the iterated PGFE derivative given by $u \in S_1^{(0)}(T_h)$ and those post-processed by the Richardson extrapolation.

Example 2. This group of numerical results are presented to show the effectiveness of the a-posteriori error estimators based on the higher order approximations generated by the Richardson extrapolation applied to the PGFE solutions or the iterated PGFE derivatives.

Table 4 contains the actual errors of the PGFE solutions in the space $S_1^{(0)}(T_{h/2})$ and their estimates generated by $(u^{1/2}-u)/3$. Obviously, the difference between the actual error and its estimate given by the a-posteriori estimator approaches zero quickly. In fact, the linear regression shows that the difference listed in the 3rd column of Table 4 goes to zero at a rate $O(h^4)$ as predicted by Theorem 4.3. Similar results for PGFE solutions in the space $S_2^{(0)}(T_h)$ are given in Table 5, and we note that, for the quadratic PGFE solutions, the difference between the actual error and its estimate given by the a-posteriori estimator approaches zero so quickly that it has reached the machine accuracy for $h \leq 1/48$.

Similar results are also obtained for the iterated PGFE derivatives, see Table 6 and Table 7. In this situation, we again see that the a-posteriori error estimators based on the Richardson extrapolation work very well. In particular, the error estimator for the iterated PGFE derivative given by the quadratic PGFE solution seems to reach the machine accuracy very quickly as the mesh size becomes small.

h	$\left\|u^{1/2} - y\right\|_{\infty,h}$	$\left\|(u^{1/2} - u)/3\right\|_{\infty,h}$	difference
1/24	4.77201802482341e-005	4.7767115065452e-005	4.69348172179237e-008
1/48	1.19278544185031e-005	1.1930775276577e-005	2.92085807392614e-009
1/96	2.98182683644566e-006	2.98200919401914e-006	1.82357573486058e-010
1/192	7.45448163419482e-007	7.45459557675391e-007	1.13942559091268e-011
1/384	1.86361506782085e-007	1.86362218879133e-007	7.12097047994575e-013

Table 4. Actual errors of the PGFE solutions in the space $S_1^{(0)}(T_{h/2})$ and their estimates given by the a-posteriori estimator based on the Richardson extrapolation.

h	$\left\|u^{1/2} - y\right\|_{\infty,h}$	$\left\|(u^{1/2} - u)/15\right\|_{\infty,h}$	difference
1/24	9.37920169308271e-009	9.37929074777226e-009	8.90546895493713e-014
1/48	5.88760595832127e-010	5.88786708277667e-010	2.61124455392044e-014
1/96	3.67973429504787e-011	3.67975501921099e-011	2.07241631260347e-016
1/192	2.29893881709131e-012	2.29990841186615e-012	9.6959477483933e-016
1/384	1.40887301824932e-013	1.43914509938744e-013	3.02720811381126e-015

Table 5. Actual errors of the PGFE solutions in the space $S_2^{(0)}(T_{h/2})$ and their estimates given by the a-posteriori estimator based on the Richardson extrapolation.

h	$\left\|(u_{it,1}^{1/2})' - y'\right\|_{\infty,h}$	$\left\|\left((u_{it,1}^{1/2})' - u'_{it,1}\right)/3\right\|_{\infty,h}$	difference
1/24	0.0001101614448904	0.000110195804884287	3.43599938871646e-008
1/48	2.75981521045132e-005	2.76021443858336e-005	3.99228132034674e-009
1/96	6.9030291446337e-006	6.90322540049711e-006	1.96255863412112e-010
1/192	1.72595892078409e-006	1.72597291392407e-006	1.3993139980073e-011
1/384	4.31505404074883e-007	4.31506224455684e-007	8.20380800347329e-013

Table 6. Actual errors of the iterated PGFE derivatives produced by $u \in S_1^{(0)}(T_{h/2})$ and their estimates given by the a-posteriori estimator based on the Richardson extrapolation.

h	$\left\|(u_{it,2}^{1/2})' - y'\right\|_{\infty,h}$	$\left\|\left((u_{it,2}^{1/2})' - u_{it,2}'\right)/15\right\|_{\infty,h}$	difference
1/24	1.96517233597149e-008	1.96593437233143e-008	7.62036359939493e-012
1/48	1.23651611172448e-009	1.23670978643039e-009	1.93674705902457e-013
1/96	7.72794050973857e-011	7.72824471084732e-011	3.04201108747551e-015
1/192	4.83080242474898e-012	4.8332301124295e-012	2.42768768051389e-015
1/384	2.97317725994617e-013	3.02343335552753e-013	5.02560955813657e-015

Table 7. Actual errors of the iterated PGFE derivatives produced by $u \in S_2^{(0)}(T_{h/2})$ and their estimates given by the a-posteriori estimator based on the Richardson extrapolation.

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