Journal of Computational Mathematics, Vol.19, No.6, 2001, 583–590.

# PICARD ITERATION FOR NONSMOOTH EQUATIONS*1)

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**Abstract**

This paper presents an analysis of the generalized Newton method, approximate Newton methods, and splitting methods for solving nonsmooth equations from Picard iteration viewpoint. It is proved that the radius of the weak Jacobian (RGJ) of Picard iteration function is equal to its least Lipschitz constant. Linear convergence or superlinear convergence results can be obtained provided that RGJ of the Picard iteration function at a solution point is less than one or equal to zero. As for applications, it is pointed out that the approximate Newton methods, the generalized Newton method for piecewise $C^1$ problems and splitting methods can be explained uniformly with the same viewpoint.

*Key words*: Nonsmooth equations, Picard iteration, Weak Jacobian, Convergence.

## 1. Introduction

Consider the following nonsmooth equations

$$F(x) = 0 \tag{1}$$

where $F : R^n \to R^n$ is Lipschitz continuous. A lot of work has been done and is beng done to deal with (1). It is basicly a generalization of the classic Newton method [8,10,11,14], Newton-like methods[1,18] and quasi-Newton methods [6,7]. As it is discussed in [7], the latter, quasi-Newton methods, seem to be limited when applied to nonsmooth case in that a bound of the deterioration of updating matrix can not be maintained without smoothness assumption of $F$ at solution points. Therefore, more efforts have been made to discuss the former. The discussions are mainly on the local and global convergence under the assumption that $F$ is semismooth.

An interesting discovery is that a majority of nonsmooth equations discussed in the recent years are almost either piecewise $C^1$ or well structured. The former mainly originate from nonlinear complementarity problems, variational inequality and nonlinear programming problems, see [4,12,15-17], while the latter from nonsmooth partial differential equations [5]. This encourages one to extend discussions on some specific problems which are either more than semismooth or well structured but non-semismooth.

In this paper, we are motivated to discuss these problems. We give a unified analysis on the generalized Newton method, approximate Newton methods and splitting methods from Picard iteration viewpoint.

In section 2, we simply review some basic definitions and results related to nonsmooth equations. A kind of generalized norm is introduced for the convex set-valued family. In section 3, we set up a relationship between the Lipschitz constant and the radius of the weak Jacobian of Picard iteration function. This is a generalization of the classic results in the smooth case. In

* Received December 9, 1997; Final revised January 15, 2001.
1)The research was partly supported by NNSFC(No. 19771047) and NSF of Jiangsu Province(BK97059).

section 4, we try to explain the approximate Newton methods, the generalized Newton method for piecewise $C^1$ problems and the splitting methods from Picard iteration viewpoint.

## 2. Generalized Jacobian

we use $\|\cdot\|$ to denote 2-norm of vectors in $R^n$ and induced norm of matrices in $n\times n$ matrix space $L(R^n)$. We denote the set of points of $R^n$ at which $F$ is differentiable by $D_F$. We let $S(x,\delta)$ denote a closed ball in $R^n$ with center $x$ and radius $\delta$.

We assume throughout that $F$ is Lipschitz continuous in $R^n$ in the sence that for every $x\in R^n$, there exist $L>0$, and $\delta>0$, such that

$$\| F(y)-F(z) \|\leq L \| y-z \| \tag{2}$$

for all $y,z\in S(x,\delta)$. Here $L$ is called Lipschitz constant of $F$ at $x$.

According to the Rademacher theorem, $F$ is differentiable almost everywhere in $R^n$, and the generalized Jacobian of $F$ at $x$ was defined by Clarke [2] as follows:

$$\partial F(x)=conv[\lim_{x_i\to x, x_i\in D_F}\nabla F(x_i)].$$

Here and later, $\nabla F(x)$ denotes the Jacobian of $F$ at $x\in D_F$, and "conv" denotes convex hull.

**Proposition 2.1.** *(Proposition 2.6.2, Clarke [2]) $\partial F(x)$ is compact and upper semicontinuous in the sence that for every $\varepsilon>0$, there exists $\delta>0$, such that for all $y\in S(x,\delta)$,*

$$\partial F(y)\subset\partial F(x)+B,$$

*where $B$ denotes an open unit ball in $L(R^n)$.*

A useful subset of $\partial F(x)$ was defined by Qi[13] as follows:

$$WF(x)=\lim_{x_i\to x, x_i\in D_F}\nabla F(x_i)$$

Here, we call it weak Jacobian. It is clear that

$$\partial F(x)=convWF(x)$$

**Proposition 2.2.** *(Proposition 2.1[18]) $WF(x)$ is compact and upper semicontinuous in the sense that for every $\varepsilon>0$, there exists $\delta>0$, such that for all $y\in S(x,\delta)$,*

$$WF(y)\subset WF(x)+\varepsilon B,$$

*where $B$ denotes an open unit ball in $L(R^n)$.*

Now consider convex sets in $L(R^n)$. Let $A$, $B$ be two convex sets in $L(R^n)$, $\alpha$ is a scalar, define the operations

$$A+B=[c=a+b: a\in A, b\in B];$$

$$\alpha A=[c=\alpha a: a\in A].$$

Then, all these sets form a convex family with scalar multiplication and addition, we denote it by $\Gamma$. Taking an arbitrary element $A\in\Gamma$, we define

$$\| A \|=\sup_{\alpha\in A}\| \alpha \|.$$

We call $\| \partial F(x) \|$ the radius of the generalized Jacobian of $F$ at $x$ (RGJ in brief). Similarly, we can define the norm of $WF(x)$ by

$$\| WF(x) \|=\sup_{W\in WF(x)}\| W \|.$$

**Proposition 2.3.** *For every $x\in R^n$, $\| \partial F(x) \|=\| WF(x) \|$.*

*Proof.* It suffices to prove that

$$\| \partial F(x) \|\leq\| WF(x) \|.$$

From the Caratheodory theorem, it follows that for every $V \in \partial F(x)$, there exists an integer $\overline{m}$, real numbers $\alpha_i \in [0,1], \Sigma_{i\in\overline{m}}\alpha_i = 1$, and matrices $W_i \in WF(x), i \in \overline{m}$, such that $V = \Sigma_{i\in\overline{m}}\alpha_i W_i$. Clearly,

$$\| V \| \leq \sum_{i\in\overline{m}} \alpha_i \| W_i \| \leq \sup_{W\in WF(x)} \| W \|$$

The proof is complete.

## 3. Picard Iteration

Actually, Newton iteration is only a subclass of the Picard iteration. In this section, we give an analysis on a nonsmooth version of the classic Picard iteration for nonsmooth equations.

The classic Picard iteration takes a form as

$$x_{k+1} = \phi(x_k) \tag{3}$$

where $\phi : R^n \to R^n$ is called iteration function.

From now on, we suppose $\phi$ is Lipschitz continuous satisfying (2). Let $x^*$ denote a solution of

$$x = \phi(x) \tag{4}$$

It is well known that $\{x_k\}$ produced by (3) converges to $x^*$ locally provided that $\phi(x)$ is contractive in the neighborhood of $x^*$.

Let $L(x,\alpha)$ be the Lipschitz constant of $\phi$ in $S(x,\alpha)$ in the sense that

$$\| \phi(y) - \phi(z) \| \leq L(x,\alpha) \| y - z \| \tag{5}$$

for all $y, z \in S(x,\alpha)$.

**Definition 3.1.** *We call $l(x,\alpha)$ the exact Lipschitz constant of $\phi$ in $S(x,\alpha)$ if $l(x,\alpha) \leq L(x,\alpha)$ for all $L(x,\alpha)$ satisfying (5).*

**Proposition 3.1.** *The limit $\lim_{\alpha\to 0} l(x,\alpha)$ exists, denote $\lim_{\alpha\to 0} l(x,\alpha) = l(x,0)$.*

**Theorem 3.1.** *For every $x \in R^n$,*

$$\| W\phi(x) \| = l(x,0).$$

*Proof.* By definition, for every $W \in W\phi(x)$ there exists a sequence $\{x_k\} \to x, x_k \in D_\phi \bigcap S(x,\alpha)$, such that $\nabla\phi(x_k) \to W$, as $k \to \infty$ (We may use a subsequence if necessary). For every fixed $h$, $\| h \| = 1$,

$$\| \nabla\phi(x_k)h \| = \lim_{t\to 0} \| \phi(x_k + th) - \phi(x_k) \| / t \leq l(x,\alpha),$$

thereby $\| \nabla\phi(x_k) \| \leq l(x,\alpha)$, and $\| W \| \leq l(x,\alpha)$. Letting $\alpha \to 0$, we have $\| W \| \leq l(x,\alpha)$, which implies $\| W\phi(x) \| \leq l(x,0)$.

Conversely, since $\phi$ is Lipschitz continuous, it follows from the Rademacher theorem that, for arbitrary $y, z \in S(x,\alpha)$,

$$\phi(y) - \phi(z) = \int_0^1 \nabla\phi(z + t(y-z))(y-z)dt$$

By Proposition 2.2, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$W\phi(y) \subset W\phi(x) + \varepsilon B$$

for all $y \in S(x,\delta)$. Thereby for $\alpha < \delta, \nabla\phi(z + t(y-z)) \in W\phi(x) + \varepsilon B$, and consequently,

$$\| \phi(y) - \phi(z) \| \leq (\| W\phi(x) \| + \varepsilon \| B \|) \| y - z \| .$$

Let $L(x, \varepsilon) = \| W\phi(x) \| + \varepsilon \| B \|$. It is clear that

$$\lim_{\varepsilon \to 0} L(x, \varepsilon) = \| W\phi(x) \|,$$

which implies

$$l(x, 0) \leq \| W\phi(x) \| .$$

This completes the proof.

**Corollary 3.1.** $\| \partial\phi(x) \| = l(x, 0)$.

The corollary says that RGJ of $\phi$ at $x$ is equal to the least Lipschitz constant of $\phi$ at $x$. This is a generalization of the results in the smooth case.

**Corollary 3.2.** *The following statements are equivalent.*

a) *$\phi$ is strictly differentiable at $x$ with $\nabla\phi(x) = 0$;*

b) *$\partial\phi(x) = [0]$;*

c) *$W\phi(x) = [0]$;*

d) *$l(x, 0) = 0$.*

**Definition 3.2.** *We say $\phi$ is strongly consistent Lipschitz continuous in the neighborhood of $x$ if $l(x, 0) = 0$.*

**Theorem 3.2.** *Let $x^*$ be a solution point of (4). Then for $\alpha$ sufficiently small and arbitrary $x_0 \in S(x^*, \alpha), \{x_k\}$ produced by (3) is well defined and converges to $x^*$ at least linearly in the sense that*

$$\| x_{k+1} - x^* \| \leq l(x^*, \alpha) \| x_k - x^* \|$$

*if $l(x^*, \alpha) < 1$. If, in addition, $\phi$ is strongly consistent Lipschitz continuous in the neighborhood of $x^*$, then $\{x_k\}$ converges to $x^*$ superlinearly.*

We omit the proof since it is obvious.

**Theorem 3.3.** *Let $x^*$ be a solution point of (4). $\{x_k\}$ is produced by (3). Then*

a) *if $\| W\phi(x^*) \| < 1$, then $\{x_k\}$ converges to $x^*$ locally and linearly;*

b) *if, in addition, $\| W\phi(x^*) \| = 0$, then $\{x_k\}$ converges to $x^*$ locally and superlinearly.*

*Proof.* It is a direct consequence of Theorem 3.1 and Theorem 3.2.

## 4. Applications

First we consider the generalized Newton iteration of Qi and Sun [11]. The iteration function may be written as:

$$\phi_{GN}(x) = x - V(x)^{-1} F(x), V(x) \in \partial F(x). \tag{6}$$

We find it difficult to apply Theorem 3.1-3.2 to $\phi_{GN}(x)$ in that $V(x)$ is not necessarily Lipschitz continuous. However, if $V(x)$ is replaced by a Lipschitz continuous invertible matrix, then $\phi_{GN}(x)$ is Lipschitz continuous.

### 4.1. Approximate Newton Methods

Let $J(x)$ be an approximation of $V(x)$, $J(x)$ is Lipschitz continuous and nonsingular. We consider

$$x_{k+1} = \phi_J(x_k) \tag{7}$$

where

$$\phi_J(x) = x - J(x)^{-1} F(x).$$

Iteration (7) is a simplified form of approximate Newton methods.

Let $D_J$ denote the set of points at which $J(\cdot)$ is differentiable. Then

$$\nabla\phi_J(x) = I - \nabla(J(x)^{-1}) \times F(x) - J(x)^{-1} \nabla F(x)$$

for $x \in D_J \bigcap D_F$, where

$$\nabla(J(x)^{-1}) \times F(x) = \sum_{i \in \overline{n}} f_i \nabla(J(x)^{-1} e_i),$$

$e_i$ is i-th unit vector in $R^n$. $f_i$ is the i-th component of $F(x)$.

**Assumption 4.1.** (A4.1) There exists a constant $\gamma(x^*, \alpha) > 0$, such that

$$\| \nabla(J(x)^{-1}) \times F(x) \| \leq \gamma(x^*, \alpha)$$

for $x \in S(x^*, \alpha) \bigcap D_J$.

**Theorrm 4.1.** *Let $x^*$ be a solution point of (4). Suppose that A4.1 holds. then*

a) *if there exists $\rho(x^*, \alpha) > 0$ such that*

$$\| I - J(x)^{-1} \nabla F(x) \| \leq \rho(x^*, \alpha)$$

*for all $x \in S(x^*, \alpha) \bigcap D_J \bigcap D_F$ and there exist $\alpha_0 > 0, \beta \in [0, 1)$, such that $\gamma(x^*, \alpha) + \rho(x^*, \alpha) < 1 - \beta$, for all $0 < \alpha < \alpha_0$. Then the iterates generated by (7) are locally and linearly convergent to $x^*$;*

b) *if, in addition,*

$$\lim_{\alpha \to 0} \gamma(x^*, \alpha) + \rho(x^*, \alpha) = 0$$

*then the iterates produced by (7) are locally and superlinearly convergent to $x^*$.*

*Proof.* It follows from Theorem 3.1 that for $0 < \alpha < \alpha_0, l(x^*, \alpha) < \gamma(x^*, \alpha) + \rho(x^*, \alpha) < 1 - \beta$. The rest follows directly from Theorem 3.3. The proof is complete.

Generally, We suppose that $\lim_{\alpha \to 0} \gamma(x^*, \alpha) = 0$. Hence, the behavior of $\{x_k\}$ generated by $\phi_J(x)$ depends mainly on $I - J(x)^{-1} \nabla F(x)$. The more $J(x)$ approximates $\nabla F(x)$ in the neighborhood of $x^*$, the faster the iterates $\{x_k\}$ generated by (7) converges to $x^*$.

**Remark 4.1.** a) It seems that the simple generalized Newton iteration does not work, since in general

$$\|I - \nabla F(x_0)^{-1} \nabla F(x)\| \not\leq 1, \forall x \in S(x^*, \alpha) \bigcap D_F.$$

b) We may extend our discussion to more general case where $J(x)^{-1}$ is replaced by a Lipschitz continuous matrix $M(x)$, *i.e.*

$$\phi_M(x) = x - M(x) F(x) \tag{8}$$

Similar conclusion can be reached. For smooth case, (8) was discussed by J.E.Dennis Jr.[3].

c) Theorem 4.1 is established without an assumption of semismoothness. Actually we extend our discussion from different point of view of Qi[12].

The assumption for iteration function (6) is strong, *i.e.*, the nonsingularity assumption of $\partial F(x)$ can not usually be satisfied. Qi[13] and Xu[18] suggested using the following iteration

$$\phi_{WN}(x) = x - W(x)^{-1} F(x), W(x) \in WF(x), \tag{9}$$

where $WF(x)$ is the weak Jacobian of $F$ at $x$. Locally superlinear results were obtained in [13], and [18].

### 4.2. Generalized Newton Methods for Piecewise $C^1$ Equations

As it is pointed by X. Chen and L.Qi in [1], a majority of nonsmooth equations arise originally either from nonlinear complementarity, variational inequality and mathematical programming problems, or from nonsmooth partial differential equations. For the former, the functions involved are to some extent piecewise $C^1$; for the latter, the problems usually have some special structure. In this section, we extend our discussion on the application of the theory developed in Section 3 to the former.

A strict definition of a piecewise $C^1$ function was first given by Kojima and Shindo [6] as follows:

**Definition 4.1.** *Let $F : R^n \to R^n$ be a continuous function. $F$ is a $PC^1$ function if there exists a countable family $[U_i : i \in \Lambda]$ of closed subsets of $R^n$ such that*

(a) *$cl(intU_i) = U_i$ for every $i \in \Lambda$;*

(b) *$(intU_i)\bigcap(intU_j) = \emptyset$ whenever $i, j \in \Lambda$ and $i \neq j$;*

(c) *$\bigcup_{i\in\Lambda} U_i = R^n$;*

(d) *$[U_i : i \in \Lambda]$ has locally finite property, i.e., for any $x \in R^n$, there exists an open neighborhood $V$ of $x$ such that $[i : V \bigcap U_i \neq \emptyset]$ is finite;*

(e) *for each $i \in \Lambda$ the restriction $F|U_i$ is a $C^1$ function. More precisely, there exists $C^1$ function $F_i$ from an open neighborhood of $U_i$ into $R^n$ such that $F(x) = F_i(x)$ for any $x \in U_i$. The family $[U_i : i \in \Lambda]$ is called a subdivision of $R^n$ and each $U_i$ a piece. We call all boundaries kinks.*

Let $I(x) = [i \in \Lambda, F(x) = F_i(x)]$.

**Assumpion 4.2.** (A4.2) Let $F : R^n \to R^n$ be a $PC^1$ function. $x^*$ is a solution of (1). Suppose that

(a) $\nabla F(x)$ is Lipschitz continuous in $U_i$, for every $i \in \Lambda$;

(b) $\nabla F_i(x^*)$ is nonsingular for every $i \in I(x^*)$.

**Proposition 4.1.** *Under A4.2, there exists $\alpha_0 > 0$, such that $\nabla F_i(x)^{-1}$ exists and is Lipschitz continuous for $x \in S(x^*, \alpha_0)\bigcap U_i, i \in I(x^*)$.*

*Proof.* We only show that there exists $\alpha_0 > 0$, such that $\nabla F_i(x)^{-1}$ is Lipschitz continuous in $S(x^*, \alpha_0)\bigcap U_i, i \in I(x^*)$.

By assumption, $\nabla F_i(x^*)^{-1}$ exists for $i \in I(x^*)$. Let $C$ be a positive constant such that

$$\| \nabla F_i(x^*)^{-1} \| \leq C, \text{ for } i \in I(x^*).$$

From Banach perturbation theorem, it follows that $\alpha_0$ sufficiently small, $\nabla F_i(x)^{-1}$ exists and

$$\| F_i(x)^{-1} \| \leq 2C, \text{ for all } x \in S(x^*, \alpha_0)\bigcap U_i, i \in I(x^*).$$

Now let $x, y$ be arbitrary in $S(x^*, \alpha_0)\bigcap U_i, i \in I(x^*)$, then

$$\| \nabla F_i(x)^{-1} - \nabla F_i(y)^{-1} \| \leq 4C^2 \| \nabla F_i(x) - \nabla F_i(y) \| \leq \overline{C} \| x - y \|,$$

where $\overline{C}$ is a constant. The proof is complete.

Suppose that A4.2 holds. Then follows from Proposition 4.1 that $\nabla F(x)^{-1}$ is almost everywhere differentiable in $S(x^*, \alpha_0)\bigcap intU_i, i \in I(x^*)$.

For the sake of convenience, we consider only column of $\nabla F(x)^{-1}$, $\nabla F(x)^{-1}e_1$ say. We denote it by $h(x)$. Then for $x \in intU_i \bigcap S(x^*, \alpha_0), i \in \Lambda$, the Clarke's generalized Jacobian of $h(x)$ exists. When $x$ belongs to kinks, we define

$$\partial h(x) = conv[\lim_{x_i \in D_h; x_i \to x} \nabla h_i(x)]$$

Under A4.2, $\partial h(x)$ is a nonempty compact set.

**Theorem 4.2.** *Let $x^*$ be a solution point of (1). Suppose A4.2 holds. Then there exists a positive constant $\alpha_0$, such that if $x_0 \in S(x^*, \alpha_0)$, then the sequence generated by (9) is well defined and converges superlinearly to $x^*$.*

*Proof.* By Theorem 3.3, it suffices to prove that $\| \partial\phi_{WN}(x^*) \| = 0$. By Proposition 4.1, there exists $\alpha_0 > 0$ such that $\nabla F(x)^{-1}$ exists and Lipschitz continuous for $x \in S(x^*, \alpha_0)\bigcap U_i, i \in I(x^*)$. Thereby, $\phi_{WN}$ is Lipschitz continuous at $x \in S(x^*, \alpha_0)\bigcap U_i, i \in I(x^*)$, and

$$\nabla\phi_{WN}(x) = I - \nabla(\nabla F(x)^{-1})F(x) - \nabla F(x)^{-1}\nabla F(x)$$

$$= -\nabla(\nabla F(x)^{-1})F(x) = -\sum_{i=1}^{n} f_i(x)\nabla h_i(x)$$

for $x \in S(x^*, \alpha_0) \bigcap intU_i \bigcap D_{\nabla F}, i \in I(x^*)$, where $h_i(x) = \nabla F(x)^{-1} e_i$. As a consequence,

$$\partial \phi_{WN}(x^*) = conv[\lim_{x_i \in D; x_i \to x^*} \sum_{i=1}^{n} f_i(x) \nabla h_i(x)] = 0,$$

where $D = S(x^*, \alpha_0) \bigcap intU_i \bigcap D_{\nabla F}, i \in I(x^*)$. The rest follows directly from Theorem 3.3. The proof is complete.

### 4.3. Splitting Methods

We now turn to discuss nonsmooth equation derived from partial differential equations. Consider the nonsmooth partial differential equations,

$$\begin{cases} -\triangle u + T(u) = P(x,y), & in \quad the \quad domain \quad \Omega \subset R^2 \\ u = Q(x,y), & on \quad the \quad boundary \quad \partial\Omega \end{cases}$$

Where $T(u)$ is not differentiable. Using finite difference method or finite element method to discretize the above equations, we may obtain a system of

$$F(x) = Ax + T(x) = 0, x \in R^n,$$

where $A$ belongs to $L(R^n)$. Using Krasnoselskii-Zincenko iteration, we have

$$x_{k+1} = x_k - A^{-1}F(x_k) = -A^{-1}T(x_k). \tag{10}$$

**Theorem 4.3.** *Let $\overline{x} \in R^n$. Suppose that $T(x)$ is Lipschitz continuous in the neighborhood of $\overline{x}$ and*

$$\| A^{-1}T(\overline{x}) \| < 1.$$

*then there exists $\alpha_0 > 0$, such that for $x_0 \in S(\overline{x}, \alpha_0)$ the iterates generated by (10) converge to $\overline{x}$ linearly. In addition, if $\| A^{-1}T(\overline{x}) \| = 0$, then the iterates converge to $\overline{x}$ superlinearly.*

We shall not present a proof since it is a direct consequence of Theorem 3.3.

More generally, if $F(x)$ can be splitted into the following form

$$F(x) = A(x) + T(x) = 0, x \in R^n \tag{11}$$

where $A(x)$ is nonsingular and continuously differentiable, while $T(x)$ is Lipschitz continuous. Then using Krasnoselskii-Zincenko iteration for (11), we have

$$x_{k+1} = x_k - A(x_k)^{-1}F(x_k) = -A(x_k)^{-1}T(x_k). \tag{12}$$

Convergence results for (12) can be easily obtained.

An important application of the splitting methods is in nonlinear complementarity problem.

**Example 4.1.** (nonlinear complementarity problem)

(NCP) find $x \in R^n$ such that $x \geq 0, F(x) \geq 0, x^T F(x) = 0$,
where $F : R^n \to R^n$ belongs to $C^2$.

Pang [8-9] transformed (NCP) into the following equation problem:

$$H(x) = min(x, F(x)) = 0, \tag{13}$$

where "min" denote componentwise minimum.

With a slight transformation of (13), we get a splitting from

$$H(x) = x + min(0, F(x) - x) = 0. \tag{14}$$

Let $T(x) = min(0, F(x) - x)$. It is easy to verify that $T(x)$ is Lipschitz continuous satisfying (2). Using Krasnoselskii-Zincenko iteration to (14), we have

$$x_{k+1} = x_k - H(x_k) = -T(x_k),$$

which is a special case of (10).

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