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A CLASS OF NEW PARALLEL HYBRID ALGEBRAIC MULTILEVEL ITERATIONS*[1)]

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Abstract

For the large sparse system of linear equations with symmetric positive definite block coefficient matrix resulted from suitable finite element discretization of the second-order self-adjoint elliptic boundary value problem, by making use of the algebraic multilevel iteration technique and the blocked preconditioning strategy, we construct preconditioning matrices having parallel computing function for the coefficient matrix and set up a class of parallel hybrid algebraic multilevel iteration methods for solving this kind of system of linear equations. Theoretical analyses show that, besides much suitable for implementing on the high-speed parallel multiprocessor systems, these new methods are optimal-order methods. That is to say, their convergence rates are independent of both the sizes and the levels of the constructed matrix sequence, and their computational workloads are bounded by linear functions in the order number of the considered system of linear equations, respectively.

Key words: Elliptic boundary value problem, System of linear equations, Symmetric positive definite matrix, Multilevel iteration, Parallel method.

1. Introduction

Consider the large sparse system of linear equations

$$Ax = b, \tag{1.1}$$

where, for a fixed positive integer α, $A \in L(R^n)$ is a symmetric positive definite (SPD) matrix, having the blocked form

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1\alpha} \\ \vdots & \ddots & \vdots \\ A_{\alpha 1} & \cdots & A_{\alpha\alpha} \end{pmatrix}, \quad A_{ij} \in L(R^{n_j}, R^{n_i}), \quad i,j = 1,2,\cdots,\alpha; \tag{1.2}$$

$x, b \in R^n$ are the unknown and the known vectors, respectively, having the corresponding blocked forms

$$\begin{cases} x = (x_1^T, x_2^T, \cdots, x_\alpha^T)^T, & x_i \in R^{n_i}, \\ b = (b_1^T, b_2^T, \cdots, b_\alpha^T)^T, & b_i \in R^{n_i}, \end{cases} \qquad i = 1,2,\cdots,\alpha; \tag{1.3}$$

$n_i (n_i \le n; i = 1,2,\cdots,\alpha)$ are α given positive integers, satisfying $\sum_{i=1}^{\alpha} n_i = n$. This system of linear equations often arises in suitable finite element discretizations of many second-order

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self-adjoint elliptic boundary value problems. For details, we refer to [3-6]. Therefore, to study efficient numerical methods for getting the approximate solution of (1.1) has both theoretical and applicable meanings.

There has been a vast amount of research literature on iterative methods for solving the system of linear equatons (1.1). Among these methods, the recently developed algebraic multilevel iteration (AMLI) methods (see [3-5, 7, 9-10, 15-17]) are considerably applicable and efficient, because they are optimal-order methods in the sense that their convergence rates are independent of both the sizes and the level numbers of the grids, and their computational workloads are bounded by linear functions about the stepsizes of the finest grids.

To suit the requirement of the parallel high-speed multiprocessor systems, the authors of [6, 8, 18] presented several parallel algebraic multilevel iteration methods for solving the system of linear equations (1.1), which are, substantially, the parallelized variants of the existing AMLI methods originally presented by Axelsson and Vassilevski [3-5] and further studied by Vassilevski [17] and Bai [7], respectively. These parallel algebraic multilevel iteration methods not only inherit the intrinsic advantages of the AMLI methods, but also have nice parallelism. Therefore, they are very suitable for solving the system of linear equations (1.1) on the parallel computing environments.

Through reasonable combinations of the AMLI technique and the block strategy, in this paper we establish a class of new parallel hybrid AMLI methods based upon the abovementioned existing results. Compared with the parallel AMLI methods in [6, 8, 18], these novel ones have less computational complexities. Hence, they can achieve higher parallel computational efficiency. In a careful way, we estimate the relative condition numbers of the new parallel preconditioners and calculate the computational workloads of the resulted parallel hybrid algebraic multilevel preconditioning methods. We demonstrate that the new parallel algebraic multilevel iteration methods are optimal-order methods for both two-dimensional (2-D) and three-dimensional (3-D) problem domains. That is to say, their computational workloads are proportional to the dimension of the linear system (1.1), and their relative condition numbers of the preconditioners are not only independent of the regularity of the solution, but also bounded uniformly with respect to possible jumps of the coefficients of the second-order self-adjoint elliptic boundary value problem as long as these jumps occur only across edges (faces in 3-D) of elements from the coarsest triangulation. At last, we also formulate adaptive procedures for the new parallel hybrid algebraic multilevel iteration methods in order to construct the involved polynomials after each group of fixed recursion steps of the preconditioners. Therefore, these polynomials can vary from one group of the fixed recursion steps to the next.

2. Constructions of the New Methods

Denote $\Lambda = \{1, 2, \cdots, \alpha\}$. For a fixed positive integer l, starting from the blocked matrix $A \in L(R^n)$ in (1.2), we construct a matrix sequence $\{A^{(k)}\}_{k=0}^{l}$ in accordance with the following rule:

$$\begin{cases} A^{(l)} = A, \quad A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & \cdots & A_{1\alpha}^{(k)} \\ \vdots & \ddots & \vdots \\ A_{\alpha 1}^{(k)} & \cdots & A_{\alpha\alpha}^{(k)} \end{pmatrix}, \qquad A_{ij}^{(k)} \in L(R^{n_j^{(k)}}, R^{n_i^{(k)}}), \\ i, j = 1, 2, \cdots, \alpha; \quad k = 0, 1, \cdots, l, \end{cases}$$

where

$$A_{ii}^{(k)} = \begin{pmatrix} C_{ii}^{(k)} & E_{ii}^{(k)} \\ E_{ii}^{(k)^T} & A_{ii}^{(k-1)} \end{pmatrix}, \quad A_{ij}^{(k)} = \begin{pmatrix} C_{ij}^{(k)} & E_{ij}^{(k)} \\ F_{ij}^{(k)} & A_{ij}^{(k-1)} \end{pmatrix}, \qquad i \neq j, \quad i, j \in \Lambda,$$

and $\{n_i^{(k)}\}_{k=0}^{l} (i = 1, 2, \cdots, \alpha)$ are α positive integer sequences satisfying

$$n_i = n_i^{(l)} > n_i^{(l-1)} > \cdots > n_i^{(0)}, \qquad i = 1, 2, \cdots, \alpha.$$

Evidently, $A_{ij}^{(k)^T} = A_{ji}^{(k)} (i \neq j, i, j \in \Lambda)$, and $A^{(k)}$ and $A_{ii}^{(k)} (i \in \Lambda)$ are SPD. For each $k \in \{1, 2, \cdots, l\}$ and $i \in \Lambda$, denote the Schur decomposition of $A_{ii}^{(k)}$ as

$$A_{ii}^{(k)} = \begin{pmatrix} C_{ii}^{(k)} & 0 \\ E_{ii}^{(k)^T} & S_{ii}^{(k)} \end{pmatrix} \begin{pmatrix} I & C_{ii}^{(k)^{-1}} E_{ii}^{(k)} \\ 0 & I \end{pmatrix},$$

with $S_{ii}^{(k)} = A_{ii}^{(k-1)} - E_{ii}^{(k)^T} C_{ii}^{(k)^{-1}} E_{ii}^{(k)}$ the Schur complement of $A_{ii}^{(k)}$. Let $B_{ii}^{(k)}$ be SPD matrices, which approximate the matrices $C_{ii}^{(k)}$, respectively, and $p_{\nu_i}^{(i)}(t)$ be a polynomial of order ν_i, which satisfies $0 \le p_{\nu_i}^{(i)}(t) < 1$ $(0 < t \le 1)$ and $p_{\nu_i}^{(i)}(0) = 1$. If we define

$$Q_{\nu_i - 1}^{(i)}(t) = (1 - p_{\nu_i}^{(i)}(t))/t, \tag{2.1}$$

then $Q_{\nu_i-1}^{(i)}(t)$ is a polynomial of order $(\nu_i - 1)$ satisfying

$$Q_{\nu_i-1}^{(i)}(t) > 0 \quad (0 < t \le 1), \quad Q_{\nu_i-1}^{(i)}(0) = -\frac{d}{dt} p_{\nu_i}^{(i)}(t)\Big|_{t=0}. \tag{2.2}$$

With the above preparations, the new parallel preconditioners can be readily constructed through integrating the sub-preconditioners with respect to the block diagonal matrices of the matrix sequence $\{A^{(k)}\}_{k=0}^{l}$.

First of all, for each $i \in \Lambda$ we introduce an auxiliary matrix sequence $\{R_{ii}^{(k)}\}_{k=0}^{l}$, based on $\{B_{ii}^{(k)}\}_{k=1}^{l}$ and $\{A_{ii}^{(k)}\}_{k=0}^{l}$, in accordance with either of the following two methods:

Method (I). $R_{ii}^{(0)} = A_{ii}^{(0)}$ *and* $\{R_{ii}^{(k)}\}_{k=1}^{l}$ *is recursively defined by*

$$R_{ii}^{(k)} = \begin{pmatrix} B_{ii}^{(k)} & E_{ii}^{(k)} \\ E_{ii}^{(k)^T} & R_{ii}^{(k-1)} \end{pmatrix}, \qquad k = 1, 2, \cdots, l.$$

Method (II). $R_{ii}^{(0)} = A_{ii}^{(0)}$ *and* $\{R_{ii}^{(k)}\}_{k=1}^{l}$ *is recursively defined by*

$$R_{ii}^{(k)} = \begin{pmatrix} B_{ii}^{(k)} & E_{ii}^{(k)} \\ E_{ii}^{(k)^T} & \widetilde{R}_{ii}^{(k-1)} \end{pmatrix}, \qquad k = 1, 2, \cdots, l,$$

with

$$\begin{cases} \widetilde{R}_{ii}^{(k-1)} = \begin{cases} A_{ii}^{(k-1)}, & \text{if } k = sk_0, \\ R_{ii}^{(k-1)}, & \text{otherwise}, \end{cases} \\ k = 1, 2, \cdots, l; \quad s = 1, 2, \cdots, l(k_0), \end{cases}$$

where $l(k_0) = l/k_0 - 1$.

Then, by utilizing these two kinds of approximate matrix sequences, we define the new parallel hybrid algebraic multilevel preconditioners $\{M^{(k)}\}_{k=0}^{l}$,

$$M^{(k)} = diag(M_{11}^{(k)}, M_{22}^{(k)}, \cdots, M_{\alpha\alpha}^{(k)}), \qquad k = 0, 1, 2, \cdots, l, \tag{2.3}$$

with respect to the matrices $\{A^{(k)}\}_{k=0}^{l}$ as follows:

$$\begin{cases} M_{ii}^{(0)} = A_{ii}^{(0)}, \quad M_{ii}^{(k)} = \begin{pmatrix} B_{ii}^{(k)} & 0 \\ E_{ii}^{(k)^T} & \widetilde{M}_{ii}^{(k-1)} \end{pmatrix} \begin{pmatrix} I & B_{ii}^{(k)^{-1}} E_{ii}^{(k)} \\ 0 & I \end{pmatrix}, \\ i = 1, 2, \cdots, \alpha; \quad k = 0, 1, \cdots, l, \end{cases} \tag{2.4}$$

where for the fixed positive integer k_0,

$$\begin{cases} \widetilde{M}_{ii}^{(k-1)} = \begin{cases} \widehat{M}_{ii}^{(k-1)}, & \text{if } k = sk_0 + 1, \\ M_{ii}^{(k-1)}, & \text{otherwise}, \end{cases} \\ i = 1, 2, \cdots, \alpha; \quad k = 1, 2, \cdots, l; \quad s = 1, 2, \cdots, l(k_0), \end{cases} \tag{2.5}$$

while

$$\widehat{M}_{ii}^{(k-1)} = \begin{cases} \widetilde{S}_{ii}^{(k)} [I - p_{\nu_i}^{(i)} (M_{ii}^{(k-1)^{-1}} \widetilde{S}_{ii}^{(k)})]^{-1}, & \text{version (i)} \\ R_{ii}^{(k-1)} [I - p_{\nu_i}^{(i)} (M_{ii}^{(k-1)^{-1}} R_{ii}^{(k-1)})]^{-1}, & \text{version (ii)} \end{cases} \tag{2.6}$$

with

$$\widetilde{S}_{ii}^{(k)} = R_{ii}^{(k-1)} - E_{ii}^{(k)^T} B_{ii}^{(k)^{-1}} E_{ii}^{(k)} \tag{2.7}$$

being either the Schur complement of the matrix $R_{ii}^{(k)}$ according to Method (I) or the approximated Schur complement of the matrix $R_{ii}^{(k)}$ according to Method (II).

These well-defined preconditioners immediately result in the following new parallel hybrid AMLI methods for solving the system of linear equations (1.1):

Parallel Hybrid AMLI Methods.

Given an initial vector $x^{(0)} \in R^n$.

FOR $p = 0, 1, 2, \cdots$ *until* $\{x^{(p)}\}$ *convergence DO*

CoBegin proc(1), proc(2), $\cdots$*, proc(*α*).*

proc(i), $i = 1, 2, \cdots, \alpha$*:*

Begin

$r_i^{(p)} = b_i - \sum_{j=1}^{\alpha} A_{ij} x_j^{(p)}$;

$\Delta x_i^{(p)} = M_{ii}^{-1} r_i^{(p)}$;

$x_i^{(p+1)} = x_i^{(p)} + \omega \Delta x_i^{(p)}$

End

CoEnd.

Here, $\omega \in (0, +\infty)$ *is a relaxation factor, and proc(i)(*$i = 1, 2, \cdots, \alpha$*) are the* α *processors that make up of the referred multiprocessor system.*

Evidently, the parallel hybrid AMLI methods are considerably suitable for executing on the high-speed parallel multiprocessor systems. Moreover, with the simple notation $M = M^{(l)}$, they can be briefly rewrite as

$$x^{(p+1)} = x^{(p)} + \omega M^{-1}(b - Ax^{(p)}), \qquad p = 0, 1, 2, \cdots.$$

Clearly, the convergence behaviours of these methods are closely related to the largest and the smallest eigenvalues of the matrix $M^{-1}A$.

3. Preliminary Analyses

The following assumptions are essential for estimating the relative condition numbers of the matrices $\{A^{(k)}\}_{k=0}^{l}$ with respect to their corresponding preconditioners $\{M^{(k)}\}_{k=0}^{l}$ defined in the last section.

Assumption $(\mathbf{A_1})$. *For $k \in \{0, 1, \cdots, l-k_0\}$ and $i \in \Lambda$, it holds that*

$$y_{i2}^T A_{ii}^{(k)} y_{i2} \le \eta_i(k_0) y_i^T A_{ii}^{(k+k_0)} y_i, \qquad y_i = (y_{i1}^T, y_{i2}^T)^T \in R^{n_i^{(k+k_0)}}, \quad y_{i2} \in R^{n_i^{(k)}}.$$

The functions $\eta_i = \eta_i(k_0)$ are monotone increasing functions of k_0 independent of k. More precisely, either of the following asymptotic behaviors holds: Case (a) $\eta_i(k_0) = C_i k_0$; Case (b) $\eta_i(k_0) = C_i \mu_i^{k_0}$. The constants $\mu_i \ge 2$ are upper bounds of the ratios of the dimensional numbers $n_i^{(k)}$ and $n_i^{(k+1)}$ of two consecutive levels, that is, $\mu_i \ge \max\limits_{0 \le k \le l-1} \frac{n_i^{(k+1)}}{n_i^{(k)}}$. The constants C_i are independent of possible jumps of the coefficients of the bilinear form of the variational formulation corresponding to the original second-order self-adjoint elliptic boundary value problem as long as they are discontinuous only across edges (faces) of elements from the initial triangulation.

Assumption $(\mathbf{A_2})$. *For $k \in \{0, 1, \cdots, l-k_0\}$ and $i \in \Lambda$, $B_{ii}^{(k)}$ are SPD matrices satisfying*

$$z_i^T C_{ii}^{(k)} z_i \le z_i^T B_{ii}^{(k)} z_i \le (1+\beta_i^{(k)}) z_i^T C_{ii}^{(k)} z_i, \quad z_i \in R^{n_i^{(k)} - n_i^{(k-1)}},$$

where $\beta_i^{(k)}$ are nonnegative constants allowing the asymptotic behaviours $\beta_i^{(k)} \le \beta_i q_i^{l-k}$ for some $\beta_i > 0$ and $q_i \in (0,1)$.

Assumption $(\mathbf{A_3})$. *For $i \in \Lambda$, $p_{\nu_i}^{(i)}(t)$ are monotone decreasing polynomials of degrees ν_i, respectively, such that $p_{\nu_i}^{(i)}(0) = 1$ and $p_{\nu_i}^{(i)}(t) \in (0,1]$ for all $t \in (0,1]$.*

Assumption $(\mathbf{A_4})$. *For $k \in \{0, 1, \cdots, l\}$ and $i, j \in \Lambda$ with $i \ne j$, there exist constants $\gamma_{ij} \ge 0$ satisfying $\gamma_{ij} = \gamma_{ji}$ and $\gamma_i = \sum\limits_{\substack{j=1 \\ j \ne i}}^{n} \gamma_{ij} \in [0,1)$, such that*

$$|z_i^T A_{ij}^{(k)} z_j| \le \gamma_{ij} (z_i^T A_{ii}^{(k)} z_i)^{\frac{1}{2}} (z_j^T A_{jj}^{(k)} z_j)^{\frac{1}{2}}, \qquad z_i \in R^{n_i^{(k)}}.$$

These assumptions immediately result in the following lemmas.

Lemma 1. (see [8]) *Let Assumption (A_1) be satisfied. Then it holds that*

$$\left| y_i^T E_{ii}^{(k)} z_i \right| \le \gamma_{ii} (y_i^T C_{ii}^{(k)} y_i)^{1/2} (z_i^T A_{ii}^{(k-1)} z_i)^{1/2}, \qquad y_i \in R^{n_i^{(k)} - n_i^{(k-1)}}, \quad z_i \in R^{n_i^{(k-1)}}$$

for $i \in \Lambda$ and $k \in \{1, 2, \cdots, l\}$, where $\gamma_{ii} = \sqrt{1 - 1/\eta_i(1)} < 1$ holds uniformly in $k = 1, 2, \cdots, l$.

Lemma 2. (see [8]) *Let Assumption (A_1) be satisfied. Then it holds that*

(1) $y_i^T C_{ii}^{(k)} y_i \le \eta_i(1) w_i^T A_{ii}^{(k)} w_i, \qquad w_i = (y_i^T, z_i^T)^T \in R^{n_i^{(k)}}$;

(2) $z_i^T E_{ii}^{(k)^T} C_{ii}^{(k)^{-1}} E_{ii}^{(k)} z_i \le \gamma_{ii}^2 z_i^T A_{ii}^{(k-1)} z_i$;

(3) $(1 - \gamma_{ii}^2) z_i^T A_{ii}^{(k-1)} z_i \le z_i^T S_{ii}^{(k)} z_i \le z_i^T A_{ii}^{(k-1)} z_i$,

for $y_i \in R^{n_i^{(k)} - n_i^{(k-1)}}, z_i \in R^{n_i^{(k-1)}}$ and $i \in \Lambda$, $k \in \{1, 2, \cdots, l\}$.

Lemma 3. *Let Assumptions $(A_1) - (A_2)$ be satisfied. Then for $k \in \{1, 2, \cdots, l\}$, $i \in \Lambda$ and $y_i \in R^{n_i^{(k-1)}}$, it holds that*

$$y_i^T S_{ii}^{(k)} y_i \le y_i^T \widehat{S}_{ii}^{(k)} y_i \le (y_i^T S_{ii}^{(k)} y_i + \beta_i^{(k)} y_i^T A_{ii}^{(k-1)} y_i)/(1 + \beta_i^{(k)}),$$

where $\widehat{S}_{ii}^{(k)} = A_{ii}^{(k-1)} - E_{ii}^{(k)^T} B_{ii}^{(k)^{-1}} E_{ii}^{(k)}$.

Proof. It is a direct conclusion of Assumption (A_2).

Lemma 4. *Let Assumptions $(A_1) - (A_2)$ be satisfied. Then for $k \in \{1, 2, \cdots, k_0\}$ and $s \in \{0, 1, \cdots, l(k_0)\}$, it holds that*

$$1 \le \frac{y_i^{(sk_0+k)^T} R_{ii}^{(sk_0+k)} y_i^{(sk_0+k)}}{y_i^{(sk_0+k)^T} A_{ii}^{(sk_0+k)} y_i^{(sk_0+k)}} \le 1 + \eta_i^{(s+1)}(k_0), \qquad i = 1, 2, \cdots, \alpha,$$

where $y_i^{(sk_0+k)} \in R^{n_i^{(sk_0+k)}}$, $\eta_i(0) \equiv 1$, *and*

$$\begin{cases} \eta_i^{(s+1)}(k_0) = \begin{cases} q_i^{l-k_0}\phi_i(k_0)\eta_i(k_0)^s \sum_{j=0}^{s} \left[q_i^{k_0}\eta_i(k_0)\right]^{-j} := \eta_{i,s+1}^{(I)}(k_0), & \textit{Method (I)} \\ q_i^{l-(s+1)k_0}\phi_i(k_0) := \eta_{i,s+1}^{(II)}(k_0), & \textit{Method (II)} \end{cases} \\ \phi_i(k_0) := \beta_i\eta_i(1)\sum_{j=0}^{k_0} q_i^j\eta_i(j). \end{cases}$$

Moreover, $\eta_i^{(s)}(k_0)(i = 1, 2, \cdots, \alpha; s = 1, 2, \cdots, l/k_0)$ *are increasing functions about both* k_0 *and* s *for both Method (I) and Method (II), and they can be bounded uniformly with respect to* $s = 1, 2, \cdots, l/k_0$ *from above by* $\overline{\eta}_i(k_0)(i = 1, 2, \cdots, \alpha)$*, respectively, with*

$$\overline{\eta}_i(k_0) = \begin{cases} \psi_i(k_0) := \frac{\phi_i(k_0)}{1-q_i^{k_0}\eta_i(k_0)}, & \textit{Method (I)} \\ \phi_i(k_0), & \textit{Method (II)} \end{cases}$$

provided $q_i^{k_0}\eta_i(k_0) < 1(i = 1, 2, \cdots, \alpha)$ *hold for Method (I).*

Proof. For $i \in \Lambda$ and $k \in \{1, 2, \cdots, l\}$, take $v_i^{(k)} = (v_{i1}^{(k)^T}, v_{i2}^{(k)^T})^T \in R^{n_i^{(k)}}$, $v_{i2}^{(k)} \in R^{n_i^{(k-1)}}$ and $v_i^{(k-1)} = v_{i2}^{(k)}$. Since for $\{R_{ii}^{(k)}\}_{k=0}^{l}$ defined by Method (I),

$$v_i^{(k)^T}(R_{ii}^{(k)} - A_{ii}^{(k)})v_i^{(k)} = v_{i1}^{(k)^T}(B_{ii}^{(k)} - C_{ii}^{(k)})v_{i1}^{(k)} + v_i^{(k-1)^T}(R_{ii}^{(k-1)} - A_{ii}^{(k-1)})v_i^{(k-1)}, \tag{3.1}$$

while for $\{R_{ii}^{(k)}\}_{k=0}^{l}$ defined by Method (II),

$$v_i^{(k)^T}(R_{ii}^{(k)} - A_{ii}^{(k)})v_i^{(k)} = v_{i1}^{(k)^T}(B_{ii}^{(k)} - C_{ii}^{(k)})v_{i1}^{(k)} + v_i^{(k-1)^T}(\widetilde{R}_{ii}^{(k-1)} - A_{ii}^{(k-1)})v_i^{(k-1)}, \tag{3.2}$$

by Assumption (A_2) we can inductively get

$$v_i^{(k)^T}(R_{ii}^{(k)} - A_{ii}^{(k)})v_i^{(k)} \geq 0, \qquad i = 1, 2, \cdots, \alpha; \quad k = 0, 1, \cdots, l \tag{3.3}$$

for the matrix sequences $\{R_{ii}^{(k)}\}_{k=0}^{l}(i = 1, 2, \cdots, \alpha)$ defined by either Method (I) or Method (II). Hence, we have demonstrated the left-hand sides of the inequalities in Lemma 4.

Now, for $i \in \Lambda$ and $k \in \{sk_0 + 1, sk_0 + 2, \cdots, (s + 1)k_0\}$, by recursively using (3.1)-(3.2) we can derive the estimates:

$$\begin{cases} v_i^{(k)^T}(R_{ii}^{(k)} - A_{ii}^{(k)})v_i^{(k)} \leq v_i^{(k-j)^T}(R_{ii}^{(k-j)} - A_{ii}^{(k-j)})v_i^{(k-j)} + \sum_{m=k-j+1}^{k} v_{i1}^{(m)^T}(B_{ii}^{(m)} - C_{ii}^{(m)})v_{i1}^{(m)}, \\ j \in \{1, 2, \cdots, k - sk_0\}; \quad s = 0, 1, \cdots, l(k_0). \end{cases}$$

In particular, for $i \in \Lambda$, $k \in \{1, 2, \cdots, k_0\}$ and $s \in \{0, 1, \cdots, l(k_0)\}$, it holds that

$$\begin{aligned} v_i^{(sk_0+k)^T}(R_{ii}^{(sk_0+k)} - A_{ii}^{(sk_0+k)})v_i^{(sk_0+k)} \leq & v_i^{(sk_0)^T}(R_{ii}^{(sk_0)} - A_{ii}^{(sk_0)})v_i^{(sk_0)} \\ & + \sum_{m=sk_0+1}^{sk_0+k} v_{i1}^{(m)^T}(B_{ii}^{(m)} - C_{ii}^{(m)})v_{i1}^{(m)}. \end{aligned} \tag{3.4}$$

By Assumption (A_2), Lemma 2(1) and Assumption (A_1), we have

$$\begin{aligned}
\sum_{m=sk_0+1}^{sk_0+k} v_{i1}^{(m)^T}(B_{ii}^{(m)}-C_{ii}^{(m)})v_{i1}^{(m)} &\le \sum_{m=sk_0+1}^{sk_0+k} \beta_i^{(m)} v_{i1}^{(m)^T} C_{ii}^{(m)} v_{i1}^{(m)} \\
&\le \sum_{m=sk_0+1}^{sk_0+k} \beta_i^{(m)} \eta_i(1) v_i^{(m)^T} A_{ii}^{(m)} v_i^{(m)} \\
&= \sum_{m=1}^{k} \beta_i^{(sk_0+m)} \eta_i(1) v_i^{(sk_0+m)^T} A_{ii}^{(sk_0+m)} v_i^{(sk_0+m)} \\
&\le \sum_{m=1}^{k} \beta_i^{(sk_0+m)} \eta_i(1)\eta_i(k-m) v_i^{(sk_0+k)^T} A_{ii}^{(sk_0+k)} v_i^{(sk_0+k)} \\
&\le \sum_{m=1}^{k} \beta_i q_i^{l-(sk_0+m)} \eta_i(1)\eta_i(k-m) v_i^{(sk_0+k)^T} A_{ii}^{(sk_0+k)} v_i^{(sk_0+k)} \\
&= \sum_{m=0}^{k-1} \beta_i q_i^{l-(sk_0+k-m)} \eta_i(1)\eta_i(m) v_i^{(sk_0+k)^T} A_{ii}^{(sk_0+k)} v_i^{(sk_0+k)}.
\end{aligned}$$

Substituting this estimate into (3.4) we obtain

$$\begin{aligned}
&v_i^{(sk_0+k)^T}(R_{ii}^{(sk_0+k)}-A_{ii}^{(sk_0+k)})v_i^{(sk_0+k)} \\
&\le v_i^{(sk_0)^T}(R_{ii}^{(sk_0)}-A_{ii}^{(sk_0)})v_i^{(sk_0)} + \beta_i\eta_i(1)q_i^{l-sk_0-k}\sum_{m=0}^{k-1} q_i^m \eta_i(m) v_i^{(sk_0+k)^T} A_{ii}^{(sk_0+k)} v_i^{(sk_0+k)},
\end{aligned} \tag{3.5}$$

for all $i \in \Lambda$, $k \in \{1,2,\cdots,k_0\}$ and $s \in \{0,1,\cdots,l(k_0)\}$.

Based on (3.5) we can assert

$$\begin{cases} v_i^{(sk_0+k)^T}(R_{ii}^{(sk_0+k)}-A_{ii}^{(sk_0+k)})v_i^{(sk_0+k)} \le \overline{\eta}_i^{(s+1)}(k_0) v_i^{(sk_0+k)^T} A_{ii}^{(sk_0+k)} v_i^{(sk_0+k)}, \\ i=1,2,\cdots,\alpha; \quad k=1,2,\cdots,k_0; \quad s=0,1,\cdots,l(k_0), \end{cases} \tag{3.6}$$

where

$$\begin{cases} \overline{\eta}_i^{(1)}(k_0) = q_i^{l-k_0}\phi_i(k_0), \\ \overline{\eta}_i^{(s+1)}(k_0) = \begin{cases} q_i^{l-(s+1)k_0}\phi_i(k_0) + \eta_i(k_0)\overline{\eta}_i^{(s)}(k_0), & \text{Method (I)} \\ q_i^{l-(s+1)k_0}\phi_i(k_0), & \text{Method (II)} \end{cases} \\ i=1,2,\cdots,\alpha; \quad s=1,2,\cdots,l(k_0). \end{cases} \tag{3.7}$$

In fact, it follows from Method (II) that

$$\begin{aligned}
&v_i^{(sk_0+k)^T}(R_{ii}^{(sk_0+k)}-A_{ii}^{(sk_0+k)})v_i^{(sk_0+k)} \\
&\le v_{i1}^{(sk_0)^T}(B_{ii}^{(sk_0)}-C_{ii}^{(sk_0)})v_{i1}^{(sk_0)} + v_i^{(sk_0-1)^T}(\widetilde{R}_{ii}^{(sk_0-1)}-A_{ii}^{(sk_0-1)})v_i^{(sk_0-1)} \\
&\quad + \beta_i\eta_i(1)q_i^{l-sk_0-k}\sum_{m=0}^{k-1} q_i^m\eta_i(m) v_i^{(sk_0+k)^T} A_{ii}^{(sk_0+k)} v_i^{(sk_0+k)}.
\end{aligned}$$

With the applications of Assumption (A_2) and Lemma 2(1) again, we have

$$v_{i1}^{(sk_0)^T}(B_{ii}^{(sk_0)} - C_{ii}^{(sk_0)})v_{i1}^{(sk_0)} \le \beta_i\eta_i(1)q_i^{l-sk_0-k}q_i^k\eta_i(k)v_i^{(sk_0+k)^T}A_{ii}^{(sk_0+k)}v_i^{(sk_0+k)}.$$

Therefore,

$$\begin{aligned} v_i^{(sk_0+k)^T}(R_{ii}^{(sk_0+k)} - A_{ii}^{(sk_0+k)})v_i^{(sk_0+k)} &\le \beta_i\eta_i(1)q_i^{l-sk_0-k}\sum_{m=0}^{k} q_i^m\eta_i(m)v_i^{(sk_0+k)^T}A_{ii}^{(sk_0+k)}v_i^{(sk_0+k)} \\ &\le q_i^{l-(s+1)k_0}\phi_i(k_0)v_i^{(sk_0+k)^T}A_{ii}^{(sk_0+k)}v_i^{(sk_0+k)} \end{aligned}$$

holds, and this shows the validity of (3.6) for Method (II).

Furthermore, by applying induction and Assumption (A_1), from (3.5) we know that

$$\begin{aligned} &v_i^{(sk_0+k)^T}(R_{ii}^{(sk_0+k)} - A_{ii}^{(sk_0+k)})v_i^{(sk_0+k)} \\ &\le \overline{\eta}_i^{(s)}(k_0)v_i^{(sk_0)^T}A_{ii}^{(sk_0)}v_i^{(sk_0)} + q_i^{l-(s+1)k_0}\phi_i(k_0)v_i^{(sk_0+k)^T}A_{ii}^{(sk_0+k)}v_i^{(sk_0+k)} \\ &\le [\overline{\eta}_i^{(s)}(k_0)\eta_i(k) + q_i^{l-(s+1)k_0}\phi_i(k_0)]v_i^{(sk_0+k)^T}A_{ii}^{(sk_0+k)}v_i^{(sk_0+k)} \\ &\le \overline{\eta}_i^{(s+1)}(k_0)v_i^{(sk_0+k)^T}A_{ii}^{(sk_0+k)}v_i^{(sk_0+k)}. \end{aligned}$$

Therefore, (3.6) is also true for Method (I). Now, operating (3.7) regressively, corresponding to Method (I), we can obtain

$$\begin{aligned} \overline{\eta}_i^{(s+1)}(k_0) &\le \eta_i(k_0)^s\overline{\eta}_i^{(1)}(k_0) + \sum_{m=2}^{s+1}\eta_i(k_0)^{s+1-m}q_i^{l-mk_0}\phi_i(k_0) \\ &= \eta_i(k_0)^s\overline{\eta}_i^{(1)}(k_0) + \eta_i(k_0)^{s+1}\sum_{m=2}^{s+1}[\eta_i(k_0)^m q_i^{(m-1)k_0}]^{-1}\overline{\eta}_i^{(1)}(k_0) \\ &= q_i^{l-k_0}\phi_i(k_0)\eta_i(k_0)^s\sum_{m=0}^{s}[\eta_i(k_0)q_i^{k_0}]^{-m} = \eta_{i,s+1}^{(I)}(k_0). \end{aligned}$$

This, together with (3.6)-(3.7), readily imply the validity of the first conclusion of Lemma 4.

The remainder of Lemma 4 can be demonstrated by direct and simple calculations.

Remark 3.1. For $i \in \Lambda$ and $k \in \{1, 2, \cdots, l\}$, the constraints $\beta_i^{(k)} \le \beta_i q_i^{l-k}$ in Assumption (A_2) hold provided the matrices $B_{ii}^{(k)}$ are constructed in the following way. We let the SPD matrix $G_{ii}^{(k)}$ be the incomplete triangular factorization of the matrix $C_{ii}^{(k)}$ such that $G_{ii}^{(k)}$ is a convergent splitting of $C_{ii}^{(k)}$. That is to say, the spectral radius of the matrix $I - G_{ii}^{(k)^{-1}}C_{ii}^{(k)}$ is less than one, independently of k. (i.e., $\rho(I - G_{ii}^{(k)^{-1}}C_{ii}^{(k)}) \le \widetilde{q}_i < 1$) Moreover, we let the nonzero elements in each row of the matrix $G_{ii}^{(k)}$ have the same order $(O(1))$ as those of the matrix $C_{ii}^{(k)}$, with their number being fixed. Define $\overline{G}_{ii}^{(k)} = I - G_{ii}^{(k)^{-1}}C_{ii}^{(k)}$ and $B_{ii}^{(k)} = C_{ii}^{(k)}[I - (\overline{G}_{ii}^{(k)})^{2\bar{\beta}_i^{(k)}}]^{-1}$ with $\overline{\beta}_i^{(k)} = \overline{m}_i(l-k+1)$ ($\overline{m}_i \ge 1$ is an integer). Then it is easily seen that for $v_i \in R^{n_i^{(k)}-n_i^{(k-1)}}$, $v_i^T C_{ii}^{(k)} v_i \le v_i^T B_{ii}^{(k)} v_i \le \frac{1}{1-\widetilde{q}_i^{2\bar{\beta}_i^{(k)}}} v_i^T C_{ii}^{(k)} v_i$ holds. So, Assumption (A_2) is valid with $\beta_i^{(k)} = \frac{\widetilde{q}_i^{2\bar{\beta}_i^{(k)}}}{1-\widetilde{q}_i^{2\bar{\beta}_i^{(k)}}} = \frac{\widetilde{q}_i^{2\bar{m}_i}}{1-(\widetilde{q}_i^{2\bar{m}_i})^{l-k+1}}(\widetilde{q}_i^{2\bar{m}_i})^{l-k}$. If $\widetilde{q}_i^{2\bar{m}_i} < 1$, then $\beta_i^{(k)} \le \beta_i q_i^{l-k}$ is satisfied with $\beta_i = q_i/(1-q_i)$ and $q_i = \widetilde{q}_i^{2\bar{m}_i}$.

Remark 3.2. Asymptotically, for $i = 1, 2, \cdots, \alpha$, it holds that $\max\{\phi_i(k_0), \psi_i(k_0)\} = O(1)$, provided $q_i < 1$ hold for Case (a), or $q_i < \mu_i^{-1}$ hold for Case (b), in Assumption (A_1).

Remark 3.3. In the sequel, we will use the quantites $\eta_{i,s}^{(I)}(k_0)$, $\eta_{i,s}^{(II)}(k_0)$, $\eta_i^{(s)}(k_0)$, $\overline{\eta}_i(k_0)$, $\phi_i(k_0)$ and $\psi_i(k_0)$ defined in Lemma 4 and

$$\begin{cases} \sigma_i(k_0) = (1+\phi_i(k_0))\eta_i(k_0), & \overline{\sigma}_i(k_0) = (1+\phi_i(k_0))^2\eta_i(k_0), \\ \rho_i(k_0) = (1+\psi_i(k_0))\eta_i(k_0), & \overline{\rho}_i(k_0) = (1+\psi_i(k_0))^2\eta_i(k_0), \\ \overline{\omega}_i(k_0) = \max\{\overline{\sigma}_i(k_0), \overline{\rho}_i(k_0)\}, \end{cases}$$

without further explanations. From Assumption (A_1) and Remark 3.2, it is evident that the following asymptotic behavior holds:

$$\overline{\omega}_i(k_0) = \begin{cases} O(k_0), & \text{for Case (a) and } q_i < 1, \\ O(\mu_i^{k_0}), & \text{for Case (b) and } q_i < \mu_i^{-1}. \end{cases}$$

Lemma 5. *Let Assumptions (A_1) and (A_2) be satisfied. For $i \in \Lambda$, let $\widetilde{M}_{ii}^{(k)}$ for some fixed integer $k(sk_0 < k \le (s+1)k_0, s = 0, 1, \cdots, l(k_0))$ be SPD approximations to $R_{ii}^{(k)}$ such that $\lambda(R_{ii}^{(k)^{-1}}\widetilde{M}_{ii}^{(k)}) \in [1, 1+\widetilde{\delta}_i^{(k,s)}]$ hold for some $\widetilde{\delta}_i^{(k,s)} \ge 0$. Define $M_{ii}^{(k)} = \widetilde{M}_{ii}^{(k)}$ and for $p = k+1, k+2, \cdots, k+k_0$, set*

$$M_{ii}^{(p)} = \begin{pmatrix} B_{ii}^{(p)} & 0 \\ E_{ii}^{(p)^T} & M_{ii}^{(p-1)} \end{pmatrix} \begin{pmatrix} I & B_{ii}^{(p)^{-1}}E_{ii}^{(p)} \\ 0 & I \end{pmatrix}.$$

Then, for $i = 1, 2, \cdots, \alpha$, $\lambda(R_{ii}^{(k+k_0)^{-1}}M_{ii}^{(k+k_0)}) \in [1, 1+\delta_i^{(k,s)}]$ hold with

$$\delta_i^{(k,s)} = \begin{cases} \widetilde{\delta}_i^{(k,s)}(1+\eta_{i,s+1}^{(I)}(k_0))\eta_i(k_0) + \gamma_{ii}^2 \sum\limits_{j=1}^{k_0} \eta_i(j), & \textit{Method (I)} \\ \widetilde{\delta}_i^{(k,s)}(1+\eta_{i,s+1}^{(II)}(k_0))\eta_i(k_0) + \gamma_{ii}^2 \sum\limits_{j=1}^{k_0} \eta_i(j) & \\ \quad +\eta_{i,s+1}^{(II)}(k_0)(1+\eta_{i,s+1}^{(II)}(k_0))\eta_i(k-sk_0), & \textit{Method (II)}. \end{cases}$$

Here and in the subsequent discussion, we use $\lambda(\bullet)$ to denote any eigenvalue of the corresponding matrix.

Proof. For $m = k, k+1, \cdots, p$ and $i = 1, 2, \cdots, \alpha$, let $v_i^{(m)} = (v_{i1}^{(m)^T}, v_{i2}^{(m)^T})^T \in R^{n_i^{(m)}}$ and $v_{i2}^{(m)} = v_i^{(m-1)} \in R^{n_i^{(m-1)}}$. By (3.3) and the definition of Method (II) we know that $v_i^{(m)^T}(R_{ii}^{(m)} - \widetilde{R}_{ii}^{(m)})v_i^{(m)} \ge 0$. Since $B_{ii}^{(m)}$ are all SPD matrices, we have $v_{i2}^{(m)^T}E_{ii}^{(m)^T}B_{ii}^{(m)^{-1}}E_{ii}^{(m)}v_{i2}^{(m)} \ge 0$. Additionally, by direct calculations we can obtain

$$\begin{aligned} v_i^{(p)^T}(M_{ii}^{(p)} - R_{ii}^{(p)})v_i^{(p)} =& v_{i2}^{(p)^T}E_{ii}^{(p)^T}B_{ii}^{(p)^{-1}}E_{ii}^{(p)}v_{i2}^{(p)} \\ &+ \begin{cases} v_i^{(p-1)^T}(M_{ii}^{(p-1)} - R_{ii}^{(p-1)})v_i^{(p-1)}, & \text{Method (I)} \\ v_i^{(p-1)^T}(M_{ii}^{(p-1)} - \widetilde{R}_{ii}^{(p-1)})v_i^{(p-1)}, & \text{Method (II)}. \end{cases} \end{aligned} \tag{3.8}$$

Therefore,

$$v_i^{(p)^T}(M_{ii}^{(p)} - R_{ii}^{(p)})v_i^{(p)} \ge v_i^{(p-1)^T}(M_{ii}^{(p-1)} - R_{ii}^{(p-1)})v_i^{(p-1)}$$

hold for both Method (I) and Method (II). By recursively operating this relations and considering $v_i^{(k)^T}(M_{ii}^{(k)} - R_{ii}^{(k)})v_i^{(k)} \ge 0$, we have the inequalities $v_i^{(p)^T}(M_{ii}^{(p)} - R_{ii}^{(p)})v_i^{(p)} \ge 0$. That is to say,

$$\lambda(R_{ii}^{(p)^{-1}}M_{ii}^{(p)}) \ge 1, \qquad p = k, k+1, \cdots, k+k_0; \quad i = 1, 2, \cdots, \alpha. \tag{3.9}$$

From (3.8) we have

$$v_i^{(p)^T}(M_{ii}^{(p)}-R_{ii}^{(p)})v_i^{(p)} = v_i^{(p-1)^T}(M_{ii}^{(p-1)}-R_{ii}^{(p-1)})v_i^{(p-1)} + v_{i2}^{(p)^T}E_{ii}^{(p)^T}B_{ii}^{(p)^{-1}}E_{ii}^{(p)}v_{i2}^{(p)} + \begin{cases} 0, & \text{Method (I)} \\ v_i^{(p-1)^T}(R_{ii}^{(p-1)}-\widetilde{R}_{ii}^{(p-1)})v_i^{(p-1)}, & \text{Method (II)} \end{cases} \tag{3.10}$$

for $p=k,k+1,\cdots,k+k_0$ and $i=1,2,\cdots,\alpha$. Furthermore, according to Assumption (A_2) and Lemma 2(2), for $i\in\Lambda$,

$$\begin{aligned} v_{i2}^{(p)^T}E_{ii}^{(p)^T}B_{ii}^{(p)^{-1}}E_{ii}^{(p)}v_{i2}^{(p)} &\le v_{i2}^{(p)^T}E_{ii}^{(p)^T}C_{ii}^{(p)^{-1}}E_{ii}^{(p)}v_{i2}^{(p)} \\ &\le \gamma_{ii}^2 v_i^{(p-1)^T}A_{ii}^{(p-1)}v_i^{(p-1)} \end{aligned}$$

holds. By the definition of Method (II), we have $v_i^{(p-1)^T}(R_{ii}^{(p-1)}-\widetilde{R}_{ii}^{(p-1)})v_i^{(p-1)}=0$ for $p\ne (s+1)k_0+1$. In light of Lemma 4 and (3.3) we get

$$\begin{aligned} v_i^{((s+1)k_0)^T}(R_{ii}^{((s+1)k_0)}-\widetilde{R}_{ii}^{((s+1)k_0)})v_i^{((s+1)k_0)} &= v_i^{((s+1)k_0)^T}(R_{ii}^{((s+1)k_0)}-A_{ii}^{((s+1)k_0)})v_i^{((s+1)k_0)} \\ &\le \eta_i^{(s+1)}(k_0)v_i^{((s+1)k_0)^T}A_{ii}^{((s+1)k_0)}v_i^{((s+1)k_0)} \\ &\le \eta_i^{(s+1)}(k_0)v_i^{((s+1)k_0)^T}R_{ii}^{((s+1)k_0)}v_i^{((s+1)k_0)}. \end{aligned}$$

Hence, for $i\in\Lambda$, the estimates

$$v_i^{(p)^T}(M_{ii}^{(p)}-R_{ii}^{(p)})v_i^{(p)} \le v_i^{(k)^T}(M_{ii}^{(k)}-R_{ii}^{(k)})v_i^{(k)} + \gamma_{ii}^2\sum_{j=k}^{p-1}v_i^{(j)^T}A_{ii}^{(j)}v_i^{(j)} + \begin{cases} 0, & \text{Method (I)} \\ \eta_i^{(s+1)}(k_0)v_i^{((s+1)k_0)^T}R_{ii}^{((s+1)k_0)}v_i^{((s+1)k_0)}, & \text{Method (II)} \end{cases} \tag{3.11}$$

can be obtained through regressively using (3.10). In addition, by Assumption (A_1) and (3.3) we know that

$$\begin{aligned} \sum_{j=k}^{k+k_0-1}v_i^{(j)^T}A_{ii}^{(j)}v_i^{(j)} &\le \sum_{j=k}^{k+k_0-1}\eta_i(k+k_0-j)v_i^{(k+k_0)^T}A_{ii}^{(k+k_0)}v_i^{(k+k_0)} \\ &\le \sum_{j=k}^{k+k_0-1}\eta_i(k+k_0-j)v_i^{(k+k_0)^T}R_{ii}^{(k+k_0)}v_i^{(k+k_0)} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} v_i^{((s+1)k_0)^T}R_{ii}^{((s+1)k_0)}v_i^{((s+1)k_0)} &\le (1+\eta_i^{(s+1)}(k_0))v_i^{((s+1)k_0)^T}A_{ii}^{((s+1)k_0)}v_i^{((s+1)k_0)} \\ &\le (1+\eta_i^{(s+1)}(k_0))\eta_i(k-sk_0)v_i^{(k+k_0)^T}A_{ii}^{(k+k_0)}v_i^{(k+k_0)} \\ &\le (1+\eta_i^{(s+1)}(k_0))\eta_i(k-sk_0)v_i^{(k+k_0)^T}R_{ii}^{(k+k_0)}v_i^{(k+k_0)} \end{aligned} \tag{3.13}$$

are valid for $i=1,2,\cdots,\alpha$, where we have used Lemma 4, Assumption (A_1) and (3.3) in each of the three inequalities in (3.13), successively. Now, substituting (3.12)-(3.13) into (3.11) and setting $p=k+k_0$, we immediately obtain

$$v_i^{(k+k_0)^T}(M_{ii}^{(k+k_0)}-R_{ii}^{(k+k_0)})v_i^{(k+k_0)} \le \delta_i^{(k,s)} v_i^{(k+k_0)^T} R_{ii}^{(k+k_0)} v_i^{(k+k_0)}, \quad i=1,2,\cdots,\alpha.$$

This is just the conclusion of Lemma 5. Here, the estimates

$$\begin{aligned} v_i^{(k)^T} R_{ii}^{(k)} v_i^{(k)} &\le (1+\eta_i^{(s+1)}(k_0)) v_i^{(k)^T} A_{ii}^{(k)} v_i^{(k)} \\ &\le (1+\eta_i^{(s+1)}(k_0))\eta_i(k_0) v_i^{(k+k_0)^T} A_{ii}^{(k+k_0)} v_i^{(k+k_0)} \\ &\le (1+\eta_i^{(s+1)}(k_0))\eta_i(k_0) v_i^{(k+k_0)^T} R_{ii}^{(k+k_0)} v_i^{(k+k_0)}, \end{aligned}$$

resulted from Lemma 4, Assumption (A_1) and (3.3), have been considered.

Lemma 6. (see [6, 18]) *Let Assumptions (A_1), (A_2) and (A_4) be satisfied. Then*

$$\min_{1\le i\le\alpha} \frac{1}{1+\gamma_i} \le \frac{y^T M^{(k)} y}{y^T A^{(k)} y} \le \max_{1\le i\le \alpha} \frac{\Lambda_i^{(k)}}{1-\gamma_i}, \qquad k=0,1,\cdots,l,$$

where

$$\Lambda_i^{(k)} = \sup_{\substack{z_i\in R^{n_i^{(k)}}\\ z_i\ne 0}} \frac{z_i^T M_{ii}^{(k)} z_i}{z_i^T A_{ii}^{(k)} z_i}, \qquad i=1,2,\cdots,\alpha; \quad k=0,1,\cdots,l.$$

4. Main Results

We first give a general estimate about the relative condition numbers of $M_{ii}^{(sk_0)}$ with respect to $A_{ii}^{(sk_0)}$ for $i=1,2,\cdots,\alpha$ and $s=1,2,\cdots,l(k_0)$.

Theorem 1. *Let Assumptions $(A_1)-(A_3)$ be satisfied, and define*

$$\begin{cases} \lambda_i^{(0)}=1, \qquad \lambda_i^{(s)} = \sup\limits_{v_i\ne 0} \dfrac{v_i^T M_{ii}^{(sk_0)} v_i}{v_i^T A_{ii}^{(sk_0)} v_i}, \qquad \alpha_i^{(s)} = \begin{cases} \frac{1-\gamma_{ii}^2}{\lambda_i^{(s)}}, & \text{version (i)} \\ \frac{1}{\lambda_i^{(s)}}, & \text{version (ii)} \end{cases} \\ i=1,2,\cdots,\alpha; \quad s=1,2,\cdots,l(k_0). \end{cases} \tag{4.1}$$

Then

$$\lambda_i^{(s+1)} \le (1+\eta_i^{(s+1)}(k_0))(1+\delta_i^{(sk_0,s)}), \qquad i=1,2,\cdots,\alpha; \quad s=0,1,\cdots,l(k_0)-1, \tag{4.2}$$

where

$$\delta_i^{(sk_0,s)} = \begin{cases} \frac{p_{\nu_i}^{(i)}(\alpha_i^{(s)})}{1-p_{\nu_i}^{(i)}(\alpha_i^{(s)})}[1+\eta_{i,s+1}^{(I)}(k_0)]\eta_i(k_0) + \gamma_{ii}^2 \sum\limits_{j=1}^{k_0} \eta_i(j), & \text{Method (I)} \\ \frac{p_{\nu_i}^{(i)}(\alpha_i^{(s)})}{1-p_{\nu_i}^{(i)}(\alpha_i^{(s)})}[1+\eta_{i,s+1}^{(II)}(k_0)]\eta_i(k_0) & \\ \quad +\gamma_{ii}^2 \sum\limits_{j=1}^{k_0} \eta_i(j) + \eta_{i,s+1}^{(II)}(k_0)[1+\eta_{i,s+1}^{(II)}(k_0)], & \text{Method (II).} \end{cases} \tag{4.3}$$

Proof. For $i\in\Lambda$ and $s\in\{1,2,\cdots,l(k_0)\}$, let $\overline{\lambda}_i^{(0)} = \widetilde{\lambda}_i^{(0)} = 1$ and

$$\overline{\lambda}_i^{(s)} = \sup_{v_i\ne 0} \frac{v_i^T M_{ii}^{(sk_0)} v_i}{v_i^T R_{ii}^{(sk_0)} v_i}, \qquad \widetilde{\lambda}_i^{(s)} = \sup_{v_i\ne 0} \frac{v_i^T \widetilde{M}_{ii}^{(sk_0)} v_i}{v_i^T R_{ii}^{(sk_0)} v_i}.$$

From the definitions of the matrices $\{\widetilde{S}_{ii}^{(k)}\}_{k=1}^{l}$ we know that $u_i^T\widetilde{S}_{ii}^{(sk_0+1)}u_i \le u_i^T R_{ii}^{(sk_0)}u_i$ hold for $u_i \in R^{n_i^{(sk_0)}}$. Additionally, by making use of Assumption (A_2) and Lemma 2(3) we get

$$(1-\gamma_{ii}^2)u_i^T A_{ii}^{(sk_0)}u_i \le u_i^T S_{ii}^{(sk_0+1)}u_i \le u_i^T\widetilde{S}_{ii}^{(sk_0+1)}u_i, \qquad u_i \in R^{n_i^{(sk_0)}}.$$

Now, if we denote

$$S_{ii}^{(s)} = \begin{cases} \widetilde{S}_{ii}^{(sk_0+1)}, & \text{version (i)} \\ R_{ii}^{(sk_0)}, & \text{version (ii)}, \end{cases} \tag{4.4}$$

then it clearly holds that

$$u_i^T R_{ii}^{(sk_0)}u_i \ge u_i^T S_{ii}^{(s)}u_i \ge \begin{cases} (1-\gamma_{ii}^2)u_i^T A_{ii}^{(sk_0)}u_i, & \text{version (i)} \\ u_i^T R_{ii}^{(sk_0)}u_i, & \text{version (ii)} \end{cases} \tag{4.5}$$

and

$$\widehat{M}_{ii}^{(sk_0)} = S_{ii}^{(s)}[I - p_{\nu_i}^{(i)}(M_{ii}^{(sk_0)^{-1}}S_{ii}^{(s)})]^{-1}. \tag{4.6}$$

Write

$$T_{ii}^{(s)} = S_{ii}^{(s)^{1/2}}M_{ii}^{(sk_0)^{-1}}S_{ii}^{(s)^{1/2}}, \tag{4.7}$$

we can further assert that

$$\lambda(R_{ii}^{(sk_0)^{-1}}M_{ii}^{(sk_0)}) \in [1,+\infty) \tag{4.8}$$

and

$$\lambda(T_{ii}^{(s)}) \in [\alpha_i^{(s)}, 1]. \tag{4.9}$$

As a matter of fact, remembering (3.3) and by direct calculations, we can obtain the inequalities

$$u_i^{(p)^T}(M_{ii}^{(p)} - R_{ii}^{(p)})u_i^{(p)} \ge u_i^{(p-1)^T}(M_{ii}^{(p-1)} - R_{ii}^{(p-1)})u_i^{(p-1)} + u_i^{(p-1)^T}E_{ii}^{(p)^T}B_{ii}^{(p)^{-1}}E_{ii}^{(p)}u_i^{(p-1)} \tag{4.10}$$

for $u_i^{(p)} = (u_{i1}^{(p)^T}, u_{i2}^{(p)^T})^T \in R^{n_i^{(p)}}$, $u_{i2}^{(p)} = u_i^{(p-1)} \in R^{n_i^{(p-1)}}$, $i \in \Lambda$ and $p \in \{1,2,\cdots,l\}$, from (2.4) and the definitions of Method (I) and Method (II). Based on these inequalities, (4.8) and (4.9) can be demonstrated by induction.

When $s=1$, by (2.5) and Assumption (A_2) we get

$$u_i^{(k_0)^T}(M_{ii}^{(k_0)} - R_{ii}^{(k_0)})u_i^{(k_0)} \ge u_i^{(k_0-1)^T}(M_{ii}^{(k_0-1)} - R_{ii}^{(k_0-1)})u_i^{(k_0-1)} \ge \cdots \ge 0,$$

i.e., (4.8) holds. On the other hand, from (4.5) we have

$$\sup_{u_i\neq 0}\frac{u_i^T T_{ii}^{(1)}u_i}{u_i^T u_i} \le \sup_{u_i\neq 0}\frac{u_i^T S_{ii}^{(1)}u_i}{u_i^T M_{ii}^{(k_0)}u_i} \le \sup_{u_i\neq 0}\frac{u_i^T R_{ii}^{(k_0)}u_i}{u_i^T M_{ii}^{(k_0)}u_i} \le 1, \tag{4.11}$$

and

$$
\begin{aligned}
\inf_{u_i\neq 0}\frac{u_i^T T_{ii}^{(1)} u_i}{u_i^T u_i} \geq \inf_{u_i\neq 0}\frac{u_i^T S_{ii}^{(1)} u_i}{u_i^T M_{ii}^{(k_0)} u_i} &\geq \begin{cases} (1-\gamma_{ii}^2)\inf\limits_{u_i\neq 0}\frac{u_i^T A_{ii}^{(k_0)} u_i}{u_i^T M_{ii}^{(k_0)} u_i}, & \text{version (i)}\\ \inf\limits_{u_i\neq 0}\frac{u_i^T A_{ii}^{(k_0)} u_i}{u_i^T M_{ii}^{(k_0)} u_i}, & \text{version (ii)}\end{cases}\\
&= \begin{cases} (1-\gamma_{ii}^2)/\lambda_i^{(1)}, & \text{version (i)}\\ 1/\lambda_i^{(1)}, & \text{version (ii)}\end{cases}\\
&= \alpha_i^{(1)}.
\end{aligned}
\tag{4.12}
$$

Therefore, (4.9) is valid for $s=1$, too. In accordance with (2.5)-(2.7), (4.4) and (4.6) we see from (4.10) that

$$
\begin{aligned}
u_i^{(k_0+1)^T}(M_{ii}^{(k_0+1)} - R_{ii}^{(k_0+1)})u_i^{(k_0+1)} &\geq u_i^{(k_0)^T}(\widehat{M}_{ii}^{(k_0)} - S_{ii}^{(1)})u_i^{(k_0)}\\
&= u_i^{(k_0)^T} S_{ii}^{(1)^{1/2}}[(I - p_{\nu_i}^{(i)}(T_{ii}^{(1)}))^{-1} - I]S_{ii}^{(1)^{1/2}} u_i^{(k_0)} \geq 0
\end{aligned}
$$

hold. Now, suppose that (4.8)-(4.9) hold for some $s\in\{1,2,\cdots,l(k_0)\}$, we can recursively get

$$
u_i^{(sk_0+1)^T}(M_{ii}^{(sk_0+1)} - R_{ii}^{(sk_0+1)})u_i^{(sk_0+1)} \geq 0, \qquad i=1,2,\cdots,\alpha
$$

by using (4.10). Hence

$$
u_i^{((s+1)k_0)^T}(M_{ii}^{((s+1)k_0)} - R_{ii}^{((s+1)k_0)})u_i^{((s+1)k_0)} \geq 0, \qquad i=1,2,\cdots,\alpha.
$$

That is to say, (4.8) holds for $s+1$. Moreover, applying (4.5) again we can confirm the validity of (4.9) for $s+1$ through analogous derivations to (4.11)-(4.12). By induction, (4.8)-(4.9) have been demonstrated.

By direct computations, we have

$$
\begin{aligned}
\widetilde{\lambda}_i^{(s)} &= \sup_{u_i\neq 0}\frac{u_i^T S_{ii}^{(s)}[I - p_{\nu_i}^{(i)}(M_{ii}^{(sk_0)^{-1}} S_{ii}^{(s)})]^{-1}u_i}{u_i^T R_{ii}^{(sk_0)} u_i}\\
&\leq \sup_{v_i\neq 0}\frac{v_i^T[I - p_{\nu_i}^{(i)}(T_{ii}^{(s)})]^{-1}v_i}{v_i^T v_i}\sup_{v_i\neq 0}\frac{v_i^T S_{ii}^{(s)} v_i}{v_i^T R_{ii}^{(sk_0)} v_i}\\
&\leq \sup_{t\in[\alpha_i^{(s)},1]}\frac{1}{1 - p_{\nu_i}^{(i)}(t)} = \frac{1}{1 - p_{\nu_i}^{(i)}(\alpha_i^{(s)})}.
\end{aligned}
$$

Here, we have used the inequalities in (4.5) and Assumption (A_3). According to Lemma 5, we have $\overline{\lambda}_i^{(s+1)} \leq 1 + \delta_i^{(sk_0,s)}$. Considering Lemma 4, we finally get

$$
\lambda_i^{(s+1)} \leq (1+\eta_i^{(s+1)}(k_0))\overline{\lambda}_i^{(s+1)} \leq (1+\eta_i^{(s+1)}(k_0))(1+\delta_i^{(sk_0,s)}), i=1,2,\cdots,\alpha; s=0,1,\cdots,l(k_0)-1.
$$

Up to now, the proof of this theorem is completed.

For $i=1,2,\cdots,\alpha$, we introduce the notations:

$$\begin{cases} \xi_i(k_0) = 1 + \gamma_{ii}^2 \sum\limits_{j=1}^{k_0} \eta_i(j) - (1+\psi_i(k_0))\eta_i(k_0), \\ \zeta_i(k_0) = 1 + \gamma_{ii}^2 \sum\limits_{j=1}^{k_0} \eta_i(j) + (1+\phi_i(k_0))(\phi_i(k_0) - \eta_i(k_0)), \\ \overline{\xi}_i(k_0) = (1+\psi_i(k_0))\xi_i(k_0), \\ \overline{\zeta}_i(k_0) = (1+\phi_i(k_0))\zeta_i(k_0). \end{cases}$$

These quantities will be used through the remainder of this paper.

Theorem 2. *Let Assumptions* $(A_1)-(A_3)$ *be satisfied, and assume* $q_i^{k_0}\eta_i(k_0) < 1 (i = 1,2,\cdots,\alpha)$ *hold for Method (I). If we define*

$$\begin{cases} \widehat{\lambda}_i^{(0)} = 1, \quad \widehat{\lambda}_i^{(s+1)} = (1+\overline{\eta}_i(k_0))(1+\widehat{\delta}_i^{(sk_0,s)}), \\ \widehat{\alpha}_i^{(s)} = \begin{cases} (1-\gamma_{ii}^2)/\widehat{\lambda}_i^{(s)}, & \textit{version (i)} \\ 1/\widehat{\lambda}_i^{(s)}, & \textit{version (ii)} \end{cases} \\ i = 1,2,\cdots,\alpha; \quad s = 0,1,\cdots,l(k_0)-1 \end{cases} \tag{4.13}$$

with

$$\begin{cases} \widehat{\alpha}_i^{(0)} = \alpha_i^{(0)}, \\ \widehat{\delta}_i^{(sk_0,s)} = \begin{cases} \frac{p_{\nu_i}^{(i)}(\hat{\alpha}_i^{(s)})}{1-p_{\nu_i}^{(i)}(\hat{\alpha}_i^{(s)})}\rho_i(k_0) + \gamma_{ii}^2 \sum\limits_{j=1}^{k_0} \eta_i(j), & \textit{Method (I)} \\ \frac{p_{\nu_i}^{(i)}(\hat{\alpha}_i^{(s)})}{1-p_{\nu_i}^{(i)}(\hat{\alpha}_i^{(s)})}\sigma_i(k_0) + \gamma_{ii}^2 \sum\limits_{j=1}^{k_0} \eta_i(j) + \phi_i(k_0)(1+\phi_i(k_0)), & \textit{Method (II)} \end{cases} \\ i = 1,2,\cdots,\alpha; \quad s = 0,1,\cdots,l(k_0)-1, \end{cases} \tag{4.14}$$

then, for $i = 1,2,\cdots,\alpha$, $\{\widehat{\lambda}_i^{(s)}\}_{s=0}^{l(k_0)}$ *are the majorizing sequences of* $\{\lambda_i^{(s)}\}_{s=0}^{l(k_0)}$, *respectively.*

(a) *For version (i) it holds that*

$$\begin{cases} \widehat{\alpha}_i^{(s+1)} = \begin{cases} \frac{(1-\gamma_{ii}^2)\hat{\alpha}_i^{(s)}Q_{\nu_i-1}^{(i)}(\hat{\alpha}_i^{(s)})}{\overline{\rho}_i(k_0)+\overline{\xi}_i(k_0)\hat{\alpha}_i^{(s)}Q_{\nu_i-1}^{(i)}(\hat{\alpha}_i^{(s)})}, & \textit{Method (I)} \\ \frac{(1-\gamma_{ii}^2)\hat{\alpha}_i^{(s)}Q_{\nu_i-1}^{(i)}(\hat{\alpha}_i^{(s)})}{\overline{\sigma}_i(k_0)+\overline{\zeta}_i(k_0)\hat{\alpha}_i^{(s)}Q_{\nu_i-1}^{(i)}(\hat{\alpha}_i^{(s)})}, & \textit{Method (II)} \end{cases} \\ i = 1,2,\cdots,\alpha; \quad s = 0,1,2,\cdots,l(k_0)-2. \end{cases} \tag{4.15}$$

Therefore, when

$$\begin{cases} \overline{\rho}_i(k_0) < (1-\gamma_{ii}^2)Q_{\nu_i-1}^{(i)}(0), \quad q_i^{k_0}\eta_i(k_0) < 1, & \textit{Method (I)} \\ \overline{\sigma}_i(k_0) < (1-\gamma_{ii}^2)Q_{\nu_i-1}^{(i)}(0), & \textit{Method (II)}, \end{cases} \tag{4.16}$$

each of the sequences $\{\widehat{\alpha}_i^{(s)}\}_{s=0}^{\infty}$ *defined by (4.15) satisfies*

$$\widehat{\alpha}_i^{(s)} \geq \alpha_i^*, \qquad s = 0,1,2,\cdots, \tag{4.17}$$

where $\alpha_i^* \in (0,1)$ *is the smallest positive root corresponding to the following equations:*

$$\begin{cases} (1-\gamma_{ii}^2)Q_{\nu_i-1}^{(i)}(t) - \overline{\xi}_i(k_0)tQ_{\nu_i-1}^{(i)}(t) - \overline{\rho}_i(k_0) = 0, & \textit{Method (I)} \\ (1-\gamma_{ii}^2)Q_{\nu_i-1}^{(i)}(t) - \overline{\zeta}_i(k_0)tQ_{\nu_i-1}^{(i)}(t) - \overline{\sigma}_i(k_0) = 0, & \textit{Method (II)}; \end{cases} \tag{4.18}$$

(b) *For version (ii) it holds that*

$$\begin{cases} \widehat{\alpha}_i^{(s+1)} = \begin{cases} \frac{\hat{\alpha}_i^{(s)} Q_{\nu_i-1}^{(i)}(\hat{\alpha}_i^{(s)})}{\overline{\rho}_i(k_0)+\overline{\xi}_i(k_0)\hat{\alpha}_i^{(s)} Q_{\nu_i-1}^{(i)}(\hat{\alpha}_i^{(s)})}, & \textit{Method (I)} \\ \frac{\hat{\alpha}_i^{(s)} Q_{\nu_i-1}^{(i)}(\hat{\alpha}_i^{(s)})}{\overline{\sigma}_i(k_0)+\overline{\zeta}_i(k_0)\hat{\alpha}_i^{(s)} Q_{\nu_i-1}^{(i)}(\hat{\alpha}_i^{(s)})}, & \textit{Method (II)} \end{cases} \\ i = 1, 2, \cdots, \alpha; \quad s = 0, 1, 2, \cdots, l(k_0) - 2. \end{cases} \tag{4.19}$$

Therefore, when

$$\begin{cases} \overline{\rho}_i(k_0) < Q_{\nu_i-1}^{(i)}(0), \qquad q_i^{k_0}\eta_i(k_0) < 1, & \textit{Method (I)} \\ \overline{\sigma}_i(k_0) < Q_{\nu_i-1}^{(i)}(0), & \textit{Method (II)}, \end{cases} \tag{4.20}$$

each of the sequences $\{\widehat{\alpha}_i^{(s)}\}_{s=0}^{\infty}$ *defined by (4.19) satisfies*

$$\widehat{\alpha}_i^{(s)} \geq \alpha_i^*, \qquad s = 0, 1, 2, \cdots, \tag{4.21}$$

where $\alpha_i^* \in (0,1)$ *is the smallest positive root corresponding to the following equations:*

$$\begin{cases} Q_{\nu_i-1}^{(i)}(t) - \overline{\xi}_i(k_0) t Q_{\nu_i-1}^{(i)}(t) - \overline{\rho}_i(k_0) = 0, & \textit{Method (I)} \\ Q_{\nu_i-1}^{(i)}(t) - \overline{\zeta}_i(k_0) t Q_{\nu_i-1}^{(i)}(t) - \overline{\sigma}_i(k_0) = 0, & \textit{Method (II)}. \end{cases} \tag{4.22}$$

Proof. We first inductively demonstrate that $\{\widehat{\lambda}_i^{(s)}\}_{s=0}^{l(k_0)} (i = 1, 2, \cdots, \alpha)$ are the majorizing sequences of $\{\lambda_i^{(s)}\}_{s=0}^{l(k_0)} (i = 1, 2, \cdots, \alpha)$, respectively, namely,

$$\lambda_i^{(s)} \leq \widehat{\lambda}_i^{(s)}, \qquad i = 1, 2, \cdots, \alpha; \quad s = 0, 1, \cdots, l(k_0). \tag{4.23}$$

Obviously, (4.23) is trivial for $s = 0$. Suppose that (4.23) is true for all $i \in \Lambda$ and $s \in \{0, 1, \cdots, k\}$. Then it is easy to see that $\alpha_i^{(k)} \geq \widehat{\alpha}_i^{(k)} (i = 1, 2, \cdots, \alpha)$ hold. By the monotone decreasing properties of the polynomials $p_{\nu_i}^{(i)}(t) (i = 1, 2, \cdots, \alpha)$ in $[0, 1]$, we immediately know that $p_{\nu_i}^{(i)}(\widehat{\alpha}_i^{(k)}) \geq p_{\nu_i}^{(i)}(\alpha_i^{(k)}) (i = 1, 2, \cdots, \alpha)$. Therefore, $\delta_i^{(kk_0,k)} \leq \widehat{\delta}_i^{(kk_0,k)} (i = 1, 2, \cdots, \alpha)$, and this directly implies $\lambda_i^{(k+1)} \leq \widehat{\lambda}_i^{(k+1)} (i = 1, 2, \cdots, \alpha)$. That is to say, (4.23) is also valid for all $i \in \Lambda$ and $s = k + 1$. By induction we have completed the confirmation of (4.23).

Based on (4.13)-(4.14), we easily know that $\{\widehat{\alpha}_i^{(s)}\}_{s=0}^{l(k_0)-1} (i = 1, 2, \cdots, \alpha)$ satisfy the recurrence relations (4.15) and (4.19), respectively.

In the following, we will use version (i) of Method (I) as an example to show the remainder of the proof. The other cases can be demonstrated in a quite similarly way to this one. To this end, we only need to determine that for each $i \in \Lambda$, the smallest positive root α_i^* corresponding to the first equation of (4.18) makes (4.17) uniformly hold under condition (4.16). We use the induction to complete the determination.

As a matter of fact, for $i = 1, 2, \cdots, \alpha$, if we assume that the inequalities $\widehat{\alpha}_i^{(s)} \geq \alpha_i^*$ have been obtained for some s , then, in order to get $\widehat{\alpha}_i^{(s+1)} \geq \alpha_i^*$, by the first relation of (4.15) we only need to demonstrate

$$1 - \gamma_{ii}^2 \geq \frac{\overline{\rho}_i(k_0) + \overline{\xi}_i(k_0) \alpha_i^* Q_{\nu_i-1}^{(i)}(\alpha_i^*)}{Q_{\nu_i-1}^{(i)}(\alpha_i^*)}. \tag{4.24}$$

Define a scalar function

$$f_i(t) = \frac{\overline{\rho}_i(k_0)}{Q_{\nu_i-1}^{(i)}(t)} + \overline{\xi}_i(k_0) t. \tag{4.25}$$

Then by noticing $f_i(1) > 1$, we know that (4.24) holds only if

$$1 - \gamma_{ii}^2 > \lim_{t \to 0} f_i(t). \tag{4.26}$$

Since $\lim_{t\to 0} f_i(t) = \overline{\rho}_i(k_0)/Q^{(i)}_{\nu_i - 1}(0)$, by substituting these identities into (4.26), we immediately get that there exists $\alpha_i^* \in (0,1)$ such that (4.17) holds provided the first inequality in (4.16) is satisfied. Evidently, α_i^* can be taken to be the smallest positive real numbers which make (4.24) become to equalities, respectively. This shows that α_i^* are the smallest positive roots of the first equations of (4.18), respectively.

Up to now, we have fulfilled the verification of Theorem 2.

Theorem 2 directly implies the following conclusion.

Theorem 3. *Let Assumptions* $(A_1) - (A_3)$ *be satisfied. Then*

(a) *for version (i) it holds that*

$$\lambda(A_{ii}^{(sk_0)^{-1}} M_{ii}^{(sk_0)}) \in \left[1, \frac{1-\gamma_{ii}^2}{\alpha_i^*}\right], \qquad i = 1, 2, \cdots, \alpha; \quad s = 0, 1, \cdots, l(k_0),$$

provided (4.16) is satisfied, where $\alpha_i^* \in (0,1)$ *are the smallest positive roots of the equations in (4.18);*

(b) *for version (ii) it holds that*

$$\lambda(A_{ii}^{(sk_0)^{-1}} M_{ii}^{(sk_0)}) \in \left[1, \frac{1}{\alpha_i^*}\right], \qquad i = 1, 2, \cdots, \alpha; \quad s = 0, 1, \cdots, l(k_0),$$

provided (4.20) is satisfied, where $\alpha_i^* \in (0,1)$ *are the smallest positive roots of the equations in (4.22).*

Theorem 3 and Lemma 6 directly give the estimates about the condition numbers of the matrices $A^{(sk_0)}(s = 0, 1, \cdots, l(k_0))$ with respect to the preconditioning matrices $M^{(sk_0)}(s = 0, 1, \cdots, l(k_0))$, respectively, which are independent of the level number k. Here $M^{(sk_0)}(s = 0, 1, \cdots, l(k_0))$ are defined by (2.3).

Theorem 4. *Let Assumptions* $(A_1) - (A_4)$ *be satisfied. Then*

(a) *for version (i),*

$$\lambda(A^{(sk_0)^{-1}} M^{(sk_0)}) \in \left[\min_{1\le i\le\alpha} \frac{1}{1+\gamma_i}, \quad \max_{1\le i\le\alpha} \frac{1-\gamma_{ii}^2}{(1-\gamma_i)\alpha_i^*}\right], \qquad s = 0, 1, \cdots, l(k_0)$$

hold, provided (4.16) is satisfied, where $\alpha_i^* \in (0,1)$ *are the smallest positive roots of the equations in (4.18);*

(b) *for version (ii)*

$$\lambda(A^{(sk_0)^{-1}} M^{(sk_0)}) \in \left[\min_{1\le i\le\alpha} \frac{1}{1+\gamma_i}, \quad \max_{1\le i\le\alpha} \frac{1}{(1-\gamma_i)\alpha_i^*}\right], \qquad s = 0, 1, \cdots, l(k_0)$$

hold, provided (4.20) is satisfied, where $\alpha_i^* \in (0,1)$ *are the smallest positive roots of the equations in (4.22).*

For the convenience of actual computation, we now apply the above theoretical results to two classes of concrete polynomials to obtain some special but very practical conclusions. These polynomials are the properly scaled and shifted Chebyshev polynomials

$$p_{\nu_i}^{(i)}(t) = \frac{1 + T_{\nu_i}\left(\frac{1+\alpha_i - 2t}{1-\alpha_i}\right)}{1 + T_{\nu_i}\left(\frac{1+\alpha_i}{1-\alpha_i}\right)}, \qquad \alpha_i \in (0,1), \quad i = 1, 2, \cdots, \alpha, \tag{4.27}$$

and the polynomials

$$p_{\nu_i}^{(i)}(t) = (1-t)^{\nu_i}, \qquad i = 1, 2, \cdots, \alpha, \tag{4.28}$$

where T_{ν_i} is the ν_i-th order Chebyshev polynomial.

Evidently, the polynomials given by (4.27) have the smallest local minimums in the intervals $[\alpha_i, 1](i = 1, 2, \cdots, \alpha)$, respectively, and

$$p_{\nu_i}^{(i)}(\alpha_i) = \frac{2}{1+T_{\nu_i}\left(\frac{1+\alpha_i}{1-\alpha_i}\right)}, \quad p_{\nu_i}^{(i)}(1) = \frac{1+(-1)^{\nu_i}}{1+T_{\nu_i}\left(\frac{1+\alpha_i}{1-\alpha_i}\right)}, \qquad i = 1, 2, \cdots, \alpha.$$

Note that the polynomials in (4.27)-(4.28) automatically satisfy Assumption (A_3). By making use of Theorem 4 we can obtain the following results.

Theorem 5. *Let Assumptions (A_1), (A_2) and (A_4) be satisfied, and the polynomials $p_{\nu_i}^{(i)}(t)(0 \le t \le 1)(i = 1, 2, \cdots, \alpha)$ be given by (4.27). Denote*

$$\underline{\lambda} = \min_{1\le i\le\alpha} \frac{1}{1+\gamma_i}, \qquad \overline{\lambda} = \begin{cases} \max\limits_{1\le i\le\alpha} \frac{1-\gamma_{ii}^2}{(1-\gamma_i)\alpha_i}, & \textit{version (i)} \\ \max\limits_{1\le i\le\alpha} \frac{1}{(1-\gamma_i)\alpha_i}, & \textit{version (ii).} \end{cases}$$

(a) *For version (i), if for $i = 1, 2, \cdots, \alpha$,*

$$\begin{cases} \frac{\overline{\rho}_i(k_0)}{2\nu_i} \frac{[(1+\sqrt{\alpha_i})^{\nu_i}+(1-\sqrt{\alpha_i})^{\nu_i}]^2}{\sum\limits_{j=0}^{(\nu_i-1)/2}\binom{\nu_i}{2j+1}\alpha_i^j(1-\alpha_i)^{\nu_i-1}} < 1-\gamma_{ii}^2, \quad q_i^{k_0}\eta_i(k_0) < 1, & \textit{Method (I)} \\ \frac{\overline{\sigma}_i(k_0)}{2\nu_i} \frac{[(1+\sqrt{\alpha_i})^{\nu_i}+(1-\sqrt{\alpha_i})^{\nu_i}]^2}{\sum\limits_{j=0}^{(\nu_i-1)/2}\binom{\nu_i}{2j+1}\alpha_i^j(1-\alpha_i)^{\nu_i-1}} < 1-\gamma_{ii}^2, & \textit{Method (II),} \end{cases} \tag{4.29}$$

then

$$\lambda(A^{(sk_0)^{-1}}M^{(sk_0)}) \in [\underline{\lambda}, \quad \overline{\lambda}], \qquad s = 0, 1, \cdots, l(k_0) \tag{4.30}$$

hold, where $\alpha_i \in (0,1)$ is the smallest positive root of the following equation:

$$\left[\frac{(1+\sqrt{t})^{\nu_i}+(1-\sqrt{t})^{\nu_i}}{2\sum\limits_{j=0}^{(\nu_i-1)/2}\binom{\nu_i}{2j+1}t^j}\right]^2 = \begin{cases} \frac{1-\gamma_{ii}^2-\overline{\xi}_i(k_0)t}{\overline{\rho}_i(k_0)}, & \textit{Method (I)} \\ \frac{1-\gamma_{ii}^2-\overline{\zeta}_i(k_0)t}{\overline{\sigma}_i(k_0)}, & \textit{Method (II).} \end{cases} \tag{4.31}$$

Moreover, when

$$\begin{cases} \nu_i^2 > 2\overline{\rho}_i(k_0)/(1-\gamma_{ii}^2), \qquad q_i^{k_0}\eta_i(k_0) < 1, & \textit{Method (I)} \\ \nu_i^2 > 2\overline{\sigma}_i(k_0)/(1-\gamma_{ii}^2), & \textit{Method (II),} \end{cases}$$

the smallest solution $\alpha_i \in (0,1)$ of the equation (4.31) can guarantee the validity of (4.29), and therefore, (4.30) holds.

(b) *For version (ii), if for $i = 1, 2, \cdots, \alpha$,*

$$\begin{cases} \frac{\overline{\rho}_i(k_0)}{2\nu_i} \frac{[(1+\sqrt{\alpha_i})^{\nu_i}+(1-\sqrt{\alpha_i})^{\nu_i}]^2}{\sum\limits_{j=0}^{(\nu_i-1)/2}\binom{\nu_i}{2j+1}\alpha_i^j(1-\alpha_i)^{\nu_i-1}} < 1, \quad q_i^{k_0}\eta_i(k_0) < 1, & \textit{Method (I)} \\ \frac{\overline{\sigma}_i(k_0)}{2\nu_i} \frac{[(1+\sqrt{\alpha_i})^{\nu_i}+(1-\sqrt{\alpha_i})^{\nu_i}]^2}{\sum\limits_{j=0}^{(\nu_i-1)/2}\binom{\nu_i}{2j+1}\alpha_i^j(1-\alpha_i)^{\nu_i-1}} < 1, & \textit{Method (II)}, \end{cases} \tag{4.32}$$

then

$$\lambda(A^{(sk_0)^{-1}}M^{(sk_0)}) \in [\underline{\lambda}, \quad \overline{\lambda}], \qquad s = 0, 1, \cdots, l(k_0) \tag{4.33}$$

hold, where $\alpha_i \in (0,1)$ *is the smallest positive root of the following equation:*

$$\left[\frac{(1+\sqrt{t})^{\nu_i} + (1-\sqrt{t})^{\nu_i}}{2\sum\limits_{j=0}^{(\nu_i-1)/2}\binom{\nu_i}{2j+1}t^j}\right]^2 = \begin{cases} \frac{1-\overline{\xi}_i(k_0)t}{\overline{\rho}_i(k_0)}, & \textit{Method (I)} \\ \frac{1-\overline{\zeta}_i(k_0)t}{\overline{\sigma}_i(k_0)}, & \textit{Method (II)}. \end{cases} \tag{4.34}$$

Moreover, when

$$\begin{cases} \nu_i^2 > 2\overline{\rho}_i(k_0), \quad q_i^{k_0}\eta_i(k_0) < 1, & \textit{Method (I)} \\ \nu_i^2 > 2\overline{\sigma}_i(k_0), & \textit{Method (II)}, \end{cases}$$

the smallest solution $\alpha_i \in (0,1)$ *of the equation (4.34) can guarantee the validity of (4.32), and therefore, (4.33) holds.*

Theorem 6. *Let Assumptions* (A_1), (A_2) *and* (A_4) *be satisfied, and the polynomials* $p_{\nu_i}^{(i)}(t)(0 \le t \le 1)(i = 1, 2, \cdots, \alpha)$ *be given by (4.28). Denote* $\underline{\lambda}$ *and* $\overline{\lambda}$ *as in Theorem 5.*

(a) *For version (i), if for* $i = 1, 2, \cdots, \alpha$,

$$\begin{cases} \nu_i > \overline{\rho}_i(k_0)/(1-\gamma_{ii}^2), \quad q_i^{k_0}\eta_i(k_0) < 1, & \textit{Method (I)} \\ \nu_i > \overline{\sigma}_i(k_0)/(1-\gamma_{ii}^2), & \textit{Method (II)}, \end{cases}$$

then $\lambda(A^{(sk_0)^{-1}}M^{(sk_0)}) \in [\underline{\lambda}, \quad \overline{\lambda}](s = 0, 1, \cdots, l(k_0))$ *hold, where* $\alpha_i \in (0,1)$ *is the smallest positive root of the following equation:*

$$\sum_{j=1}^{\nu_i}(-1)^j\binom{\nu_i}{j}t^{j-1} = \begin{cases} \frac{\overline{\rho}_i(k_0)}{1-\gamma_{ii}^2-\overline{\xi}_i(k_0)t}, & \textit{Method (I)} \\ \frac{\overline{\sigma}_i(k_0)}{1-\gamma_{ii}^2-\overline{\zeta}_i(k_0)t}, & \textit{Method (II)}. \end{cases}$$

(b) *For version (ii), if for* $i = 1, 2, \cdots, \alpha$,

$$\begin{cases} \nu_i > \overline{\rho}_i(k_0), \quad q_i^{k_0}\eta_i(k_0) < 1, & \textit{Method (I)} \\ \nu_i > \overline{\sigma}_i(k_0), & \textit{Method (II)}, \end{cases}$$

then $\lambda(A^{(sk_0)^{-1}}M^{(sk_0)}) \in [\underline{\lambda}, \quad \overline{\lambda}](s = 0, 1, \cdots, l(k_0))$ *hold, where* $\alpha_i \in (0,1)$ *is the smallest positive root of the following equation:*

$$\sum_{j=1}^{\nu_i}(-1)^j\binom{\nu_i}{j}t^{j-1} = \begin{cases} \frac{\overline{\rho}_i(k_0)}{1-\overline{\xi}_i(k_0)t}, & \textit{Method (I)} \\ \frac{\overline{\sigma}_i(k_0)}{1-\overline{\zeta}_i(k_0)t}, & \textit{Method (II)}. \end{cases}$$

5. Computational Complexities

In order to obtain quantitative estimates of the amount of the computational works $W_i(i = 1, 2, \cdots, \alpha)$ of the new parallel hybrid AMLI methods for each processor of the multiprocessor system at every iteration step, without loss of generality, we assume that the refinements are uniform. Then the numbers of nodes $n_i^{(k)}(i = 1, 2, \cdots, \alpha)$ at the k-th level grow in geometrical fashions, i.e., $n_i^{(k)} = n_i^{(l)}\mu_i^{d(k-l)}(i = 1, 2, \cdots, \alpha; \quad k = 1, 2, \cdots, l)$, where d are positive integers related to the dimension of the original second-order self-adjoint elliptic boundary value problem. In addition, for $i, j \in \Lambda$, $k \in \{0, 1, \cdots, l\}$, we assume that the operations of computing $A_{ij}^{(k)}y_j^{(k)}(y_j^{(k)} \in R^{n_j^{(k)}})$ are $c_1(n_i^{(k)} + n_j^{(k)})$, respectively, where c_1 is a positive constant independent of i, j and k. Denote $n^{(k)} = \sum_{i=1}^{\alpha} n_i^{(k)}(k = 0, 1, \cdots, l)$. Because for each $i \in \Lambda$ and $k \in \{0, 1, \cdots, l\}$ the approximation matrix $B_{ii}^{(k)}$ can be obtained by incomplete triangular factorization of $C_{ii}^{(k)}$, it is reasonable to assume that the amount of operations for solving the system of linear equations of coefficient matrix $B_{ii}^{(k)}$ is $c_2(n_i^{(k)} - n_i^{(k-1)})$, here c_2 is a positive constant independent of both i and k, too. Again, denote by $W_i^{(0)}$ the amount of computational work of processor i for solving the system of linear equations with coefficient matrix $A_{ii}^{(0)}$ on level 0 and, generally, by $W_i^{(k)}$ the amount of computational works of processor i for solving the system of linear equations of coefficient matrix $M_{ii}^{(k)}$ on level k. Then for $(s-1)k_0+1 \le k \le sk_0$ we have $W_i^{(k)} \le c_2(\mu_i^d - 1)n_i^{(k-1)} + W_i^{(k-1)}$. Hence,

$$W_i^{(k)} \le c_2(1 - \mu_i^{-d})n_i^{(k)}[1 + \mu_i^{-d} + \mu_i^{-2d} + \cdots + \mu_i^{-(k-j-1)d}]W_i^{(j)}$$

hold for $i \in \Lambda$ and $j \in \{(s-1)k_0 + 1, (s-1)k_0 + 2, \cdots, sk_0\}$. It is evident that from these inequalities we can directly obtain

$$\begin{cases} W_i^{(sk_0)} \le \bar{c}_i n_i^{(sk_0)} + W_i^{((s-1)k_0+1)}, & \bar{c}_i = c_2(1 - \mu_i^{-(k_0-1)d}) \\ i = 1, 2, \cdots, \alpha; \quad s = 1, 2, \cdots, l(k_0). \end{cases}$$

Because

$$W_i^{((s-1)k_0+1)} \le c_2(\nu_i - 1)(\mu_i^d - 1)n_i^{((s-1)k_0)} + \nu_i W_i^{((s-1)k_0)} \le \tilde{c}_i n_i^{(sk_0)} + \nu_i W_i^{((s-1)k_0)}$$

holds, where $\tilde{c}_i = c_2(\nu_i - 1)(\mu_i^d - 1)\mu_i^{-k_0 d}$, we know that, for $i = 1, 2, \cdots, \alpha$,

$$W_i^{((s+1)k_0)} \le \nu_i W_i^{(sk_0)} + \hat{c}_i n_i^{((s+1)k_0)},$$

where

$$\hat{c}_i = \bar{c}_i + \tilde{c}_i = c_2\{1 + \mu_i^{-k_0 d}[(\nu_i - 2)(\mu_i^d - 2) - 1]\}.$$

By recursively using the above relations, we have

$$
\begin{aligned}
W_i^{((s+1)k_0)} &= \hat{c}_i \sum_{j=0}^{s} \nu_i^j n_i^{((s-j+1)k_0)} + \nu_i^{s+1} W_i^{(0)} \\
&= \hat{c}_i \sum_{j=0}^{s} \nu_i^j \mu_i^{d(s-j+1)k_0-1} n_i^{(l)} + \nu_i^{s+1} W_i^{(0)} \\
&= \hat{c}_i n_i^{(l)} \mu_i^{d(s+1)k_0-1} \sum_{j=0}^{s} (\nu_i \mu_i^{-k_0 d})^j + \nu_i^{s+1} W_i^{(0)} \\
&\le n_i^{((s+1)k_0)} \left[\hat{c}_i \sum_{j=0}^{s} (\nu_i \mu_i^{-k_0 d})^j + \frac{W_i^{(0)}}{n_i^{(k_0)}} (\nu_i \mu_i^{-k_0 d})^{s+1} \right].
\end{aligned}
$$

If $\nu_i \mu_i^{-k_0 d} < 1$, then

$$
W_i^{((s+1)k_0)} \le \left(\hat{c}_i' + \frac{W_i^{(0)}}{n_i^{(0)}} \right) n_i^{((s+1)k_0)}.
$$

Therefore,

$$
W_i = W_i^{(l)} + c_1 \sum_{j=1}^{\alpha} (n_i^{(l)} + n_j^{(l)}) + (\alpha + 2) n_i^{(l)} \le c_i^* n_i^{(l)} + c_1 n = O(n),
$$

where $c_i^* = (c_1 + 1)\alpha + \hat{c}_i' + \frac{W_i^{(0)}}{n_i^{(0)}} + 2$. Concludingly, the asymptotic work estimation shows that the new parallel hybrid AMLI methods would be of optimal orders, provided for $i = 1, 2, \cdots, \alpha$, ν_i satisfy the inequalities

$$
Q_{\nu_i - 1}^{(i)}(0) > \begin{cases} \overline{\omega}_i(k_0)/(1 - \gamma_{ii}^2), & \text{version (i)} \\ \overline{\omega}_i(k_0), & \text{version (ii)} \end{cases}
$$

from Theorem 4, and $\nu_i \mu_i^{-k_0 d} < 1$, from the complexity requirements. More concretely, for the polynomials defined by (4.27)-(4.28), based on the asymptotic behaviors of $\overline{\omega}_i(k_0)$ (see Remark 3.3) we know that these restrictions on ν_i turn to

$$
\begin{aligned}
\mu_i^{k_0 d} > \nu_i > & \begin{cases} \sqrt{\frac{2}{1-\gamma_{ii}^2} \overline{\omega}_i(k_0)}, & \text{version (i)} \\ \sqrt{2 \overline{\omega}_i(k_0)}, & \text{version (ii)} \end{cases} \\
= & \begin{cases} O(\sqrt{k_0}), & \text{for Case (a)} \\ O(\mu_i^{k_0/2}), & \text{for Case (b)} \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_i^{k_0 d} > \nu_i > & \begin{cases} \frac{\overline{\omega}_i(k_0)}{1-\gamma_{ii}^2}, & \text{version (i)} \\ \overline{\omega}_i(k_0), & \text{version (ii)} \end{cases} \\
= & \begin{cases} O(k_0), & \text{for Case (a)} \\ O(\mu_i^{k_0}), & \text{for Case (b),} \end{cases}
\end{aligned}
$$

respectively. It is clear now that, asymptotically, for k_0 sufficiently large, the restrictions on ν_i corresponding to polynomials (4.27)-(4.28) can be satisfied for both Case (a) and Case (b). Therefore, we have the following result.

Theorem 7. *The new parallel hybrid algebraic multilevel preconditioners with the polynomials (4.27)-(4.28) give optimal order methods for k_0 sufficiently large. That is to say, they* *are spectrally equivalent to the corresponding matrices $A^{(k)}$, and the costs of evaluating the preconditioners are $O(n)$, namely, proportional to the number of the involved unknowns.*

At last, we use the following Remarks to end this section.

Remark 5.1. For the new parallel hybrid algebraic multilevel preconditioners with the properly scaled and shifted Chebyshev polynomials, we can estimate $\lambda_i^{(sk_0)}$ starting with $s=1$ and setting

$$\alpha_i^{(sk_0)} = \begin{cases} \frac{1-\gamma_{ii}^2}{\lambda_i^{(sk_0)}}, & \text{version (i)}, \\ \frac{1}{\lambda_i^{(sk_0)}}, & \text{version (ii)}, \end{cases} \qquad p_{\nu_i}^{(i)}(t) = \frac{1 + T_{\nu_i}\left(\frac{1+\alpha_i^{(sk_0)}-2t}{1-\alpha_i^{(sk_0)}}\right)}{1 + T_{\nu_i}\left(\frac{1+\alpha_i^{(sk_0)}}{1-\alpha_i^{(sk_0)}}\right)},$$

the procedures continue with $s = 0, 1, \cdots, l(k_0)$. Once an unacceptable growth of the eigenvalues $\lambda_i^{(sk_0)}$ takes place, the corresponding procedure can be restarted with a larger ν_i. Theorem 5 guarantees that a reasonable stabilization of the orders of magnitudes of the eigenvalues $\lambda_i^{(sk_0)}$ can be achieved.

Remark 5.2. $k_0(\geq 1)$ should be chosen in order to balance the arithmetic works for the estimations of the eigenvalues $\lambda_i^{(sk_0)}(i = 1, 2, \cdots, \alpha)$ and the work of polynomial acceleration at every global step, in other words, to ensure the inequalities $\nu_i < \mu_i^{k_0 d}$.

Remark 5.3. As pointed in [6, 8, 18], the system of linear equations resulted from discretizing the second-order self-adjoint elliptic boundary value problem can automatically satisfy the basic Assumptions (A_1) and (A_4) under suitable regularity assumptions on the finite element spaces and the triangulations.

6. Conclusions

To solve the large system of linear equations having symmetric positive definite coefficient matrix resulting from the discretization of many second-order self-adjoint elliptic boundary value problems by finite element method on the high-speed parallel multiprocessor systems, in this paper we propose a class of new parallel hybrid algebraic multilevel iteration methods through starting with reasonably constructed approximation matrices of the block diagonal matrices of the matrix sequence, which is obtained from successively refining the original coefficient matrix of the linear system, and through applying the blocked iteration and the multilevel preconditioning techniques in [1-10, 15, 17-18]. These methods are superior to the existing ones in several aspects such as the parallelism, the generalities, the computational costs and the constriants on the approximation matrices, etc.. It is further demonstrated that the preconditioners so derived are of optimal orders of complexities for both 2-D and 3-D problem domains, and their relative condition numbers are not only independent of the regularity of the solution, but also bounded uniformly with respect to the levels and with respect to the possible jumps of the coefficients of the original problem, provided they occur only across edges (faces in 3-D) of elements from the coarsest triangulation, and provided the triangulations are generated successively by uniform refinements. Finally, adaptive implementations of the new methods are suggested, which have the potentials to be more robust and practical in actual computations.

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