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JACOBI SPECTRAL APPROXIMATIONS TO DIFFERENTIAL EQUATIONS ON THE HALF LINE*

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Abstract

Some Jacobi approximations are investigated, which are used for numerical solutions of differential equations on the half line. The stability and convergence of the proposed schemes are proved. The main idea and techniques in this paper are also applicable to other problems on unbounded domains.

Key words: Jacobi spectral approximations, Differential equations on the half line, Stability and convergence.

1. Introduction

Many physical problems are set on unbounded domains. Some conditions at infinity are given by certain asymptotic behaviors of the solutions. For numerical simulations, we often restrict calculations to bounded domains, and impose certain artificial boundary conditions, which usually cause numerical errors . If we use spectral methods associated with orthogonal systems on unbounded domains, the mentioned troubles might be remedied. Maday, Pernaud-Thomas and Vandeven [1], Coulaud, Funaro and Kavian [2], and Funaro [3] considered Laguerre spectral approximations for linear problems on the half line. Iranzo and Falquès [4] provided some Laguerre pseudospectral schemes and Laguerre Tau schemes. Mavriplis [5] studied Laguerre spectral element method. On the other hand, Funaro and Kavian [6] proved the convergence of spectral and pseudospectral methods using Hermite functions for some linear problems on the whole line. While Guo [7] developed a spectral method using Hermite polynomials for the Burgers equation on the whole line, and Guo and Shen [8] proposed some spectral schemes using Laguerre polynomials for the Benjamin-Bona-Mahony equation and the Burgers equation on the half line. They also provided an efficient algorithm, and proved the stability and convergence of the proposed schemes . However all of these algorithms need certain quadratures on unbounded domains, which introduce errors and so weaken the merit of spectral approximations. Another approach is to use rational basis functions, see Christov [9], Boyd [10], and Iranzo and Falquès [4]. Recently, Guo [11] developed another method in which differential equations on the whole line are changed to certain problems on a finite interval. Since the coefficients of the resulting equations degenerate at both extreme points, a specific Gegenbauer approximation was

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used for their numerical solutions. Guo [11] also proved the stability and convergence of the proposed schemes .

This paper is devoted to Jacobi spectral method for differential equations on the half line. We change differential equations on the half line to certain problems on a finite interval. Since the coefficients of the resulting equations degenerate only at one of extreme points, it is not reasonable to approximate them by Gegenbauer polynomials. A natural choice is to use certain unsymmetric Jacobi approximations. In the next section, we introduce some weighted spaces and related unsymmetric Jacobi polynomials. Several weighted Poincaré inequalities and weighted inverse inequalities are obtained. In section 3, we focus on some orthogonal projections and derive the corresponding approximation results. All results in these two sections play important roles in the error analysis. The final section is for the application of unsymmetric Jacobi approximations. In particular, we take the Burgers equation as an example to show how to deal with nonlinear problems. The stability and convergence of the proposed schemes are proved. The main idea and techniques used in this paper are also useful for other problems on the half line. It is not difficult to generalize the main results in this paper to those on multiple-dimensional unbounded domains.

2. Some Jacobi Polynomials and Weighted Inequalities

Let $\Lambda = \{x| -1 < x < 1\}$ and $\chi(x)$ be a certain weight function in the usual sense. For $1 \le p \le \infty$, let

$$L^p_\chi(\Lambda) = \{v|v \text{ is measurable and } ||v||_{L^p_\chi} < \infty\}$$

where

$$||v||_{L^p_\chi} = \begin{cases} (\int_\Lambda |v(x)|^p \chi(x)dx)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Lambda} |v(x)|, & p = \infty. \end{cases}$$

In particular, $L^2_\chi(\Lambda)$ is a Hilbert space with the following inner product and norm

$$(u,v)_\chi = \int_\Lambda u(x)v(x)\chi(x)dx, \qquad ||v||_\chi = (v,v)^{\frac{1}{2}}_\chi.$$

Further, let $\partial_x v(x) = \frac{\partial}{\partial x} v(x)$, and for any non-negative integer m, define

$$H^m_\chi(\Lambda) = \{v|\partial^k_x v \in L^2_\chi(\Lambda), 0 \le k \le m\}$$

equipped with the inner product, semi-norm and norm as follows

$$(u,v)_{m,\chi} = \sum_{k=0}^m (\partial^k_x u, \partial^k_x v)_\chi,$$

$$|v|_{m,\chi} = ||\partial^m_x v||_\chi, \qquad ||v||_{m,\chi} = (v,v)^{\frac{1}{2}}_{m,\chi}.$$

For any real $r \ge 0$, we define the space $H^r_\chi(\Lambda)$ with the norm $||v||_{r,\chi}$ by space interpolation as in Adams [12]. Let $\mathcal{D}(\Lambda)$ be the set of all infinitely differentiable functions with compact support in Λ, and $H^r_{0,\chi}(\Lambda)$ be the closure of $\mathcal{D}(\Lambda)$ in $H^r_\chi(\Lambda)$. For $\chi(x) \equiv 1$, we denote the spaces $H^r_\chi(\Lambda)$ and $H^r_{0,\chi}(\Lambda)$ by $H^r(\Lambda)$ and $H^r_0(\Lambda)$. Their semi-norm and norm are denoted by $|v|_r$ and $||v||_r$, respectively. In addition, $(u,v) = (u,v)_{L^2(\Lambda)}, ||v|| = ||v||_{L^2(\Lambda)}$ and $||v||_\infty = ||v||_{L^\infty(\Lambda)}$.

We first recall some properties of the Jacobi polynomials $J_l^{(\alpha,\beta)}(x)$, defined by

$$(1-x)^\alpha (1+x)^\beta J_l^{(\alpha,\beta)}(x) = \frac{(-1)^l}{2^l l!} \partial_x^l ((1-x)^{l+\alpha}(1+x)^{l+\beta}).$$

They are the eigenfunctions of the singular Sturm-Liouville problem

$$\partial_x((1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x v(x)) + \lambda(1-x)^\alpha(1+x)^\beta v(x) = 0, \quad x \in \Lambda, \tag{2.1}$$

with the corresponding eigenvalues $\lambda_l^{(\alpha,\beta)} = l(l+\alpha+\beta+1)$. The Jacobi polynomials fulfill the recurrence relations (see Askey [13])

$$2(l+\alpha+1)J_l^{(\alpha,\beta)}(x) - 2(l+1)J_{l+1}^{(\alpha,\beta)}(x) = (2l+\alpha+\beta+2)(1-x)J_l^{(\alpha+1,\beta)}(x) \tag{2.2}$$

and

$$\partial_x J_l^{(\alpha,\beta)}(x) = \frac{1}{2}(l+\alpha+\beta+1)J_{l-1}^{(\alpha+1,\beta+1)}(x). \tag{2.3}$$

Let $\Gamma(x)$ be the Gamma function and

$$(a)_l = \frac{\Gamma(l+a)}{\Gamma(a)}.$$

We have

$$J_l^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_l}{(\alpha+\gamma+2)_l} \sum_{k=0}^{l} a_{\alpha,\beta,\gamma} J_k^{(\alpha,\gamma)}(x), \tag{2.4}$$

$$J_l^{(\alpha,\beta)}(x) = \frac{(\beta+1)_l}{(\beta+\gamma+2)_l} \sum_{k=0}^{l} b_{\alpha,\beta,\gamma} J_k^{(\gamma,\beta)}(x) \tag{2.5}$$

where

$$a_{\alpha,\beta,\gamma} = \frac{(-1)^{l-k}(\beta-\gamma)_{l-k}(\alpha+\gamma+1)_k(\alpha+\gamma+2)_{2k}(l+\alpha+\beta+1)_k}{(1)_{l-k}(\alpha+1)_k(\alpha+\gamma+1)_{2k}(l+\alpha+\gamma+2)_k},$$

$$b_{\alpha,\beta,\gamma} = \frac{(\alpha-\gamma)_{l-k}(\gamma+\beta+1)_k(\gamma+\beta+2)_{2k}(l+\alpha+\beta+1)_k}{(1)_{l-k}(\beta+1)_k(\gamma+\beta+1)_{2k}(l+\gamma+\beta+2)_k}.$$

It is noted that

$$J_l^{(\alpha,\beta)}(-x) = (-1)^l J_l^{(\beta,\alpha)}(x), \qquad J_l^{(\alpha,\beta)}(1) = \frac{\Gamma(l+\alpha+1)}{l!\Gamma(\alpha+1)}.$$

Now let

$$\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta.$$

For any real numbers $\alpha, \beta > -1$, the set $\{J_l^{(\alpha,\beta)}(x)\}$ is the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)-$orthogonal system, i.e.,

$$(J_l^{(\alpha,\beta)}, J_m^{(\alpha,\beta)})_{\chi^{(\alpha,\beta)}} = \gamma_l^{(\alpha,\beta)}\delta_{l,m} \tag{2.6}$$

where $\delta_{l,m}$ is the Kronecker function, and

$$\gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2l+\alpha+\beta+1}\frac{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}.$$

For any $v \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$v(x) = \sum_{l=0}^{\infty} \widehat{v}_l^{(\alpha,\beta)} J_l^{(\alpha,\beta)}(x)$$

where $\widehat{v}_l^{(\alpha,\beta)}$ is the Jacobi coefficient

$$\widehat{v}_l^{(\alpha,\beta)} = \frac{1}{\gamma_l^{(\alpha,\beta)}} \int_\Lambda v(x) J_l^{(\alpha,\beta)}(x) \chi^{(\alpha,\beta)}(x) dx. \tag{2.7}$$

We shall use several kinds of Jacobi polynomials. Clearly $J_l^{(0,0)}(x)$ is the Legendre polynomial $L_l(x)$ associated with

$$\chi^{(0,0)}(x) \equiv 1, \qquad h_l^* = \gamma_l^{(0,0)} = \frac{2}{2l+1}, \qquad v_l^* = \widehat{v}_l^{(0,0)}. \tag{2.8}$$

For the sake of simplicity, let

$$G_l(x) = J_l^{(1,0)}(x), \qquad \widetilde{G}_l(x) = J_l^{(0,1)}(x). \tag{2.9}$$

Accordingly,

$$\omega(x) = \chi^{(1,0)}(x) = 1 - x, \qquad h_l = \gamma_l^{(1,0)} = \frac{2}{l+1}, \qquad \hat{v}_l = \widehat{v}_l^{(1,0)}, \tag{2.10}$$

$$\tilde{\omega}(x) = \chi^{(0,1)}(x) = 1 + x, \qquad \widetilde{h}_l = \gamma_l^{(0,1)} = \frac{2}{l+1}, \qquad \widetilde{v}_l = \widehat{v}_l^{(0,1)}. \tag{2.11}$$

Moreover, set $\eta(x) = \omega^{-1}(x)$. By taking $\alpha = 2, \beta = 1$ and $\gamma = 1$ in (2.5), we get that

$$J_l^{(2,1)}(x) = \frac{1}{(l+2)(l+3)} \sum_{k=0}^{l} (k+2)(2k+3) J_k^{(1,1)}(x). \tag{2.12}$$

By taking $\alpha = 1, \beta = 1$ and $\gamma = 0$ in (2.4), we obtain

$$J_l^{(1,1)}(x) = \frac{2}{l+2} \sum_{k=0}^{l} (-1)^{l-k} (k+1) G_k(x). \tag{2.13}$$

Further, (2.3) implies that

$$\partial_x G_l(x) = \frac{1}{2}(l+2) J_{l-1}^{(2,1)}(x), \qquad \partial_x \widetilde{G}_l(x) = \frac{1}{2}(l+2) J_{l-1}^{(1,2)}(x). \tag{2.14}$$

In the sequel, we need some weighted Poincaré inequalities, stated as follows.

Lemma 2.1. *For any* $v \in H^1_\omega(\Lambda)$ *with* $v(-1) = 0$,

$$||v\omega^{\frac{1}{2}}||_\infty \le \sqrt{2}||v||_\omega |v|_{1,\omega},$$

$$||v|| \le \sqrt{2}||v||_\omega^{\frac{1}{2}} |v|_{1,\omega}^{\frac{1}{2}},$$

$$||v|| \le 2|v|_{1,\omega}.$$

Proof. For any $x \in \Lambda$,

$$v^2(x)\omega(x) = \int_{-1}^{x} \partial_y (v^2(y)\omega(y)) dy$$

and so

$$v^2(x)\omega(x) + \int_{-1}^{x} v^2(y)dy = 2\int_{-1}^{x} v(y)\partial_y(y)\omega(y)dy \le 2||v||_\omega |v|_{1,\omega}. \tag{2.15}$$

Then the first result comes immediately. Letting $x \to 1$ in (2.15), we obtain the second one. Since $||v||_\omega \le ||v||$, the last one follows.

Now let N be any positive integer, and $\mathcal{P}_N$ be the set of all algebraic polynomials of degree at most N. $\mathcal{P}_N^0 = \{v | v \in \mathcal{P}_N, v(1) = v(-1) = 0\}$. Denote by c a generic positive constant independent of any function and N.

In the numerical analysis of spectral methods, we need some weighted inverse inequalities. Let ϕ_l be an algebraic polynomial of degree l, and the set of ϕ_l be an orthogonal system in $L^2_\chi(\Lambda)$.

Lemma 2.2(see Guo[11]). *If for certain positive constant c_0 and real number δ,*

$$\|\phi_0\|_\infty \le c_0, \quad \|\phi_l\|_\infty \le c_0 l^\delta \|\phi_l\|_\chi, \ l \ge 1,$$

then for any $\phi \in \mathcal{P}_N$ and all $1 \le p \le q \le \infty$,

$$\|\phi\|_{L^q_\chi} \le c\sigma^{\frac{1}{p}-\frac{1}{q}}(N)\|\phi\|_{L^p_\chi}$$

where $\sigma(N) = N^{2\delta+1}$ for $\delta > -\frac{1}{2}$, $\sigma(N) = \ln N$ for $\delta = -\frac{1}{2}$, and $\sigma(N) = 1$ for $\delta < -\frac{1}{2}$.

Theorem 2.1. *For any $\phi \in \mathcal{P}_N$* and $1 \le p \le q \le \infty$,

$$\|\phi\|_{L^q_\omega} \le cN^{\frac{4}{p}-\frac{4}{q}}\|\phi\|_{L^p_\omega}.$$

Proof. By (2.10), $||G_l||_\omega = O(l^{-\frac{1}{2}})$. On the other hand, by Abramowitz and Stegun [14], $||G_l||_\infty \le cl$. Therefore

$$||G_l||_\infty \le cl^{\frac{3}{2}}||G_l||_\omega.$$

Finally by taking $\phi_l(x) = G_l(x), \chi(x) = \omega(x), c_0 = c, \delta = \frac{3}{2}$ in Lemma 2.2, we obtain the desired result.

Theorem 2.2. *For any $\phi \in \mathcal{P}_N$ and $r \ge 0$,*

$$||\phi||_{r,\omega} \le cN^{2r}||\phi||_\omega.$$

Proof. Let

$$\phi(x) = \sum_{l=0}^{N} \widehat{\phi}_l G_l(x), \qquad H(j,l) = \sum_{k=j}^{l}(-1)^k(2k+3).$$

By virtue of (2.12)-(2.14),

$$\begin{aligned}
\partial_x\phi(x) &= \tfrac{1}{2}\sum_{l=0}^{N-1}(l+3)\widehat{\phi}_{l+1}J_l^{(2,1)}(x)\\
&= \sum_{l=0}^{N-1}\frac{1}{l+2}\widehat{\phi}_{l+1}\Big(\sum_{k=0}^{l}(2k+3)\Big(\sum_{j=0}^{k}(-l)^{k-j}(j+1)G_j(x)\Big)\Big)\\
&= \sum_{l=0}^{N-1}\frac{1}{l+2}\widehat{\phi}_{l+1}\Big(\sum_{j=0}^{l}(-1)^j(j+1)H(j,l)G_j(x)\Big)\\
&= \sum_{j=0}^{N-1}(-1)^j(j+1)G_j(x)\Big(\sum_{l=j}^{N-1}\frac{1}{l+2}\widehat{\phi}_{l+1}H(j,l)\Big).
\end{aligned}$$

For $j, l \leq N, |H_{j,l}| \leq cN^2$. Therefore

$$||\partial_x \phi||^2_\omega \quad \leq cN^2 \sum_{j=0}^{N-1} (j+1)^2 h_j ||\phi||^2_\omega \leq cN^4 ||\phi||^2_\omega.$$

By repeating the above procedure, we find that for any non-negative integer m,

$$||\partial_x^m \phi||_\omega \leq cN^{2m} ||\phi||_\omega. \tag{2.16}$$

If $r = m+\sigma, 0 < \sigma < 1, m$ being a certain non-negative integer, then by the interpolation of Hilbert spaces (see Bergh and Löfström [15]),

$$||\phi||_{r,\omega} \leq ||\phi||^\sigma_{m+1,\omega} ||\phi||^{1-\sigma}_{m,\omega} \leq cN^{2r} ||\phi||_\omega.$$

Remark 2.1. By the Markov Theorem (see Timan[16]), for any $\phi \in \mathcal{P}_N$,

$$||\partial_x^m \phi||_\infty \leq N^{2m} ||\phi||_\infty. \tag{2.17}$$

Thus from (2.16), (2.17) and the space interpolation, we obtain that for any $\phi \in \mathcal{P}_N$, non-negative integer m and $2 \leq p \leq \infty$,

$$||\partial_x^m \phi||_{L^p_\omega} \leq cN^{2m} ||\phi||_{L^p_\omega}.$$

3. Some Jacobi Approximations

In this section, we investigate various orthogonal projections. The $L^2_\omega(\Lambda)$ -orthogonal projection $P_N : L^2_\omega(\Lambda) \to \mathcal{P}_N$ is a mapping such that for any $v \in L^2_\omega(\Lambda)$,

$$(P_N v - v, \phi)_\omega = 0, \qquad \forall \phi \in \mathcal{P}_N,$$

or equivalently,

$$P_N v(x) = \sum_{l=0}^{N} \widehat{v}_l G_l(x).$$

For technical reasons, we need some of other spaces. For non-negative integer r,

$$H^r_{\omega,A}(\Lambda) = \{v \mid v \text{ is measurable and } \|v\|_{r,\omega,A} < \infty\}$$

where

$$\|v\|_{r,\omega,A} = \begin{cases} (\sum_{k=0}^{m-1} \|(1-x^2)^{m-k} \partial_x^{2m-k} v\|^2_\omega + \|v\|^2_{m,\omega})^{\frac{1}{2}}, & \text{for} \quad r = 2m, \\ (\sum_{k=0}^{m} \|(1-x^2)^{m+\frac{1}{2}-k} \partial_x^{2m+1-k} v\|^2_\omega + \|v\|^2_{m,\omega})^{\frac{1}{2}}, & \text{for} \quad r = 2m+1. \end{cases}$$

For any real $r > 0$, the space $H^r_{\omega,A}(\Lambda)$ is defined by space interpolation. Let

$$Av(x) = -(1-x)^{-1} \partial_x ((1-x)^2 (1+x) \partial_x v(x)).$$

By induction,

$$A^m v(x) = (-1)^m (1-x^2)^m \partial_x^{2m} v(x) + \sum_{k=1}^{m-1} (1-x^2)^{m-k} p_k(x) \partial_x^{2m-k} v(x) + \sum_{k=0}^{m} q_k(x) \partial_x^k v(x)$$

where $p_k(x)$ and $q_k(x)$ are some polynomials. So A^m is a continuous mapping from $H^{2m}_{\omega,A}(\Lambda)$ to $L^2_\omega(\Lambda)$.

Next, for any even integer μ,

$$H^r_{\omega,*,\mu}(\Lambda) = \{v \mid \partial_x^\mu v \in H^{r-\mu}_{\omega,A}(\Lambda)\}, \qquad \|v\|_{r,\omega,*,\mu} = \|\partial_x^\mu v\|_{r-\mu,\omega,A}.$$

$$H^r_{\omega,**,\mu}(\Lambda) = \{v \mid v \in H^r_{\omega,*,k}(\Lambda), 0 \le k \le \mu\}, \qquad \|v\|_{r,\omega,**,\mu} = (\sum_{k=0}^{\mu} \|v\|^2_{r,\omega,*,k})^{\frac{1}{2}}.$$

For any real $\mu \ge 0$, we define the corresponding spaces by space interpolations. In particular, $||v||_{r,w,*} = ||v||_{r,\omega,*,1}$.

Theorem 3.1. *For any $v \in H^r_{\omega,A}(\Lambda)$ and $r \ge 0$,*

$$||P_N v - v||_\omega \le cN^{-r}||v||_{r,\omega,A}.$$

Proof. We first assume that $r = 2m$. By virtue of (2.1) and (2.10), we find that

$$\begin{aligned}
\widehat{v}_l &= \tfrac{1}{2}(l+1)\int_\Lambda v(x)G_l(x)\omega(x)dx \\
&= -\tfrac{l+1}{2l(l+2)}\int_\Lambda v(x)\partial_x((1-x)^2(1+x)\partial_x G_l(x))dx \\
&= \tfrac{l+1}{2l(l+2)}\int_\Lambda \partial_x v(x)(1-x)^2(1+x)\partial_x G_l(x)dx \\
&= \tfrac{l+1}{2l(l+2)}\int_\Lambda Av(x)G_l(x)\omega(x)dx \\
&= \dots \\
&= \tfrac{l+1}{2l^m(l+2)^m}\int_\Lambda A^m v(x)G_l(x)\omega(x)dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
||P_N v - v||^2_\omega &= \sum_{l=N+1}^{\infty} \widehat{v}_l^2 h_l \le cN^{-4m}\sum_{l=N+1}^{\infty} h_l\,(\frac{\int_\Lambda A^m v(x)G_l(x)\omega(x)dx}{h_l})^2 \\
&\le cN^{-2r}||A^m v||^2_\omega \le cN^{-2r}||v||^2_{r,\omega,A}.
\end{aligned}$$

We now let $r = 2m+1$. By (2.1) and (2.3),

$$\widehat{v}_l = \frac{l+1}{4l^{m+1}(l+2)^m}\int_\Lambda \partial_x A^m v(x) J^{(2,1)}_{l-1}(x)\chi^{(2,1)}(x)dx.$$

Thus

$$\begin{aligned}
\|P_N v - v\|^2_\omega &= \tfrac{1}{16}\sum_{l=N+1}^{\infty}\frac{(l+1)^2 h_l(\gamma^{(2,1)}_{l-1})^2}{l^{2m+2}(l+2)^{2m}}\,(\frac{1}{\gamma^{(2,1)}_{l-1}}\int_\Lambda \partial_x A^m v(x)J^{(2,1)}_l(x)\chi^{(2,1)}(x)dx)^2 \\
&\le cN^{-4m-2}\|\partial_x A^m v\|^2_{\chi^{(2,1)}}.
\end{aligned}$$

Moreover

$$\|\partial_x A^m v\|_{\chi^{(2,1)}} \le c\|v\|_{2m+1,\omega,A}.$$

Thus

$$\|v - P_N v\|_\omega \le cN^{-r}\|v\|_{r,\omega,A}.$$

Finally we complete the proof by using space interpolation.

Generally, $P_N\partial_x v(x) \ne \partial_x P_N v(x)$. But we have the following result.

Lemma 3.1. *For any $v \in H^r_{\omega,*}(\Lambda)\bigcap L^2_\omega(\Lambda)$ and $r \ge 1$,*

$$||P_N\partial_x v - \partial_x P_N v||_\omega \le cN^{2-r}(||v||_{r,\omega,*} + ||v||_\omega).$$

Proof. By an argument as in the proof of Theorm 2.2,

$$\partial_x P_N v(x) = \sum_{j=0}^{N-1} (-1)^j (j+1) G_j(x) (\sum_{l=j}^{N-1} \frac{1}{l+2} H_{j,l} \widehat{v}_{l+1}).$$

Similarly

$$\partial_x v(x) = \sum_{j=0}^{\infty} a_j G_j(x),$$

$$P_N \partial_x v(x) = \sum_{j=0}^{N} a_j G_j(x)$$

where

$$a_j = (-1)^j (j+1) \sum_{l=j}^{\infty} \frac{1}{l+2} H_{j,l} \widehat{v}_{l+1}.$$

Thus

$$P_N \partial_x v(x) - \partial_x P_N v(x) = \sum_{j=0}^{N} (-1)^j (j+1) G_j(x) (\sum_{l=N}^{\infty} \frac{1}{l+2} H_{j,l} \widehat{v}_{l+1}).$$

Let

$$L_N(x) = a_N \phi_N(x), \qquad \phi_N(x) = \frac{1}{N+1} \sum_{j=0}^{N} (-1)^{j+N} (j+1) G_j(x),$$

$$M_N(x) = \sum_{j=0}^{N} (-1)^j (j+1) G_j(x) (\sum_{l=N}^{\infty} \frac{1}{l+2} H_{j,N} \widehat{v}_{l+1}).$$

Since $H_{j,l} = H_{j,N} + H_{N,l}$, we have that

$$P_N \partial_x v(x) - \partial_x P_N v(x) = L_N(x) + M_N(x).$$

We now estimate $||L_N||_\omega$. By (2.10) and Theorem 3.1,

$$a_N^2 \le c h_N^{-1} ||P_N \partial_x v - \partial_x v||_\omega^2 \le c N^{3-2r} ||v||_{r,\omega,*}^2.$$

(2.10) also implies that $||\phi_N||_\omega^2 \le c$. So

$$||L_N||_\omega^2 \le c N^{3-2r} ||v||_{r,\omega,*}^2.$$

We next estimate $||M_N||_\omega$. Without losing generality, let r be any even integer. Then by (2.10) and the expression of $\widehat{v}_{l+1}$ in the proof of Theorem 3.1,

$$\begin{aligned} ||M_N||_\omega^2 &\le c \sum_{j=0}^{N} j^2 h_j (\sum_{l=N}^{\infty} H_{j,N} \widehat{v}_{l+1})^2 \\ &\le c N^2 \sum_{j=0}^{N} j (\sum_{l=N}^{\infty} l^{-1-2r}) (\sum_{l=N}^{\infty} h_l^{-1} (\int_\Lambda A^{\frac{r}{2}} v(x) G_l(x) \omega(x) dx)^2) \\ &\le c N^{4-2r} ||v||_{r,\omega,A}^2 \le c N^{4-2r} (||v||_{r,\omega,*}^2 + ||v||_\omega^2). \end{aligned}$$

Let

$$\sigma(\mu, r) = \begin{cases} 2\mu - r, & \text{for } \mu \ge 0, \\ \mu - r, & \text{for } \mu < 0. \end{cases}$$

Theorem 3.2. *For any* $v \in H^r_{\omega,**,\mu}(\Lambda), r \geq 1$ *and* $\mu \leq r$,

$$||P_N v - v||_{\mu,\omega} \leq cN^{\sigma(\mu,r)}||v||_{r,\omega,**,\mu}.$$

Proof. We use induction. Obviously Theorem 3.1 implies the desired result for $\mu = 0$. Now let $\mu > 0$. The space interpolation allows us to consider positive integer μ only. Assume that the conclusion is true for $\mu - 1$. Then

$$||P_N \partial_x v - \partial_x v||_{\mu-1,\omega} \leq cN^{\sigma(\mu-1,r-1)}||\partial_x v||_{r-1,\omega,**,\mu-1} \leq cN^{\sigma(\mu-1,r-1)}||v||_{r,\omega,**,\mu}.$$

Clearly $P_N\partial_x v(x) - \partial_x P_N v(x)$ is in the subspace $\mathcal{P}_N$. Therefore we deduce from Theorem 2.2 and Lemma 3.1 that

$$\begin{aligned} ||P_N v - v||_{\mu,\omega} &\leq ||P_N\partial_x v - \partial_x v||_{\mu-1,\omega} + ||P_N\partial_x v - \partial_x P_N v||_{\mu-1,\omega} + ||P_N v - v||_\omega \\ &\leq cN^{\sigma(\mu-1,r-1)}||v||_{r,\omega,**,\mu} + cN^{2\mu-2}||P_N\partial_x v - \partial_x P_N v||_\omega + ||P_N v - v||_\omega \\ &\leq cN^{\sigma(\mu-1,r-1)}||v||_{r,\omega,**,\mu} + cN^{\sigma(\mu,r)}(||v||_{r,\omega,*} + ||v||_\omega) + cN^{-r}||v||_{r,\omega,A}. \end{aligned}$$

Since $\sigma(\mu-1, r-1) \leq \sigma(\mu, r)$, the conclusion for $\mu > 0$ follows. Finally a duality argument leads to the result for $\mu < 0$.

We now turn to the $H^1_\omega(\Lambda)$-orthogonal projection $P^1_N : H^1_\omega(\Lambda) \to \mathcal{P}_N$. It is a mapping such that for any $v \in H^1_\omega(\Lambda)$,

$$(P^1_N v - v, \phi)_{1,\omega} = 0, \qquad \forall \phi \in \mathcal{P}_N.$$

Theorem 3.3. *For any* $v \in H^r_{\omega,*}(\Lambda)$ *and* $r \geq 1$,

$$||P^1_N v - v||_{1,\omega} \leq cN^{1-r}||v||_{r,\omega,*}.$$

Proof. Let

$$\phi(x) = \int_{-1}^x P_{N-1}\partial_y v(y)dy + \gamma$$

where γ is chosen in such a way that $\phi(-1) = v(-1)$. By the projection theorem, Lemma 2.1 and Theorem 3.1,

$$\begin{aligned} ||P^1_N v - v||_{1,\omega} &\leq ||\phi - v||_{1,\omega} \leq c|\phi - v|_{1,\omega} \\ &= c||P_{N-1}\partial_x v - \partial_x v||_\omega \leq cN^{1-r}||v||_{r,\omega,*}. \end{aligned}$$

In practical problems, we need other kinds of orthogonal projections from certain subsets of $H^1_\omega(\Lambda)$ to some subsets of $\mathcal{P}_N$. Firstly let

$$\overline{H}^1_\omega(\Lambda) = H^1_\omega(\Lambda) \bigcap \{v | v(-1) = 0\}, \qquad \overline{\mathcal{P}}_N = \mathcal{P}_N \bigcap \overline{H}^1_\omega(\Lambda),$$

and

$$a_\omega(u, v) = (\partial_x u, \partial_x v)_\omega.$$

The orthogonal projection $\overline{P}^1_N : \overline{H}^1_\omega(\Lambda) \to \overline{\mathcal{P}}_N$ is a mapping such that for any $v \in \overline{H}^1_\omega(\Lambda)$,

$$a_\omega(\overline{P}^1_N v - v, \phi) = 0, \qquad \forall \phi \in \overline{\mathcal{P}}_N.$$

Theorem 3.4. *For any* $v \in \overline{H}^1_\omega(\Lambda) \bigcap H^r_{\omega,*}(\Lambda)$ *and* $r \geq 1$,

$$||\overline{P}^1_N v - v||_{1,\omega} \leq cN^{1-r}||v||_{r,\omega,*}.$$

$$||\overline{P}^1_N v - v|| \leq cN^{1-r}||v||_{r,\omega,*}.$$

Proof. Let

$$\phi(x) = \int_{-1}^{x} P_{N-1}\partial_y v(y)dy.$$

Clearly $\phi \in \overline{\mathcal{P}}_N$. By the projection theorem, Lemma 2.1 and Theorem 3.1,

$$\begin{aligned} ||\overline{P}_N^1 v - v||_{1,\omega} &\le c|\overline{P}_N^1 v - v|_{1,\omega} \le c|\phi - v|_{1,\omega} \\ &\le c||P_{N-1}\partial_x v - \partial_x v||_\omega \le cN^{1-r}||v||_{r,\omega,*}. \end{aligned}$$

The second result comes from Lemma 2.1 and the previous result.

The $H^1_{0,\omega}(\Lambda)$ -orthogonal projection $P_N^{1,0} : H^1_{0,\omega}(\Lambda) \to \mathcal{P}_N^0$ is a mapping such that for any $v \in H^1_{0,\omega}(\Lambda)$,

$$a_\omega(P_N^{1,0} v - v, \phi) = 0, \quad \forall \phi \in \mathcal{P}_N^0.$$

Theorem 3.5. *For any* $v \in H^1_{0,\omega}(\Lambda) \bigcap H^r(\Lambda)$ *and* $r \ge 1$,

$$||P_N^{1,0} v - v||_{1,\omega} \le cN^{1-r}||v||_r,$$

$$||P_N^{1,0} v - v|| \le cN^{1-r}||v||_r.$$

Proof. Let P_N^* be the $L^2(\Lambda)$-orthogonal projection. By (2.8) and the property of Legendre approximation (see Canuto and Quarteroni [17]),

$$||P_N^* v - v|| \le \sum_{l=N+1}^{\infty} h_l^*(v_l^*)^2 \le cN^{-r}||v||_r. \tag{3.1}$$

Let

$$\phi^*(x) = \int_{-1}^{x} P_{N-1}^*\partial_y v(y)dy,$$

$$\phi(x) = \int_{-1}^{x} (P_{N-1}^*\partial_y v(y) - \frac{1}{2}\phi^*(1))dy.$$

Clearly $\phi \in \mathcal{P}_N^0$. By the projection theorem and Lemma 2.1,

$$\begin{aligned} ||P_N^{1,0} v - v||_{1,\omega} &\le c|P_N^{1,0} v - v|_{1,\omega} \le c|\phi - v|_{1,\omega} \\ &\le c||P_{N-1}^*\partial_x v - \partial_x v||_\omega + \tfrac{1}{2}|\phi^*(1)|(\textstyle\int_\Lambda \omega(x)dx)^{\frac{1}{2}}. \end{aligned}$$

Moreover, using (3.1) yields

$$\begin{aligned} |\phi^*(1)| = |v(1) - \phi^*(1)| &= |\textstyle\int_\Lambda (P_{N-1}^*\partial_x v(x) - \partial_x v(x))dx| \\ &\le c||P_{N-1}^*\partial_x v - \partial_x v|| \le cN^{1-r}||v||_r. \end{aligned}$$

This leads to the first conclusion.

Using Lemma 2.1 again, we obtain the second result.

When we use the Jacobi approximations for nonlinear problems, we also need estimations for $||P_N^{1,0} v||_{W^{r,p}(\Lambda)}$. The simplest one is given by the following theorem.

Theorem 3.6. *Let* $\delta \ge 1$. *For any* $v \in \overline{H}^1_\omega(\Lambda) \bigcap H^{1+\delta}_{\omega,*}(\Lambda) \bigcap H^\delta(\Lambda)$,

$$||\overline{P}_N^1 v\|_\infty \le c(\|v\|_{1+\delta,\omega,*} + \|v\|_\delta).$$

For any $v \in H^1_{0,\omega}(\Lambda) \cap H^{1+\delta}(\Lambda)$,

$$||P_N^{1,0} v||_\infty \le c|v|_{1+\delta}.$$

Proof. By imbedding theory,

$$||\overline{P}_N^1 v||_\infty \le ||v||_\infty + c||\overline{P}_N^1 v - v||_{\frac{\delta}{2}}.$$

Let P_N^* be the $L^2(\Lambda)$-orthogonal projection. We have that

$$||\overline{P}_N^1 v - v||_{\frac{\delta}{2}} \le ||\overline{P}_N^1 v - P_N^* v||_{\frac{\delta}{2}} + ||P_N^* v - v||_{\frac{\delta}{2}}.$$

By (3.1), Theorem 3.4 and an inverse inequality in $\mathcal{P}_N$,

$$||\overline{P}_N^1 v - P_N^* v||_{\frac{\delta}{2}} \le cN^\delta(||\overline{P}_N^1 v - v|| + ||P_N^* v - v||) \le c(||v||_{1+\delta,\omega,*} + ||v||_\delta).$$

According to an estimate in Canuto and Quarteroni [16],

$$||P_N^* v - v||_{\frac{\delta}{2}} \le c||v||_{\frac{3}{4}\delta}.$$

The combination of the above estimates implies the first result.

The second result comes from Theorem 3.5 and an argument as in the previous paragraph.

In applications of Jacobi approximations to initial-boundary value problems of partial differential equations, we need certain projections with different weight functions. We first consider the $L^2_{\widetilde{\omega}}(\Lambda)$ -orthogonal projection $\widetilde{P}_N : L^2_{\widetilde{\omega}}(\Lambda) \to \mathcal{P}_N$ such that for any $v \in L^2_{\widetilde{\omega}}(\Lambda)$,

$$(\widetilde{P}_N v - v, \phi)_{\widetilde{\omega}} = 0, \quad \forall \phi \in \mathcal{P}_N,$$

or equivalently,

$$\widetilde{P}_N v(x) = \sum_{l=0}^{N} \widetilde{v}_l \widetilde{G}_l(x).$$

Let

$$\widetilde{A}v(x) = -(1+x)^{-1}\partial_x((1-x)(1+x)^2\partial_x v(x)).$$

For technical reasons, we introduce the space $H^r_{\widetilde{\omega},\widetilde{A}}(\Lambda)$. For non-negative integer r,

$$H^r_{\widetilde{\omega},\widetilde{A}}(\Lambda) = \{v|v \text{ is measurable and } ||v||_{r,\widetilde{\omega},\widetilde{A}} < \infty\}$$

where

$$\begin{aligned}
&||v||_{r,\widetilde{\omega},\widetilde{A}} = (||(1-x^2)^{\frac{r}{2}}\partial_x^r v||^2_{\widetilde{\omega}} + ||(1-x^2)^{\frac{r}{2}-1}\partial_x^{r-1} v||^2_{\widetilde{\omega}} + ||v||^2_{r-2,\widetilde{\omega}})^{\frac{1}{2}}, \quad r \ge 2,\\
&\|v\|_{1,\widetilde{\omega},\widetilde{A}} = (\|(1-x^2)^{\frac{1}{2}}\partial_x v\|^2_{\widetilde{\omega}} + \|v\|^2_{\widetilde{\omega}})^{\frac{1}{2}},\\
&\|v\|_{0,\widetilde{\omega},\widetilde{A}} = \|v\|_{\widetilde{\omega}}.
\end{aligned}$$

For any $r \ge 0$, the space $H^r_{\widetilde{\omega},\widetilde{A}}(\Lambda)$ is defined by space interpolation. It can be shown that $\widetilde{A}$ is a continuous mapping from $H^{r+2}_{\widetilde{\omega},\widetilde{A}}(\Lambda)$ to $H^r_{\widetilde{\omega}}(\Lambda)$. Also let $H^r_{\widetilde{\omega},*}(\Lambda) = \{v|\partial_x v \in H^{r-1}_{\widetilde{\omega},\widetilde{A}}(\Lambda)\}$ and $||v||_{r,\widetilde{\omega},*} = ||\partial_x v||_{r-1,\widetilde{\omega},\widetilde{A}}$.

Theorem 3.7. *For any $v \in H^r_{\widetilde{\omega},\widetilde{A}}(\Lambda)$ and $r \ge 0$,*

$$||\widetilde{P}_N v - v||_{\widetilde{\omega}} \le cN^{-r}||v||_{r,\widetilde{\omega},\widetilde{A}}.$$

Proof. We first assume that $r = 2m$. By virtue of (2.1) and (2.11),

$$\begin{aligned}
\widetilde{v}_l &= \tfrac{1}{2}(l+1)\int_\Lambda v(x)\widetilde{G}_l(x)\widetilde{\omega}(x)dx \\
&= -\tfrac{l+1}{2l(l+2)}\int_\Lambda v(x)\partial_x((1-x)(1+x)^2\partial_x\widetilde{G}_l(x))dx \\
&= \tfrac{l+1}{2l(l+2)}\int_\Lambda \partial_x v(x)(1-x)(1+x)^2\partial_x\widetilde{G}_l(x)dx \\
&= \tfrac{l+1}{2l(l+2)}\int_\Lambda \widetilde{A}v(x)\widetilde{G}_l(x)\widetilde{\omega}(x)dx \\
&= \dots \\
&= \tfrac{l+1}{2l^m(l+2)^m}\int_\Lambda \widetilde{A}^m v(x)\widetilde{G}_l(x)\widetilde{\omega}(x)dx.
\end{aligned}$$

Thus

$$||\widetilde{P}_N v - v||^2_\omega = \sum_{l=N+1}^{\infty} \widetilde{v}_l^2 \widetilde{h}_l \le cN^{-2r}||v||^2_{r,\widetilde{\omega},\widetilde{A}}.$$

The above estimate is also valid for $r = 2m+1$. Finally we complete the proof by space interpolation.

Now let

$$L^2_{0,\eta}(\Lambda) = \{v|v \in L^2_\eta(\Lambda), v(1) = v(-1) = 0\}$$

and

$$g_l(x) = (1-x^2)J_l^{(1,2)}(x), \quad 0 \le l \le N-2.$$

The set $\{g_l\}$ is the $L^2_{0,\eta}(\Lambda)-$orthogonal system, since

$$(g_l, g_m)_\eta = (J_l^{(1,2)}, J_m^{(1,2)})_{\chi^{(1,2)}} = \gamma_l^{(1,2)}\delta_{l,m} = \frac{8(l+1)}{(l+2)(l+3)}\delta_{l,m}.$$

For any $v \in L^2_{0,\eta}(\Lambda)$,

$$v(x) = \sum_{l=0}^{\infty} \widehat{\widehat{v}}_l g_l(x)$$

where

$$\widehat{\widehat{v}}_l = \frac{1}{\gamma_l^{(1,2)}}\int_\Lambda v(x)g_l(x)\eta(x)dx.$$

The $L^2_{0,\eta}(\Lambda)-$orthogonal projection $\widehat{P}_N : L^2_{0,\eta}(\Lambda) \to \mathcal{P}^0_N$ is a mapping such that for any $v \in L^2_{0,\eta}(\Lambda)$,

$$(\widehat{P}_N v - v, \phi)_\eta = 0, \quad \forall \phi \in \mathcal{P}^0_N.$$

Let

$$\widehat{A}v(x) = (1+x)^{-1}\partial_x(v(x)(1+x))$$

and

$$H^r_{\widetilde{\omega},\widehat{A}}(\Lambda) = \{v|v \in L^2_{0,\eta}(\Lambda) \ \text{ and } ||v||_{r,\widetilde{\omega},\widehat{A}} < \infty\}, \quad ||v||_{r,\widetilde{\omega},\widehat{A}} = ||\widehat{A}v||_{r-1,\widetilde{\omega},\widetilde{A}}.$$

Theorem 3.8. *For any* $v \in L^2_{0,\eta}(\Lambda)\bigcap H^r_{\widetilde{\omega},\widehat{A}}(\Lambda)$ *and* $r \ge 0$,

$$||\widehat{P}_N v - v||_\eta \le cN^{-r}||v||_{r,\widetilde{w},\widehat{A}}.$$

Proof. Let a_l be the coefficient of expansion of function $\widehat{A}v(x)$ in terms of $\widetilde{G}_l(x)$. By (2.3),

$$\begin{aligned}\widehat{\widehat{v}}_l &= \tfrac{1}{\gamma^{(1,2)}}\int_\Lambda v(x)(1+x)J_l^{(1,2)}(x)dx\\ &= \tfrac{2}{(l+3)\gamma_l^{(1,2)}}\int_\Lambda v(x)(1+x)\partial_x\widetilde{G}_{l+1}(x)dx\\ &= -\tfrac{2}{(l+3)\gamma_l^{(1,2)}}\int_\Lambda \partial_x(v(x)(1+x))\widetilde{G}_{l+1}(x)dx\\ &= -\tfrac{2}{(l+3)\gamma_l^{(1,2)}}\int_\Lambda (1+x)^{-1}\partial_x(v(x)(1+x))\widetilde{G}_{l+1}(x)\widetilde{\omega}(x)dx\\ &= -\tfrac{2h_l}{(l+3)\gamma_l^{(1,2)}}a_{l+1}.\end{aligned}$$

Therefore Theorem 3.7 yields

$$||\widehat{P}_N v - v||_\eta^2 = \sum_{l=N+1}^{\infty}\gamma_l^{(1,2)}\widehat{\widehat{v}}_l^2 \le cN^{-2}\sum_{l=N+1}^{\infty}\widetilde{h}_l a_l^2 \le cN^{-2r}||v||^2_{r,\widetilde{\omega},\widehat{A}}.$$

For $r \ge 0$, let

$$W^r(\Lambda) = H^{r+2}_{\widetilde{\omega},\widehat{A}}(\Lambda) + H^{r+3}_{\omega,*}(\Lambda)$$

with the norm

$$\|v\|_{W^r(\Lambda)} = (\|v\|^2_{r+2,\widetilde{\omega},\widehat{A}} + \|v\|^2_{r+3,\omega,*})^{\frac{1}{2}}.$$

Theorem 3.9. *For any $v \in L^2_{0,\eta}(\Lambda)\bigcap W^r(\Lambda)$ and $r \ge 0$,*

$$\|\hat{P}_N v - v\|_{1,\omega} \le cN^{-r}\|v\|_{W^r(\Lambda)}.$$

Proof. We have from Lemma 2.1 that

$$\begin{aligned}\|\hat{P}_N v - v\|_{1,\omega} &\le c|\hat{P}_N v|_{1,\omega}\\ &\le |\hat{P}_N v - P_N^1 v|_{1,\omega} + |P_N^1 v - v|_{1,\omega}.\end{aligned}$$

By virtue of Theorems 2.2, 3.3 and 3.8,

$$\begin{aligned}|\hat{P}_N v - P_N^1 v|_{1,\omega} &\le cN^2\|\hat{P}_N - P_N^1 v\|_\omega\\ &\le cN^2(\|\hat{P}_N v - v\| + \|P_N^1 v - v\|_\omega) \le cN^{-r}\|v\|_{W^r(\Lambda)}.\end{aligned}$$

Also by Theorem 3.3,

$$|P_N^1 - v|_{1,\omega} \le cN^{-r}\|v\|_{r+1,\omega,*}.$$

The proof is complete.

Theorem 3.10. *For any $v \in L^2_{0,\eta}(\Lambda)\bigcap H^\delta_{\widetilde{\omega},\widehat{A}}\bigcap H^\delta(\Lambda)$ and $\delta > 1$,*

$$\|\hat{P}_N v\|_\infty \le c(\|v\|_{\delta,\widetilde{\omega},\widehat{A}} + \|v\|_\delta).$$

Proof. By the imbedding theory

$$\|\hat{P}_N v\|_\infty \le c(\|v\|_{\frac{\delta}{2}} + \|\hat{P}_N v - P_N^* v\|_{\frac{\delta}{2}} + \|P_N^* v - v\|_{\frac{\delta}{2}}).$$

By (3.1), Theorem 3.8 and an inverse inequality in $\mathcal{P}_N$,

$$\begin{aligned}\|\hat{P}_N v - P_N^* v\|_{\frac{\delta}{2}} &\le cN^\delta\|\hat{P}_N v - P_N^* v\|\\ &\le cN^\delta(\|\hat{P}_N v - v\|_\eta + \|P_N^* v - v\|)\\ &\le c(\|v\|_{\delta,\widetilde{\omega},\widehat{A}} + \|v\|_\delta).\end{aligned}$$

Moreover,

$$\|P_N^* v - v\|_{\frac{\delta}{2}} \le c\|v\|_{\frac{3}{4}\delta}.$$

4. Some Applications

As pointed out in the first section, the main motivation of studying unsymmetric Jacobi approximations in this paper is to consider their use for numerical solutions of differential equations on the half line. We first consider a simple example as follows,

$$\begin{cases} -\partial_y^2 V(y) = F(y), & 0 < y < \infty, \\ \partial_y V(y) \to 0, & \text{as } y \to \infty, \\ V(0) = d. \end{cases} \tag{4.1}$$

Let

$$y(x) = -2\ln(1-x) + 2\ln 2. \tag{4.2}$$

Then $y(-1) = 0, y(1) = \infty$ and for $x \in \Lambda$,

$$\frac{dx}{dy} = (\frac{dy}{dx})^{-1} = \frac{1}{2}(1-x) > 0.$$

Let $U(x) = V(y(x)), f(x) = F(y(x))$ and for simplicity, $d = 0$. Then

$$\begin{cases} -\partial_x(\omega(x)\partial_x U(x)) = 4f(x)\eta(x), & x \in \Lambda, \\ \partial_x U(x)\omega(x) \to 0, & \text{as } x \to 1, \\ U(-1) = 0. \end{cases} \tag{4.3}$$

A weak formulation of (4.3) is to find $U \in \overline{H}^1_w(\Lambda)$ such that

$$a_\omega(U, v) = 4(f, v)_\eta, \quad \forall v \in \overline{H}^1_\omega(\Lambda). \tag{4.4}$$

If $f \in \left(\overline{H}^1_\omega(\Lambda)\right)'$, then (4.4) has a unique solution. Let $u_N \in \overline{\mathcal{P}}_N$, be the numerical solution satisfying

$$a_\omega(u_N, \phi) = 4(f, \phi)_\eta, \quad \forall \phi \in \overline{\mathcal{P}}_N. \tag{4.5}$$

Then $u_N = \overline{P}_N U$. According to Theorem 3.4, we know that

$$\|U - u_N\| + ||U - u_N||_{1,\omega} \le cN^{1-r}||U||_{r,\omega,*}.$$

Remark 4.1. In actual computation, a suitable choice of base functions is important. We can take the base functions in (4.5) as

$$\phi_l(x) = \sqrt{\frac{2}{l}} J_l^{(0,-1)}(x), \quad 1 \le l \le N.$$

Clearly $\phi_l \in \overline{\mathcal{P}}_N$. Moreover (2.3) implies that

$$a_\omega(\phi_l, \phi_m) = (G_{l-1}, G_{m-1})_\omega = \delta_{l,m}$$

which leads to the identity matrix for the Jacobi coefficients of unknown function $u_N(x)$.

Remark 4.2. If in addition $V(y) \to 0$ as $y \to \infty$ in (4.1), then $U(1) = 0$ in (4.3). In this case, the numerical solution $u_N \in \mathcal{P}_N^0$ satisfies

$$a_\omega(u_N, \phi) = 4(f, \phi)_\eta, \quad \forall \phi \in \mathcal{P}_N^0.$$

On use of Theorem 3.5, we assert that

$$||U - u_N|| + ||U - u_N||_{1,\omega} \leq cN^{1-r}||U||_r.$$

Now we turn to a more complicated example. As we know, the Burgers equation plays an important role in the description of one-dimensional fluid flow. The simplest initial-boundary value problem of Burgers equation is as follows

$$\begin{cases} \partial_t V(y,t) + V(y,t)\partial_y V(y,t) - \mu\partial_y^2 V(y,t) = F(y,t), & 0 < y < \infty, 0 < t \leq T, \\ V(y,t) \to 0, & y \to \infty, 0 \leq t \leq T, \\ V(0,t) = d(t), & 0 \leq t \leq T, \\ V(y,0) = V_0(y), & 0 \leq y < \infty \end{cases} \tag{4.6}$$

where $\mu > 0$ is the kinetic viscosity, $d(0) = V_0(0)$ and $V_0(y) \to 0, F(y,t) \to 0$ as $y \to \infty$. We use the variable transformation (4.2), and put $U(x,t) = V(y(x),t), f(x,t) = F(y(t),t)$ and $U_0(x) = V_0(y(x))$. For simplicity, let $d(t) \equiv 0$. Then (4.6) becomes

$$\begin{cases} \eta(x)\partial_t U(x,t) + \frac{1}{2}U(x,t)\partial_x U(x,t) - \frac{1}{4}\mu\partial_x(\omega(x)\partial_x U(x,t)) & \\ \quad = \eta(x)f(x,t), & x \in \Lambda, 0 < t \leq T, \\ U(1,t) = U(0,t) = 0, & 0 \leq t \leq T, \\ U(x,0) = U_0(x), & x \in \overline{\Lambda}. \end{cases} \tag{4.7}$$

A weak formulation of (4.7) is to find $U \in L^2(0,T;H^1_{0,\omega}(\Lambda)) \cap L^\infty(0,T;L^2_\eta(\Lambda))$ such that

$$\begin{cases} (\partial_t U(t), v)_\eta - \frac{1}{4}(U^2(t), \partial_x v) + \frac{1}{4}\mu a_\omega(U(t), v) & \\ \quad = (f(t), v)_\eta, & \forall v \in H^1_{0,\omega}(\Lambda), 0 < t \leq T, \\ U(0) = U_0. \end{cases} \tag{4.8}$$

The solution of (4.8) satisfies the following equality,

$$||U(t)||^2_\eta + \frac{1}{2}\mu\int_0^t |U(\xi)|^2_{1,\omega}d\xi = ||U_0||^2_\eta + 2\int_0^t (f(\xi), U(\xi))_\eta d\xi. \tag{4.9}$$

Let u_N be the numerical approximation to U. The Jacobi spectral scheme for (4.8) is to find $u_N \in \mathcal{P}^0_N$ for all $0 \leq t \leq T$ such that

$$\begin{cases} (\partial_t u_N(t), \phi)_\eta - \frac{1}{4}(u_N^2(t), \partial_x\phi) + \frac{1}{4}\mu a_\omega(u_N(t), \phi) & \\ \quad = (f(t), \phi)_\eta, & \forall \phi \in \mathcal{P}^0_N, 0 < t \leq T, \\ u_N(0) = u_{N,0} = \widehat{P}_N U_0. \end{cases} \tag{4.10}$$

By taking $\phi = u_N$ in (4.10), it follows that

$$||u_N(t)||^2_\eta + \frac{1}{2}\mu\int_0^t |u_N(\xi)|^2_{1,\omega}d\xi = ||u_{N,0}||^2_\eta + 2\int_0^t (f(\xi), u_N(\xi))_\eta d\xi.$$

This is a reasonable analogy of (4.9).

We now turn to analysis of the stability of scheme (4.10). Since this is a nonlinear problem, it is not possible to be stable in the sense of Courant, Friedrichs and Lewy [18]. But it might be stable in the sense of Guo [19, 20] and Stetter [21]. To this end, suppose that f and $u_{N,0}$ are disturbed by $\widetilde{f}$ and $\widetilde{u}_{N,0}$, respectively. Accordingly, the numerical solution u_N has the error $\widetilde{u}_N$. Then we get from (4.10) that

$$\begin{cases} (\partial_t \widetilde{u}_N(t), \phi)_\eta - \frac{1}{4}(\widetilde{u}_N^2(t) + 2u_N(t)\widetilde{u}_N(t), \partial_x \phi) \\ \quad + \frac{1}{4}\mu a_\omega(\widetilde{u}_N(t), \phi) = (\widetilde{f}(t), \phi)_\eta, \qquad \forall \phi \in \mathcal{P}_N^0, 0 < t \le T, \\ \widetilde{u}_N(0) = \widetilde{u}_{N,0}. \end{cases} \tag{4.11}$$

By taking $\phi = 2\widetilde{u}_N$ in (4.11), it follows that

$$\frac{d}{dt}||\widetilde{u}_N(t)||_\eta^2 + F(t) + \frac{1}{2}\mu|\widetilde{u}_N(t)|_{1,\omega}^2 = 2(\widetilde{f}(t), \widetilde{u}_N(t))_\eta \tag{4.12}$$

where

$$F(t) = -(u_N(t)\widetilde{u}_N(t), \partial_x \widetilde{u}_N(t)).$$

It is easy to see that

$$\begin{aligned} |F(t)| &\le ||u_N(t)||_\infty ||\widetilde{u}_N(t)||_\eta |\widetilde{u}_N(t)|_{1,\omega} \\ &\le \frac{1}{\mu}||u_N(t)||_\infty^2 ||\widetilde{u}_N(t)||_\eta^2 + \frac{1}{4}\mu|\widetilde{u}_N(t)|_{1,\omega}^2. \end{aligned} \tag{4.13}$$

To describe the errors, let

$$E(v,t) = ||v(t)||_\eta^2 + \frac{1}{4}\mu \int_0^t |v(\xi)|_{1,\omega}^2 d\xi,$$

$$\rho(w_1, w_2, t) = ||w_1||_\eta^2 + \int_0^t ||w_2(\xi)||_\eta^2 d\xi.$$

Also let $|||v|||_\infty = \sup\limits_{0 \le t \le T} ||v(t)||_\infty$, and

$$M(v) = 1 + \frac{1}{\mu}|||v|||_\infty.$$

By substituting (4.13) into (4.12) and integrating the resulting inequality for t, we find that

$$\begin{aligned} E(\widetilde{u}_N, t) &\le M(u_N) \textstyle\int_0^t ||\widetilde{u}_N(\xi)||_\eta^2 d\xi + \rho(\widetilde{u}_{N,0}, \widetilde{f}, t) \\ &\le M(u_N) \textstyle\int_0^t E(\widetilde{u}_N, \xi) d\xi + \rho(\widetilde{u}_{N,0}, \widetilde{f}, t). \end{aligned} \tag{4.14}$$

Finally we reach the following conclusion.

Theorem 4.1. *Let u_N be the solution of (4.10), and $\widetilde{u}_N$ be its error induced by $\widetilde{u}_{N,0}$ and $\widetilde{f}$. Then for all $0 \le t \le T$,*

$$E(\widetilde{u}_N, t) \le \rho(\widetilde{u}_{N,0}, \widetilde{f}, t) e^{M(u_N)t}.$$

The previous theorem shows that the error of the numerical solution is bounded by the error of the initial data and the average error of the source term. In other words, scheme (4.10) satisfies the generalized stability of Guo [19, 20], or the restricted stability of Stetter [21].

We next deal with the convergence. Let $U_N = \widehat{P}_N U$. Then we derive from (4.8) that

$$\begin{cases} (\partial_t U_N(t), \phi)_\eta - \frac{1}{4}(U_N^2(t), \partial_x \phi) + \frac{1}{4}\mu a_\omega(U_N(t), \phi) \\ \quad + G_0(\phi, t) + G_1(\phi, t) + G_2(\phi, t) = (f(t), \phi)_\eta, \qquad \forall \phi \in \mathcal{P}_N^0, 0 < t \le T, \\ U_N(0) = \widehat{P}_N U_0 \end{cases} \tag{4.15}$$

where

$$G_0(\phi, t) = \frac{1}{4}\mu a_\omega(U(t) - U_N(t), \phi),$$

$$G_1(\phi,t)=(\partial_t U(t)-\partial_t U_N(t),\phi)_\eta,$$
$$G_2(\phi,t)=\frac{1}{4}(U_N^2(t)-U^2(t),\partial_x\phi).$$

Further put $\widetilde{U}_N=u_N-U_N$. We obtain from (4.10) and (4.15) that

$$\begin{cases}(\partial_t\widetilde{U}_N(t),\phi)_\eta-\frac{1}{4}(\widetilde{U}_N^2(t)+2U_N(t)\widetilde{U}_N(t),\partial_x\phi)+\frac{1}{4}\mu a_\omega(\widetilde{U}_N(t),\phi)\\ \quad =G_0(\phi,t)+G_1(\phi,t)+G_2(\phi,t), \qquad\qquad \forall\phi\in\mathcal{P}_N^0, 0<t\le T,\\ \widetilde{U}_N(0)=0.\end{cases} \tag{4.16}$$

Comparing (4.16) with (4.11), we derive an error estimate like (4.14). But $u_N,\widetilde{u}_N$, and $M(u_N)$ in (4.14) are now replaced by $U_N,\widetilde{U}_N$, and $M(U_N)$. Therefore it remains to estimate $|G_j(\widetilde{U}_N,t)|$.

We first use Theorem 3.9 to obtain

$$|G_0(\widetilde{U}_N,t)|\le\frac{\mu}{4}|U(t)-U_N(t)|_{1,\omega}|\widetilde{U}_N(t)|_{1,\omega}\le cN^{-2r}\|U(t)\|^2_{W^r(\Lambda)}+\frac{1}{16}\mu|\widetilde{U}_N(t)|^2_{1,\omega}.$$

According to Theorem 3.8, we obtain the following result.

$$\begin{aligned}|G_1(\widetilde{U}_N,t)| &\le ||\widetilde{U}_N(t)||_\eta^2+\tfrac{1}{4}||\partial_t U(t)-\partial_t U_N(t)||_\eta^2\\ &\le ||\widetilde{U}_N(t)||_\eta^2+cN^{-2r}||\partial_t U||^2_{r,\widetilde{\omega},\widehat{A}}.\end{aligned}$$

By virtue of Theorem 3.10 ,

$$\begin{aligned}|G_2(\widetilde{U}_N,t)| &\le \tfrac{1}{2}||U_N(t)+U(t)||_\infty||\widetilde{U}_N(t)||_\eta|\widetilde{U}_N(t)|_{1,\omega}\\ &\le 4(||U(t)||^2_{\delta,\widetilde{\omega},\widehat{A}}+||U(t)||^2_\delta)||\widetilde{U}_N(t)||_\eta^2+\tfrac{1}{16}\mu|\widetilde{U}_N(t)|^2_{1,\omega}.\end{aligned}$$

Using Theorem 3.10 again yields

$$M(U_N)\le c(1+|||U_N|||)\le c(1+||U||_{L^\infty(0,T;H^\delta_{\widetilde{\omega},\widehat{A}}(\Lambda)\cap H^\delta(\Lambda))}).$$

Inserting the above estimates into an inequality like (4.14), we obtain the following result.

Theorem 4.2. *Let U and U_N be the solutions of (4.6) and (4.10), respectively. Assume that for $r\ge 0$ and $\delta>1$,*

$$U\in L^\infty(0,T;L^2_{0,\eta}(\Lambda))\bigcap H^\delta_{\widetilde{\omega},\widehat{A}}(\Lambda)\bigcap H^\delta(\Lambda))\bigcap L^2(0,T;W^r(\Lambda))\bigcap H^1(0,T;H^r_{\widetilde{\omega},\widehat{A}}(\Lambda)).$$

Then for all $0\le t\le T$,

$$E(U-u_N,t)\le M^*(U)N^{-2r}$$

where $M^*(U)$ is a positive constant depending only on μ and the norms of U and U_0 in the mentioned spaces.

Remark 4.4. If $\partial_y V(y)\to\infty$ as $y\to\infty$ in (4.6), then in (4.7), $\partial_x U(x)\omega(x)\to 0$ as $x\to 1$. In this case, the Jacobi spectral scheme for (4.6) is to find $u_N(t)\in\overline{\mathcal{P}}_N$ for all $0\le t\le T$ such that

$$\begin{cases}(\partial_t u_N(t),\phi)_\eta+\frac{1}{2}(u_N(t)\partial_x u_N(t),\phi)+\frac{1}{4}\mu a_\omega(u_N(t),\phi)\\ \quad =(f,\phi)_\eta, \qquad\qquad \forall\phi\in\overline{\mathcal{P}}_N, 0<t\le T,\\ u_N(0)=u_{N,0}.\end{cases}$$

The corresponding stability and convergence can also be derived.

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