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MAXIMUM NORM ERROR ESTIMATES OF CROUZEIX-RAVIART NONCONFORMING FINITE ELEMENT APPROXIMATION OF NAVIER-STOKES PROBLEM*

Qing-ping Deng
(*Department of Mathematics, University of Tennessce, U.S.A*)

Xue-jun Xu
(*LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, P.O.Box 2719, Beijing 100080, China*)

Shu-min Shen
(*Department of Mathematics, University of Suzhou 215006, China*)

Abstract

This paper deals with Crouzeix-Raviart nonconforming finite element approximation of Navier-Stokes equation in a plane bounded domain, by using the so-called velocity-pressure mixed formulation. The quasi-optimal maximum norm error estimates of the velocity and its first derivatives and of the pressure are derived for nonconforming C-R scheme of stationary Navier-Stokes problem. The analysis is based on the weighted inf-sup condition and the technique of weighted Sobolev norm. By the way, the optimal L^2-error estimate for nonconforming finite element approximation is obtained.

Key words: Navier-Stokes problem, P1 nonconforming element, Maximum Norm.

1. Introduction

There are many research works on finite element approximation of Navier-Stokes problem in the case of lower Reynold number, by using the so-called velocity-pressure mixed formulations,e.g.[12,15,16,17,23,26]. Various sorts of conforming finite element schemes(the discrete space of velocity belongs to C^0) and nonconforming finite element schemes(the discrete space of velocity does not belong to C^0) have been discussing. It is the main reasons that the discrete spaces of velocity and pressure can not be chosen independently, they must satisfy discrete inf-sup condition (i.e. LBB condition)[1,3]. Thus, to construct conforming finite element scheme usually need some techniques for satisfying the compatibility between the discrete space of velocity and pressure and for obtaining the optimal energy norm error estimates of both velocity and pressure. Therefore, simple nonconforming finite element schemes have received considerable attention from both a theoretical and applied point of view. Nonconforming finite element scheme for stationary Stokes problem was first studied by Crouzeix and Raviart[3](this scheme is called C-R scheme). In the paper, the compatibility between the nonconforming piecewise linear triangle element and piecewise constant was shown, and the optimal energy norm error estimates of both velocity and pressure were obtained. Moreover,this scheme(i.e. C-R scheme) was extended to stationary Navier-Stokes problem

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by Teman [26], the optimal results were also obtained. The other nonconforming finite element schemes for Navie-Stokes problem may be found in [4,8,9,14,15,23,26]. But so far, maximum norm error estimates for any nonconforming finite element schemes were not considered.

Recently, the quasi-optimal maximum norm error estimates of the velocity and its first derivatives and of the pressure for conforming finite element schemes of stationary Stokes problem were studied by Duran, Nechotto and Wang[6] and Duran and Nechotto[5]. Their analyses were based on the techniques of regularized Green's functions and weighted inf-sup condition respectively. But these analyses relied on the continuity of discrete space of velocity, and did not include any nonconforming finite element schemes. In addition, L^∞ error estimates (i.e. maximum norm error estimates) for C-R scheme of nonstationary Navier-Stokes problem were consided by Rannacher[23], but its results were not quasi-optimal. The main aim of this paper is to study maximum error estimates of the velocity and its first derivatives and of the pressure for C-R scheme of stationary Navier-Stokes problem. The quasi-optimal maximum norm error estimate results for stationary problem are showen with the similar technique of weighted Sobolev norms introduced by Duran, Nechotto[5] and Rannnacher[8]. By the way, the optimal lower-norm (i.e. L^2-norm) error estimate is also shown.

The plan of this paper is the following. In section 2 we state the notations of Navier-Stokes problem and its nonconforming finite element approximations. Section 3 contains some discussion of the nonconforming finite spaces and their properties. The optimal L^2-error estimate for noncnforming finite elemet scheme is proved in section 4. In section 5 we introduce weighted Sobolev norm and give a weighted priori estimates for the solution of Stokes problem using basic solutions, and present several useful lemmas. Section 6 deals with the L^∞-error estimates for Navier-Stokes problem.

2. Notations and Prelimiaries

Let Ω be a convex polygonal domain in R^2. We consider the stationary Navier-Stokes problem for incompressible flows:

$$\begin{cases} -\gamma\triangle U + U\cdot\nabla U + \nabla p = f, & (\Omega) \\ divU = 0, & (\Omega) \\ U = 0, & (\partial\Omega) \end{cases} \tag{2.1}$$

where γ denotes the constant inverse Reynolds number. $U = (u_1, u_2)$ represents the velocity of the fluid, p its pressure and $f = (f_1, f_2)$ a diven external force. In order to write problem (2.1) in a weak form we introduce the notations:

$$X = (H_0^1(\Omega))^2, \quad M = L_0^2(\Omega), \quad L_0^2(\Omega) = \{q | q \in L^2(\Omega), \int_\Omega q dx = 0\},$$

$$a(U,V) = \gamma\int_\Omega \nabla U\cdot\nabla V dx = \gamma(\nabla U, \nabla V),$$

$$\tilde{b}(U,V,W) = \int_\Omega U\cdot\nabla V\cdot W dx = \sum_{i,j}\int_\Omega u_j\partial_j v_i w_i dx.$$

Here $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and $(L(\Omega))^2$. Then the variational form of problem (2.1) reads: Find $(U,p) \in X \times M$ such that

$$a(U,V) + \tilde{b}(U,U,V) - (p, divV) = (f,V), \quad \forall V \in X, \tag{2.2}$$
$$(divU, q) = 0, \qquad \forall q \in M. \tag{2.3}$$

Throughout this paper, let $W^{m,p}(\Omega)$ and $H^m(\Omega)$ denote the Sobolev spaces on Ω with the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and $\|\cdot\|_m$ respectively ($\|\cdot\|_{0,\Omega} = \|\cdot\|$ denotes L^2-norm over Ω), $H_0^1(\Omega)$ denotes the closure of the space $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

Let Γ_h be a regular and quasi-uniform family of decompositions of Ω into element $\tau \in \Gamma_h$ with $\bar{\Omega} = \bigcup_{\tau \in T_h} \bar{\tau}$. We will work with nonconforming finite element space X_h and M_h, where $X_h \subset (L^2(\Omega))^2, M_h \subset M$, and $X_h \not\subset X$. (i.e. X_h is a nonconforming finite element space). But for every $V_h \in X_h$. We have $V_h|_\tau \in (H^1(\tau))^2, \tau \in \Gamma_h$. We assume the norm on X_h by

$$\|\nabla_h V_h\| = \{\sum_\tau \|\nabla V_h\|_{0,\tau}^2\}^{1/2}.$$

Then the nonconforming finite element approximate problem(2.2),(2.3) can be defined: Find $(U_h, p_h) \in X_h \times M_h$, such that

$$a_h(U_h, V_h) + b_h(U_h, U_h, V_h) - (p_h, div_h V_h) = (f, V_h), \quad \forall V_h \in X_h, \tag{2.4}$$
$$(div_h U_h, q_h) = 0, \qquad \forall q_h \in M_h. \tag{2.5}$$

where

$$a_h(U,V) = \sum_\tau \gamma \int_\tau \nabla U \cdot \nabla V dx = \sum_\tau \gamma(\nabla_h U, \nabla_h V),$$
$$\tilde{b}_h(U,V,W) = \sum_\tau \int_\tau U \cdot \nabla V \cdot W dx,$$
$$b_h(U,V,W) = \frac{1}{2}(\tilde{b}_h(U,V,W) - \tilde{b}_h(U,W,V)),$$

where $\nabla_h V$ and $div_h V$ may be defined L^2-vector function and L^2-function by elementwise calculation respectively. It is well known that X_h and M_h cannot be chosen independently. X_h and M_h must satisfy the following discrete inf-sup condition (i.e. LBB condition):

$$\sup_{0 \neq V_h \in X_h} \frac{(div_h V_h, q_h)}{\|\nabla_h V\|} \geq \beta \|q_h\|, \qquad \forall Q_h \in M_h, \tag{2.6}$$

where β is a positive number independent of h. Equivalently, there exists a projection operator $\pi_h : X \to X_h$ such that

$$\begin{cases} \|\nabla_h \pi_h V\| \leq C\|\nabla V\|, & \forall V \in X, \\ (div_h(V - \pi_h V), q_h) = 0, & \forall V \in X, \quad q_h \in M_h. \end{cases} \tag{2.7}$$

From[15,26], if we denote

$$X_h^* = \{V_h \in X_h | (div_h V_h, q_h) = 0, \quad \forall q_h \in M_h, \},$$
$$N_h = \sup_{\Phi_h, V_h, W_h \in X_h^*} \frac{|b_h(\Phi_h, V_h, W_h)|}{\|\nabla_h \Phi_h\| \ \|\nabla_h V_h\| \ \|\nabla_h W_h\|},$$
$$\|f\|_h^* = \sup_{v_h \in X_h^*} \frac{|(f, V_h)|}{\|\nabla_h V_h\|},$$

and under the following condition:

$$\frac{N_h \|f\|_h^*}{\gamma^2} \leq 1 - \delta,$$

where $\delta \in (0,1)$ independent of h, then the discrete problem (2.4) (2.5) have unique solution $(U_h, p_h) \in X_h \times M_h$ and the following error estimates[15,26]

$$\|\nabla_h (U - U_h)\| + \|p - p_h\| \leq C[\inf_{V \in X_h, q \in M_h} [\|U - V\| + \|p - q\|]$$
$$+ \sup_{W \in X_h} \frac{|G_h(U, \pi_h U, W)|}{\|\nabla_h W\|} + \sup_{W \in X_h} \frac{|E_h(U, p, W)|}{\|\nabla_h W\|}, \tag{2.8}$$

where (U, p) and (U_h, p_h) are the solutions of problems (2.2),(2.3) and (2.4),(2.5), respectively.

$$G_h(U, V, W) = b(U, U, W) - b_h(V, V, W), \tag{2.9}$$

$$E_h(U, p, W) = \sum_\tau \int_{\partial \tau} (\gamma \frac{\partial U}{\partial n} - pn - \frac{1}{2}(U \cdot n)U) \cdot W ds, \tag{2.10}$$

where n denotes the unit outer normal vector.

3. Linear Nonconforming Finite Element Spaces and Their Properties

Let Γ_h be a triangulation of Ω. We consider the linear nonconforming finite element space S_h of $H_0^1(\Omega)$ on the triangulation Γ_h, i.e.

$$S_h = \{v_h | v_h|_\tau \in P_1, \forall \tau \in \Gamma_h, X_h \textit{ is continuous at midpiont of the edges of } \tau,$$

$$v_h(Q) = 0, \ Q \in \partial\Omega \textit{ is midpiont of the edge}\}, \tag{3.1}$$

$$X_h = S_h \times S_h. \tag{3.2}$$

Obviously, $X_h \subset (L^2(\Omega))^2$ and $X_h \not\subset X$,then X_h is a nonconforming finite element space. Moreover, it can be easily shown that X_h has the following properties[3]:

(1). $\int_{\partial\tau^- \cap \partial\tau^+} (W_h|_{\tau^-} - W_h|_{\tau^+}) ds = 0, \qquad \forall \tau^-, \tau^+ \in \Gamma_h$
are two adjacent elements, $\forall W_h \in X_h$

(2). $\int_{\partial\tau \cap \partial\Omega} W_h|_\tau ds = 0, \quad \forall \tau \in \Gamma_h. \quad \forall W_h \in X_h.$

Furthermore, there exists operator $\pi_h : X \oplus X_h \to X_h$ such that

(3). $\pi_h W_h = W_h, \quad \forall W_h \in X_h.$

(4). $\int_{L_{i\tau}} \pi_h V ds = \int_{L_{i\tau}} V ds, \quad \forall V \in X, L_{i\tau} (i = 1, 2, 3)$ is the edge of τ

$(5).\|W-\pi_h W\|_{L^p}+h\|\nabla_h(W-\pi_h W)\|_{L^p}\le ch^2\|\nabla^2 W\|_{L^p},$

$$\forall W\in X\cap(H^2(\Omega))^2,\quad \forall 1\le p\le +\infty.$$

$(6).If\ V\in X\ and\ divV=0,\ then\ div_h(\pi_h V)=0.$

For given $v\in H^1(\tau),\tau\in\Gamma_h$, we denote $L_{i\tau}(i=1,2,3)$ by the three edges of τ and define $M_\tau v$ by

$$M_\tau v=\frac{1}{meas(L_{i\tau})}\int_{L_{i\tau}} vds \qquad on \quad L_{i\tau}. \tag{3.3}$$

Then we have

Lemma 3.1. *If $v\in H^1(\Omega)$ and $w_h\in S_h$, then [3]*

$$\sum_{\tau\in\Gamma_h}\int_{\partial\tau} vw_h n_i ds\le Ch\|\nabla v\|\|\nabla_h w_h\|,\quad i=1,2, \tag{3.4}$$

here $n=(n_1,n_2)$ is the unit outer normal vector of the side of τ.

In addition, we have the following discrete sobolev inequalities (cf. [26,28,27] for details):

$$\|w_h\|\le C\|\nabla_h w_h\|. \tag{3.5}$$

$$\|w_h\|_{L^4}\le C\|\nabla_h w_h\|. \tag{3.6}$$

Now we define

$$P_h=\{q_h|q_h|_\tau=const,\forall\tau\in\Gamma_h\}. \tag{3.7}$$

$$M_h=\{q_h\in P_h|\int_\Omega q_h dx=0\}. \tag{3.8}$$

Moreover there exists a interpolation operator[4] $I_h:L^2(\Omega)(M)\to P_h(M_h)$ such that

$$(7).\|q-I_h q\|\le Ch\|\nabla q\|,\qquad \forall q\in M\cap H^1(\Omega). \tag{3.9}$$

For Navier-Stokes problem(2.2)(2.3), if X_h and M_h are defined by (3.2) and (3.8), then discrete problem (2.4)(2.5) can be called C-R scheme. Notice that $X_h\times M_h$ satifies the conditions (2.6) and(2.7)(here π_h of (2.7) is also the interpolation operator), then it follows from (2.8) that we have the following energy norm estimate[4]:

Theorem 3.1. *Let $(U,p)\in X\times M$ and $(U_h,p_h)\in X_h\times M_h$ be the solutions of problem (2.2)(2.3) and (2.4)(2.5) respectively. Then there exists a constant C independent of h, such that*

$$\|\nabla_h(U-U_h)\|+\|p-p_h\|\le Ch(\|U\|_2+\|p\|_1). \tag{3.10}$$

The error estimate (3.10) is optimal. Now we introduce the following inequalities, which are used in the following discussion.

Lemma 3.2. *For trilinear form $\tilde b_h(\cdot,\cdot,\cdot)$ and $b_h(\cdot,\cdot,\cdot)$, we have*

$$|\tilde b(\Phi,V,W)|+|b_h(\Phi,V,W)|\le C\|\nabla_h\Phi\|\ \|\nabla_h V\|\ \|\nabla_h W\|,\qquad \forall\Phi,V,W\in X_h\oplus X. \tag{3.11}$$

$$|b_h(\Phi,V,W)\le C\|\Phi\|_{L^\infty(\Omega)}\|\nabla_h V\|\|\nabla_h W\|,\qquad \forall\Phi\in(L^\infty(\Omega))^2,V,W\in X_h\oplus X. \tag{3.12}$$

$$|b_h(\Phi,V,W)| \le C\|\Phi\|\|V\|_{W^{1,\infty}(\Omega)}\|\nabla_h W\|, \qquad \forall V \in (W^{1,\infty}(\Omega))^2, \Phi, W \in X_h \oplus X. \tag{3.13}$$

and if $\Phi \in W^{1,\infty}(\Omega) \cap X, div\Phi = 0, V, W \in X_h \oplus X$, *then*

$$|b_h(\Phi,V,W)| \le C\|\Phi\|_{W^{1,\infty}(\Omega)}\|\nabla_h W\|(\|V\| + h\|\nabla_h V\|). \tag{3.14}$$

Proof. It follows from (3.5)(3.6) and holder inequality that we obtain clearly (3.11)(3.12)and (3.13). From Green's formula and $div\Phi = 0$, we have

$$\begin{aligned} b_h(\Phi,V,W) &= \frac{1}{2}(\sum_{\tau\in\Gamma_h}\int_\tau \Phi\cdot\nabla V\cdot W dx - \sum_{\tau\in\Gamma_h}\int_\tau \Phi\cdot\nabla W\cdot V dx) \\ &= \frac{1}{2}\sum_{\tau\in\Gamma_h}\int_{\partial\tau}(\Phi\cdot n)(V\cdot W)ds - \sum_{\tau\in\Gamma_h}\int_\tau \Phi\cdot\nabla W\cdot V dx. \end{aligned} \tag{3.15}$$

Application of Lemma 3.1 gives

$$\begin{aligned} &\sum_{\tau\in\Gamma_h}\int_{\partial\tau}(\Phi\cdot n)[V\cdot W]ds \\ &\le Ch\sum_{\tau^-,\tau^+\in\Gamma_h}\{\|\nabla(\Phi\cdot n_L)\cdot W^+\|_{\tau^+}\cdot\|\nabla_h V\|_{\tau^-\cup\tau^+} \\ &+\|\nabla(\Phi\cdot n_L)\cdot W^-\|_{\tau^-}\cdot\|\nabla_h W\|_{\tau^-\cup\tau^+}\} \\ &\le Ch\|\Phi\|_{W^{1,\infty}(\Omega)}\|\nabla_h V\|\|\nabla_h W\| \end{aligned}$$

Combining (3.15),we obtain (3.14). □

Lemma 3.3. *If U_h satisfies (2.5), then*

$$(div_h U_h, q_h) = 0, \qquad \forall q_h \in P_h \tag{3.16}$$

Proof. Setting $\tilde{q}_h = \frac{1}{meas(\Omega)}\int_\Omega q_h dx$, we have $q_h - \tilde{q}_h \in M_h$, It following (2.5) and Green' formula that

$$(div_h U_h, q_h) = (div_h U_h, \tilde{q}_h) = \sum_{\tau\in\Gamma_h}\int_{\partial\tau} U_h\cdot\tilde{q}_h\cdot n ds$$

Hence, from $U_h \in X_h$ and property(1)(2), we obtain(3.16).□

4. L^2 -Error Estimate

Now we derive the optimal error estimate in L^2 for the velocity by the duality argument. This result is also useful below.

Theorem 4.1. *Let $(U,p) \in X \times M \cap ((H^2(\Omega)) \times H^1(\Omega))$ be the solution of problem (2.2), (2.3), and let $(U_h, p_h) \in X_h \times M_h$ be the solution of problem (2.4), (2.5), then we have*

$$\|U - U_h\| \le Ch^2(\|U\|_2 + \|p\|_1). \tag{4.1}$$

Proof. We introduce the following auxiliary problem for $\Phi = (\phi_1, \phi_2)$ and ψ:

$$\begin{cases} -\gamma\nabla\phi_i - \sum_{j=1}^{2}[u_j\frac{\partial\phi_i}{\partial x_j} - \frac{1}{2}\phi_j\frac{\partial u_i}{\partial x_i} + \frac{1}{2}u_j\frac{\partial\phi_j}{\partial x_i}] + \frac{\partial\psi}{\partial x_i} = g_i, \qquad i = 1,2,(\Omega), \\ div\Phi = 0, \qquad \Omega, \\ \Phi = 0, \qquad \partial\Omega. \end{cases} \tag{4.2}$$

the weak form is

$$\left\{\begin{array}{ll} a(\Phi,W)+b(U,W,\Phi)+b(W,U,\Phi)-(\psi,divW)=(g,W), & \forall W\in X, \\ (div\Phi,q)=0, \qquad \forall q\in M. \end{array}\right. \tag{4.3}$$

For $g=(g_1,g_2)\in(L^2(\Omega))^2$, the dual problem (4.2) exists a unique solution $(\Phi,\psi)\in X\cap(H^2(\Omega))^2\times M\cap H^1(\Omega)$ [13],[8],[19] and moreover

$$\|\Phi\|_2+\|\psi\|_1\le C\|g\| \tag{4.4}$$

Now we let $g=U-U_h$ in (4.2) and let $U-U_h$ multiply by(4.2). Thus, noting that $(U-U_h)\in(L^0(\Omega))^2$ and $divU=0$, we then have

$$\begin{array}{l} \|U-U_h\|^2=(U-U_h,U-U_h)=(g,U-U_h) \\ =a_h(\Phi,U-U_h)-(\psi,div_h(U-U_h))+b_h(U,U-U_h,\Phi) \\ +b_h(U-U_h,U,\Phi)+\sum\limits_{\tau\in\Gamma_h}\int_{\partial\tau}(\gamma\frac{\partial\Phi}{\partial n}-\psi n-\frac{1}{2}(U\cdot n)\Phi)\cdot(U-U_h)ds. \end{array} \tag{4.5}$$

From (2.1)(2.4) and(2.5), we have

$$a_h(U-U_h,W_h)-(p-p_h,div_hW_h)=G_h(U,U_h,W_h)+E_h(U,p,W_h),\forall W\in M_h. \tag{4.6}$$

and

$$(div_h(U-U_h),q_h)=0, \qquad q_h\in M_h. \tag{4.7}$$

Taking (4.6) into the right of (4.5), we obtain

$$\begin{array}{l} \|U-U_h\|^2=a_h(U-U_h,\Phi-W_h)-(\psi,div_h(U-U_h)) \\ +(p-p_h,div_hW_h)+b_h(U,U-U_h,\Phi)+b_h(U-U_h,U,\Phi) \\ +\sum\limits_{\tau\in\Gamma_h}\int_{\partial\tau}(\gamma\frac{\partial\Phi}{\partial n}-\psi n-\frac{1}{2}(U\cdot n)\Phi)\cdot(U-U_h)ds \\ +G_h(U,U_h,W_h)+E_h(U,p,W_h). \end{array}$$

Setting $W_h=\pi_h\Phi\in X_h$ and noting that(4.7) and $div\Phi=0$, moreover noting that

$$G_h(U,U_h,W_h)=b_h(U,U,W_h)-b_h(U_h,U_h,W_h), \tag{4.8}$$

so we have

$$\begin{array}{l} \|U-U_h\|^2=a_h(U-U_h,\Phi-\pi_h\Phi)-(\psi-I_h\psi,div_h(U-U_h)) \\ +(p-p_h,div_h(\pi_h\Phi-\Phi))+b_h(U-U_h,U-U_h,\pi_h\Phi) \\ +b_h(U,U-U_h,\Phi-\pi_h\Phi)+b_h(U-U_h,U,\Phi-\pi_h\Phi) \\ +\sum\limits_{\tau\in\Gamma_h}\int_{\partial\tau}(\gamma\frac{\partial\Phi}{\partial n}-\psi n-\frac{1}{2}(U\cdot n)\Phi)\cdot(U-U_h)ds+E_h(U,p,\pi_h\Phi). \end{array} \tag{4.9}$$

Using Lemma 3.1, we easily obtain

$$\begin{array}{l} \sum\limits_{\tau\in\Gamma_h}\int_{\partial\tau}(\gamma\frac{\partial\Phi}{\partial n}-\psi n-\frac{1}{2}(U\cdot n)\Phi)\cdot(U-U_h)ds \\ \le Ch(\|\Phi\|_2+\|\psi\|_1+\|\nabla U\|\ \|\Phi\|)\|\nabla_h(U-U_h)\|, \end{array}$$

$$E_h(U,p,\pi\Phi)=E_h(U,p,\pi_h\Phi-\Phi)\le Ch(\|U\|_2+\|p\|_1)\|\nabla_h(\Phi-\pi_h\Phi)\|,$$

where we have used $E_h(U,p,\Phi) = 0$. Hence, using(3.11), (2.7), (4.4) and property(3)(5)(7), we soon have

$$\begin{aligned}
&\|U - U_h\|^2 \\
&\le C\{\|\nabla_h(U - U_h)\| \cdot \|\nabla_h(\Phi - \pi_h\Phi)\| + \|\psi - I_h\psi\| \cdot \|\nabla_h(U_h - U)\| \\
&\quad +\|p - p_h\| \cdot \|\nabla_h(\Phi - \pi_h\Phi)\| + \|\nabla_h(U - U_h)\|^2\|\nabla_h\pi_h\Phi\| \\
&\quad +\|\nabla U\|\|\nabla_h(U - U_h)\|\|\nabla_h(\Phi - \pi_h\Phi)\| + h\|\Phi\|_2\|\nabla_h(U - U_h)\|\|\nabla U\| \\
&\quad +h(\|U\|_2 + \|p\|_1)\|\nabla_h(\Phi - \pi_h\Phi)\| \\
&\quad +Ch(\|\Phi\|_2 + \|\psi\|_1 + \|\nabla U\|\|\Phi\|)\|\nabla_h(U - U_h)\|\} \\
&\le Ch(\|\nabla_h(U - U_h)\| + \|p - p_h\|)(\|\Phi\|_2 + \|\psi\|_1 + \|\nabla U\|\|\Phi\|_2) \\
&\quad +C\|\nabla_h(U - U_h)\|^2\|\nabla\Phi\| + Ch^2(\|U\|_2 + \|p\|_1)\|\Phi\|_2) \\
&\le C\|U - U_h\|\{h(\|\nabla_h(U - U_h)\| + \|p - p_h\|)(1 + \|\nabla U\|) \\
&\quad +h^2(\|U\|_2 + \|p\|_1) + \|\nabla_h(U - U_h)\|^2\}.
\end{aligned}$$

Finally, it follows from (3.10) that we soon have(4.1).□

5. Sobolev Weighted Norms and Some Lemmas

To discuss the maximum norm error estimates, we recall the definition of the usual weight function along with some of its fundametal properties.

The weight function σ is defined by

$$\sigma(x) := (|x - x_0|^2 + \theta^2)^{\frac{1}{2}} \quad \forall x, x_0 \in \Omega, \tag{5.1}$$

where $\theta >> h$ is a small parameter to be determined later on. It is well known that σ satisfies the following properties[2,22,21]

$$\max_{\tau\in\Gamma_h}[\max_{x\in\tau}\sigma(x)/\min_{x\in\tau}\sigma(x)] \le C, \tag{5.2}$$

$$|\partial^k\sigma^\alpha(x)| \le C(k,\alpha)\sigma^{\alpha-k}(x), \quad \forall x \in \Omega, \tag{5.3}$$

where $k \in R$, C ic a constant and independent of h and x_0, and $\partial^j f$ denotes jth derivatives of f. Moreover, a simple calculation shows that [2][5]

$$\int_\Omega \sigma^{-\alpha}(x)dx \le \begin{cases} C\theta^{2-\alpha}, & \alpha > 2, \\ C|ln\theta|, & \alpha = 2. \end{cases} \tag{5.4}$$

For $\alpha \in R$ and $k \in N$, the weighted sobolev seminorms are defined by

$$\|\nabla^k V\|_{(\alpha)} = (\sum_{\tau\in\Gamma_h}\sum_{|\xi|=k}\int_\tau \sigma^\alpha|D^\xi V|^2dx)^{\frac{1}{2}}, \tag{5.5}$$

where $D^\xi f$ denotes the tensor of kth ($k = |\xi|$) derivaties of f. The same notation will be used for vector valued functions. Then we have the following results [2],[22],[20],[25],

$$\|v\|_{(-\alpha)} \le C\|v\|_{L^\infty(\Omega)} \begin{cases} \theta^{(2-\alpha)/2}, & \alpha > 2, \\ |ln\theta|^{\frac{1}{2}}, & \alpha = 2. \end{cases} \tag{5.6}$$

$$\|w\|_{L^\infty(\Omega)} \le C(\theta^\alpha/h^2)^{\frac{1}{2}}\|w\|_{(-\alpha)}, \qquad \forall\alpha \in R, \qquad w \in S_h(or M_h). \tag{5.7}$$

where we assume that x_0 in (5.1) such that $|w(x_0)| = \|w\|_{L^\infty}$. Moreover, the following interpolation properties and supper approximation properties hold [2],[5]:

$$\|W - \pi_h W\|_{(\alpha)} + h\|\nabla_h(W - \pi_h W)\|_{(\alpha)} \le Ch\|\nabla^2 W\|_{(\alpha)}, \qquad \forall \alpha \in R. \tag{5.8}$$

$$\|q - \pi q\|_{(\alpha)} \le Ch\|\nabla q\|_{(\alpha)}, \qquad \forall \alpha \in R. \tag{5.9}$$

and

$$\begin{aligned} &\|\nabla_h(\sigma^{(-2)}(W - W_h) - \pi_h(\sigma^{(-2)}(W - W_h)))\|_{(\alpha)} \le Ch/\theta^{(1-\frac{\alpha}{2})}\|\nabla^2 W\|_{(-2)} \\ &+Ch/\theta^{2-\frac{\alpha}{2}}(\|W - W_h\|_{(-4)} + \|\nabla(W - W_h)\|_{(-2)}), \qquad \alpha = 0, 2. \end{aligned} \tag{5.10}$$

Moreover, we have

$$\|\sigma^{(-2)}(q - q_h) - I_h(\sigma^{(-2)}(q - q_h))\|_{(2)} \le C(h/\theta\|q - q_h\|_{(-2)} + h\|\nabla q\|_{(-2)}). \tag{5.11}$$

where $W \in (H^2(\Omega))^2$, $q \in H^1(\Omega), W_h \in X_h, q_h \in M_h(P_h)$.

To proceed further, we need a priori estimates for the stokes problem. Let $g \in (L^2(\Omega))^2, \eta \in H_0^1(\Omega) \cap M$. Then there exists a unique solution $(V, \lambda) \in X \times M$ of the generalized stokes problem [12],[18],

$$\begin{cases} -\triangle V + \nabla\lambda = g, & (\Omega), \\ divV = \eta, & (\Omega), \\ V = 0, & (\partial\Omega). \end{cases}$$

Which satisfies

$$\|V\|_2 + \|\lambda\|_1 \le C(\|g\| + \|\nabla\eta\|). \tag{5.12}$$

Lemma 5.1. *Let $(V, \lambda) \in X \times M$ be the solution of the following Stokes problem*

$$\begin{cases} -\triangle V + \nabla\lambda = g, & (\Omega), \\ divV = 0, & (\Omega), \\ V = 0, & (\partial\Omega). \end{cases} \tag{5.13}$$

where $g \in (L^2(\Omega))^2$, then we have [5], [10], [11]

$$\|\nabla^2 V\|_{(2)} + \|\nabla\lambda\|_{(2)} \le C|ln\theta|^{\frac{1}{2}}/\theta\|g\|_{(4)}. \tag{5.14}$$

Now we introduce the weighted inf-sup condition [5].

Lemma 5.2. *There exists a constant $\beta > 0$ independent of h, such that*

$$\sup_{V \in X} \frac{(divV, q)}{\|\nabla V\|_{(2)}} \ge \beta|ln\theta|^{1/2}\|q\|_{(-2)}, \qquad \forall q \in M \tag{5.15}$$

Lemma 5.3. *For $E_h(U, p, W_h)$ defined by (2.10) and $E'_h(U, p, W_h)$ defind by*

$$E'_h(U, p, W_h) = \sum_{\tau \in \Gamma_h} \int_{\partial\tau} (\gamma\frac{\partial U}{\partial n} - pn) \cdot W_h ds, \tag{5.16}$$

then, for $\forall W_h \in X \oplus X_h$, we have

$$|E_h(U, p, W_h)| \le Ch(\|\nabla^2 U\|_{(-\alpha)} + \|\nabla U^2\|_{(-\alpha)} + \|\nabla p\|_{(-\alpha)})\|\nabla_h W_h\|_{(\alpha)}, \quad \forall \alpha \in R, \tag{5.17}$$

$$|E_h'(U,p,W_h)| \leq Ch(\|\nabla^2 U\|_{(-\alpha)} + \|\nabla p\|_{(-\alpha)})\|\nabla_h W_h\|_{(\alpha)}, \quad \alpha \in R. \tag{5.18}$$

Proof. It suffices only to deal with (5.18). Using property (1),(2),(3.4) and (5.2), then we have

$$|E_h(U,p,W_h)| = |\sum_{\tau\in\Gamma}(\int_L \frac{\partial U}{\partial n} - pn - \frac{1}{2}(U\cdot n)U)\cdot W_h ds|$$
$$= |\sum_{\tau\in\Gamma_h}\sum_{L\in\partial\tau}\int_L [(\frac{\partial U}{\partial n} - pn - \frac{1}{2}(U\cdot n)U)\cdot W_h ds|$$
$$= |\sum_{\tau\in\Gamma_h}\sum_{L\in\partial\tau}\int_L [(\frac{\partial U}{\partial n} - pn - \frac{1}{2}(U\cdot n)U)$$
$$-M_\tau(\frac{\partial U}{\partial n} - pn - \frac{1}{2}(U\cdot n)U)][W_h - M_\tau W_h]ds|$$
$$\leq Ch\|\nabla^2 U + \nabla p + \frac{1}{2}\nabla U^2\|_{(-\alpha)}\|\nabla_h W_h\|_{(\alpha)}$$
$$\leq Ch(\|\nabla^2 U\|_{(-\alpha)} + \|\nabla p\|_{(-\alpha)} + \|\nabla U^2\|_{(-\alpha)})\|\nabla_h W_h\|_\alpha, \quad \forall\alpha \in R.$$

6. Maximum Error Estimates

This section is devoted to the maximum error estimates for C-R nonconforming finite element approximation of stationary Navier-Stokes problem, and obtain the following qusi-optimal results.

Theorem 6.1. *Let $(U,p) \in X\times M$ and $(U_h,p_h) \in X_h \times M_h$ be the problem (2.2)(2.3) and (2.4)(2.5) respectively, and let $U \in (W^{2,\infty}(\Omega))^2$, $p \in W^{1,\infty}(\Omega)$. Then there exists a constant $C > 0$ independent of h, such that*

$$\|U - U_h\|_{L^\infty} \leq Ch^2|lnh|^{5/2}(\|U\|_{W^{2,\infty}} + \|p\|_{W^{1,\infty}}). \tag{6.1}$$

$$\|\nabla_h(U - U_h)\|_{L^\infty} \leq Ch|lnh|^2(\|U\|_{W^{2,\infty}} + \|p\|_{W^{1,\infty}}). \tag{6.2}$$

$$\|p - p_h\|_{L^\infty} \leq Ch|lnh|^2(\|U\|_{W^{2,\infty}} + \|p\|_{W^{1,\infty}}). \tag{6.3}$$

Proof. The proof of this Theorem will be carried out in four steps.

Step1. First we prove

$$\|U - U_h\|_{(-4)} \leq C\frac{h}{\theta}|ln\theta|^{1/2}\{\|\nabla_h(U - U_h)\|_{(-2)} + h(K + L)\}. \tag{6.4}$$

Where $K := \|\nabla^2 U\|_{(-2)} + \|\nabla U^2\|_{(-2)} + \|\nabla p\|_{(-2)}$, $L := \|U\|_2^2 + \|p\|_1^2 + \|U\|_{W^{1,\infty}}^2$.

Taking $g = \sigma^{-4}(U - U_h)$ in (5.13) and using (5.13)(5.14), then we have

$$\|\nabla^2 V\|_{(2)} + \|\nabla\lambda\|_{(2)} \leq C|ln\theta|^{\frac{1}{2}}/\theta\|U - U_h\|_{(-4)}, \tag{6.5}$$

but

$$\begin{aligned}
\|U - U_h\|_{(-4)}^2 &= (U - U_h, g) = (U - U_h, -\Delta V + \nabla\lambda) \\
&= a_h(U - U_h, V) - (\lambda, div_h(U - U_h)) + E_h'(V,\lambda,U - U_h) \\
&= a_h(U - U_h, \pi_h V) - a_h(U - U_h, V - \pi_h V_h) \\
&\quad -(\lambda - I_h\lambda, div_h(U - U_h)) + E_h'(V,\lambda,U - U_h).
\end{aligned}$$

Setting $W_h = \pi_h V$ in (4.6), we then have

$$\begin{aligned}
&\|U - U_h\|^2_{(-4)} \\
&= a_h(U - U_h, V - \pi_h V) + (p - p_h, div_h \pi_h V) - (\lambda - I_h\lambda, div_h(U - U_h)) \\
&+E'_h(V, \lambda, U - U_h) + E_h(U, p, \pi_h V) + G_h(U, U_h, , \pi_h V) \\
&= a_h(U - U_h, V - \pi_h V) + (p - I_h p, div_h \pi_h V) - (\lambda - I_h\lambda, div_h(U - U_h)) \\
&+E'_h(V, \lambda, U - U_h) + E_h(U, p, \pi_h V - V) + G_h(U, U_h, \pi_h V)
\end{aligned}$$

where we have used $divU = divV = 0$, (2.5) and $E_h(U, p, V) = 0$. Thus, from (5.8) (5.9) (5.17) (5.18), we obtain

$$\begin{aligned}
&\|U - U_h\|^2_{(-4)} \\
&\leq Ch\|\nabla(U - U_h)\|_{(-2)}\|\nabla^2 V\|_2 \\
&+Ch^2\|\nabla p\|_{(-2)}\|\nabla^2 V\|_{(2)} + Ch\|\nabla\lambda\|_{(2)}\|\nabla_h(U - U_h)\|_{(-2)} \\
&+Ch(\|\nabla^2 V\|_{(2)} + \|\nabla\lambda\|_{(2)})\|\nabla_h(U - U_h)\|_{(-2)} + |G_h(U, U_h, \pi_h V)|.
\end{aligned}$$

i.e.

$$\begin{aligned}
&\|U - U_h\|^2_{(-4)} \leq Ch^2 K\|\nabla^2 V\|_{(2)} + Ch(\|\nabla^2 V\|_{(2)} + \|\nabla\lambda\|_{(2)}) \\
&\cdot\|\nabla(U - U_h)\|_{(-2)} + G_h(U, U_h, \pi_h V). \qquad (6.6)
\end{aligned}$$

For the last term of right side of (6.6), we have

$$\begin{aligned}
|G_h(U, U_h, \pi_h V)| &\leq |b_h(U_h - U, U_h - U, \pi_h V)| + |b_h(U_h - U, U, \pi_h V)| \\
&\quad +|b_h(U, U - U_h, \pi_h V)| \\
&= I_1 + I_2 + I_3. \qquad (6.7)
\end{aligned}$$

Using(3.11),(2.7) and(3.10), we clearly have

$$\begin{aligned}
I_1 &\leq C\|\nabla_h(U - U_h)\|^2 \cdot \|\nabla_h \pi_h V\| \\
&\leq C\|\nabla_h(U - U_h)\|^2 \cdot \|\nabla V\| \leq Ch^2 L\|\nabla V\|. \qquad (6.8)
\end{aligned}$$

From(3.13), (2.7) and (4.1), we have

$$\begin{aligned}
I_2 &\leq C\|U - U_h\| \cdot \|U\|_{W^{1,\infty}}\|\nabla_h \pi_h V\| \\
&\leq Ch^2 L\|\nabla V\|. \qquad (6.9)
\end{aligned}$$

It follows from (3.14)(2.7)(3.10)(4.1) that

$$\begin{aligned}
I_3 &\leq C\|U\|_{W^{1,\infty}}\|\nabla_h \pi_h V\|(\|U - U_h\| + h\|\nabla_h(U - U_h)\|) \\
&\leq Ch^2 L\|\nabla V\| \qquad (6.10)
\end{aligned}$$

Combining (6.7)(6.8)(6.9)(6.10), we soon obtain

$$|G_h(U, U_h, \pi_h V)| \leq Ch^2 L\|\nabla V\| \qquad (6.11)$$

Thus, it follows from (6.5)(6.6) that (6.4).

Step2. Second we prove

$$\|\nabla_h(U-U_h)\|_{(-2)} \le (\frac{C^*h}{\theta}+\gamma)\|p-p_h\|_{(-2)} + C\gamma^{-1}(\|U-U_h\|_{(-4)} + h(K+L)), \quad (6.12)$$

where C^* is a determined constant, $0 < \gamma < 1$ is a small parameter to be selected.

Noting that

$$\begin{aligned} &\|\nabla(U-U_h)\|^2_{(-2)} && (6.13)\\ &= (\nabla_h(U-U_h), \sigma^{-2}\nabla_h(U-U_h))\\ &= a_h(U-U_h, \sigma^{-2}(U-U_h)) - (\nabla_h(U-U_h), (U-U_h)\cdot\nabla\sigma^{-2}) && (6.14)\end{aligned}$$

Setting $\eta = \sigma^{-2}(U-U_h)$, $\pi_h\eta \in X_h$, then from (4.6), we have

$$\begin{aligned} &a_h(U-U_h, \sigma^{-2}(U-U_h)) && (6.15)\\ &= a_h(U-U_h, \eta-\pi_h\eta) + a_h(U-U_h, \pi_h\eta)\\ &= a_h(U-U_h, \eta-\pi_h\eta) + (p-p_h, div_h(\pi_h\eta-\eta))\\ &\quad +(p-p_h, div_h\eta) + G_h(U, U_h, \pi_h\eta) + E_h(U, p, \pi_h\eta)\\ &\hat{=} J_1 + J_2 + J_3 + J_4 + J_5. && (6.16)\end{aligned}$$

Using (5.8)(5.9)(5.3)(5.10), we obtain

$$\begin{aligned} J_1 &\le C\|\nabla_h(U-U_h)\|_{(-2)}\|\nabla_h(\eta-\pi_h\eta)\|_{(2)}\\ &\le C\|\nabla_h(U-U_h)\|_{(-2)}[h\|\nabla^2U\|^2_{(-2)} + \frac{h}{\theta}(\|U-U_h\|_{(-4)} + \|\nabla_h(U-U_h)\|_{(-2)})]\\ &\le (C\frac{h}{\theta}+\varepsilon_1)\|\nabla_h(U-U_h)\|^2_{(-2)} + C(h^2\|\nabla^2U\|^2_{(-2)} + \|U-U_h\|^2_{(-4)}).\\ J_2 &\le C\|p-p_h\|_{(-2)}\|\nabla_h(\eta-\pi_h\eta)\|_{(2)}\\ &\le C\|p-p_h\|_{(-2)}(h\|\nabla^2U\|_{(-2)} + \frac{h}{\theta}(\|U-U_h\|_{(-4)} + \|\nabla_h(U-U_h)\|_{(-2)}))\\ &\le (C\frac{h}{\theta}+\frac{1}{2}\gamma^2)\|p-p_h\|^2_{(-2)} + C(\frac{h}{\theta}\|\nabla_h(U-U_h)\|^2_{(-2)}\\ &\quad +\|U-U_h\|^2_{(-4)}) + C\gamma^{-2}h^2K^2.\end{aligned}$$

Noting that $I_h(\sigma^{-2}(p-p_h)) \in P_h$, $div_h\eta = \sigma^{-2}div_h(U-U_h) + \nabla\sigma^{-2}\cdot(U-U_h)$ and using Lemma 3.3 and (5.11), then we obtain

$$\begin{aligned} J_3 &= (\sigma^{-2}(p-p_h), div_h(U-U_h)) + (p-p_h, \nabla\sigma^{-2}\cdot(U-U_h))\\ &= (\sigma^{-2}(p-p_h) - I_h\sigma^{-2}(p-p_h), div_h(U-U_h)) + (p-p_h, \nabla\sigma^{-2}\cdot(U-U_h))\\ &\le C\|\sigma^{-2}(p-p_h) - I_h(\sigma^{-2}(p-p_h))\|_{(2)}\|\nabla_h(U-U_h)\|_{(-2)}\\ &\quad +C\|p-p_h\|_{(-2)}\|U-U_h\|_{(-4)}\\ &\le C\|\nabla_h(U-U_h)\|_{(-2)}(\frac{h}{\theta}\|p-p_h\|_{(-2)} + h\|p\|_{(-2)})\\ &\quad +C\|p-p_h\|_{(-2)}\|U-U_h\|_{(-4)}\\ &\le (C\frac{h}{\theta}+\varepsilon_2)\|\nabla_h(U-U_h)\|^2_{(-2)} + (C\frac{h}{\theta}+\gamma^2)\|p-p_h\|^2_{(-2)}\\ &\quad +Ch^2K^2 + C\gamma^{-2}\|U-U_h\|^2_{(-4)}.\end{aligned}$$

Similarly, using (3.11)(3.13)(3.14) and the proof method of (6.7), we have

$$J_4 \le Ch^2 L\|\nabla_h \pi_h \eta\|$$

It follows from (5.10) that

$$\begin{aligned}
\|\nabla_h \pi_h \eta\| &= \|\nabla_h \sigma^{-2}(U-U_h)\| + \|\nabla_h(\eta - \pi_h \eta)\| \\
&\le \frac{C}{\theta}(\|\nabla_h(U-U_h)\|_{(-2)} + \|U-U_h\|_{(-4)}) + C\frac{h}{\theta}\|\nabla^2 U\|_{(-2)} \\
&\quad + C\frac{h}{\theta^2}(\|\nabla_h(U-U_h)\|_{(-2)} + \|U-U_h\|_{(-4)}) \\
&\le C\frac{h}{\theta}\|\nabla_h U\|_{(-2)} + \frac{C}{\theta}(\|\nabla_h(U-U_h)\|_{(-2)} + \|U-U_h\|_{(-4)})
\end{aligned}$$

Hence

$$\begin{aligned}
J_4 &\le Ch^2 L(\frac{h}{\theta}\|\nabla^2 U\|_{(-2)} + \frac{1}{\theta}(\|\nabla_h(U-U_h)\|_{(-2)} + \|U-U_h\|_{(-4)})) \\
&\le Ch^2(K^2+L^2) + \varepsilon_3\|\nabla_h(U-U_h)\|^2_{(-2)} + \|U-U_h\|^2_{(-4)}.
\end{aligned}$$

It follows from (5.17) that

$$\begin{aligned}
J_5 &\le ChK\|\nabla \pi_h \eta\|_{(2)} \le ChK(\|\nabla_h \sigma^{-2}(U-U_h)\|_{(2)} + \|\nabla_h(\eta - \pi_h\eta)\|_{(2)}) \\
&\le Ch^2 K(\|\nabla^2 U\|_{(-2)} + \frac{1}{\theta}(\|\nabla_h(U-U_h)\|_{(-2)} + \|U-U_h\|_{(-4)})) \\
&\le Ch^2 K^2 + \varepsilon_4\|\nabla_h(U-U_h)\|^2_{(-2)} + C\|U-U_h\|^2_{(-4)}.
\end{aligned}$$

Thus, combining with (6.14), we have

$$\begin{aligned}
& a_h(U-U_h, \sigma^2(U-U_h)) \\
& \le (C\frac{h}{\theta} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)\|\nabla_h(U-U_h)\|^2_{(-2)} \\
& \quad + (C\frac{h}{\theta} + \gamma^2)\|p-p_h\|^2_{(-2)} + C\gamma^{-2}(h^2(L^2+K^2)).
\end{aligned}$$

Noting that

$$\begin{aligned}
& (\nabla_h(U-U_h), (U-U_h)\nabla^2\sigma^{-2}) \le C\|\nabla_h(U-U_h)\|_{(-2)}\|U-U_h\|_{(-4)} \\
& \le \varepsilon_5\|\nabla_h(U-U_h)\|_{(-2)} + C\|U-U_h\|^2_{(-4)},
\end{aligned}$$

and combining with (6.13), we obtain

$$\begin{aligned}
& \|\nabla_h(U-U_h)\|^2_{(-2)} \le (C\frac{h}{\theta} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)\|\nabla_h(U-U_h)\|^2_{(-2)} \\
& + (C\frac{h}{\theta} + \gamma^2)\|p-p_h\|^2_{(-2)} + C\gamma^{-2}(h^2 + \|U-U_h\|^2_{(-4)}).
\end{aligned}$$

Therefore, taking θ, such that

$$C\frac{h}{\theta} \le \frac{1}{4} \tag{6.17}$$

and taking $\varepsilon_1 - \varepsilon_5$ sufficent small, then we soon obtain (6.12), where C^* is a determined constant independent of h.

Step 3. We prove

$$\|p - p_h\|_{(-2)} \leq C|lnh|(\|U - U_h\|_{(-4)} + h(K + L)) \tag{6.18}$$

It follows from (2.7) that

$$\|\nabla_h \pi_h V\|_{(2)} \leq C\|\nabla V\|_{(2)}$$

$$\frac{(div_h \pi_h V, q_h)}{\|\nabla_h \pi_h V\|_{(2)}} \geq \frac{(div_h \pi_h V, q_h)}{\|\nabla V\|_{(2)}} = \frac{(div V, q_h)}{\|\nabla V\|_{(2)}} \qquad q_h \in M_h \subset M \tag{6.19}$$

Thus, it follows from Lemma 5.2 that

$$\sup_{V_h \in X_h} \frac{(div_h V_h, q_h)}{\|\nabla_h V_h\|_{(2)}} \geq \tilde{\beta}|ln\theta|^{-1/2}\|q_h\|_{(-2)}, \qquad q_h \in M_h. \tag{6.20}$$

Where $\tilde{\beta}$ is a constant independent of h Moreover, from (6.18) we have

$$\begin{aligned} \|(p - p_h)\|_{(-2)} &\leq \|p - q_h\|_{(-2)} + \|q_h - p_h\|_{(-2)} \\ \leq \quad & \|p - q_h\|_{(-2)} + \tilde{\beta}^{-1}|ln\theta|^{1/2} \sup_{V_h \in X_h} \frac{(div_h V_h, p_h - q_h)}{\|\nabla_h V_h\|_{(2)}}, \quad \forall q_h \in M_h. \end{aligned} \tag{6.21}$$

Using (4.6),(5.17) and (6.11), we obtain

$$\begin{aligned} &(div_h V_h, p_h - q_h) = (div_h V_h, p - q_h) + (div_h V_h, p_h - p) \\ &= (div_h V_h, p - q_h) - a_h(U - U_h, V_h) + G_h(U, U_h, V_h) + E_h(U, p, V_h) \\ &G_h(U, U_h, V_h) \leq Ch^2 L\|\nabla_h V_h\| \leq C(h^2/\theta)L\|\nabla_h V_h\|_{(2)} \leq ChL\|\nabla_h V_h\|_{(2)}. \end{aligned}$$

So

$$(div_h V_h, p_h - q_h) \leq (\|p - q_h\|_{(-2)} + \|\nabla(U - U_h)\|_{(-2)} + h(K + L))\|\nabla_h V_h\|_{(2)}.$$

Therefore

$$\begin{aligned} \sup_{V_h \in X_h} \frac{(div_h V_h, p_h - q_h)}{\|\nabla_h V_h\|_{(2)}} &\leq \|p - q_h\|_{(-2)} \\ + \|\nabla(U - U_h)\|_{(-2)} &+ h(K + L), \quad \forall q_h \in M_h. \end{aligned}$$

Taking $q_h = I_h p$ in (6.19), we obtain

$$\|p - p_h\|_{(-2)} \leq \tilde{\beta}^{-1}|ln\theta|^{1/2}(\|\nabla_h(U - U_h)\|_{(-2)} + h(K + L)). \tag{6.22}$$

Now we select

$$\theta = kh|lnh|, \quad \gamma = (k|lnh|)^{1/2}, \tag{6.23}$$

where $k >> 1$, such that (6.15) and (6.21)

$$\tilde{\theta}^{-1}|ln\theta|^{1/2}((\frac{C^* h}{\theta})^{1/2} + \gamma) \leq \frac{1}{2}. \tag{6.24}$$

Now, combining with (6.12)(6.20), we obtain(6.16)

Step 4. Finally, we prove (6.1)-(6.3).

Using inverse estimates and (5.6)(5.7), we obtain

$$\begin{aligned}
&\|U-U_h\|_{L^\infty} \le \|U-\pi_h U\|_{L^\infty} + \|\pi_h U - U_h\|_{L^\infty} \\
&\le Ch^2\|\nabla^2 U\|_{L^\infty} + C\frac{\theta^2}{h}\|\pi_h U - U_h\|_{(-4)} \\
&\le CKh^2 + C\frac{\theta^2}{h}(\|\pi_h U - U_h\|_{(-4)} + \|U-U_h\|_{(-4)}) + Ch^2\|\nabla^2 U\|_{(-4)} \\
&\le CKh^2 + C\frac{\theta^2}{h}\|U-U_h\|_{(-4)} + C\theta h\|\nabla^2 U\|_{L^\infty} \\
&\le CKh(h+\theta) + C\frac{\theta^2}{h}\|U-U_h\|_{(-4)}.
\end{aligned} \tag{6.25}$$

Similarly, we also obtain

$$\|\nabla_h(U-U_h)\|_{L^\infty} \le CKh(1+|ln\theta|^{1/2}) + C\frac{\theta}{h}\|\nabla_h(U-U_h)\|_{(-2)}. \tag{6.26}$$

$$\|p-p_h\|_{L^\infty} \le CKh(1+|ln\theta|^{1/2}) + C\frac{\theta}{h}\|p-p_h\|_{(-2)}). \tag{6.27}$$

On the other hand, using (6.4)(6.12)(6.16), we have

$$\|\nabla_h(U-U_h)\|_{(-2)} \le Ck^{-\frac{1}{2}}\|\nabla_h(U-U_h)\|_{(-2)} + Ch|lnh|^{1/2}(K+L).$$

Thus, if taking $k >> 1$ large enough in (6.21), then we obtain

$$\|\nabla_h(U-U_h)\|_{(-2)} \le Ch|lnh|^{1/2}(K+L). \tag{6.28}$$

And from (6.4) and (6.16), we have

$$\|U-U_h\|_{(-4)} \le Ch(K+L). \tag{6.29}$$

$$\|p-p_h\|_{(-2)} \le Ch|lnh|^{1/2}(K+L). \tag{6.30}$$

Therefore, from (6.23)-(6.30), we obtain(6.1)-(6.3).

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