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FOURIER-LEGENDRE PSEUDOSPECTRAL METHOD FOR THE NAVIER-STOKES EQUATIONS*[1)]

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Abstract

In this paper, we construct a Fourier-Legendre pseudospectral scheme for the unsteady Navier-Stokes equations. This method easily deals with nonlinear terms and saves computational time. The strict error estimations are given.

Key Words: Navier-Stokes equations, Fourier-Legendre pseudospectral method, error estimation.

1. Introduction

The mixed spectral and pseudospectral methods are successful to numerically solve the semi-periodic problems of incompressible fluid flows (see [1-6]). This paper is devoted to the Fourier-Legendre pseudospectral method for the two-dimensional unsteady Navier-Stokes equations with semi-periodic boundary condition. This method is performed easily and has the same high accuracy as spectral method has.

Let $x = (x_1, x_2)^T$ and $\Omega = I_1 \times I_2$ where $I_1 = \{x_1 / -1 < x_1 < 1\}$, $\quad I_2 = \{x_2 / -\pi < x_2 < \pi\}$. We denote by $U(x,t)$ and $P(x,t)$ the speed and the pressure. Let $\partial_t = \frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x_j} \quad (j = 1, 2)$. We consider the Navier-Stokes equations as follows

$$\begin{cases} \partial_t U + (U \cdot \nabla)U - \nu \nabla^2 U + \nabla P = f, & \text{in} \quad \Omega \times (0, T], \\ \nabla \cdot U = 0, \qquad \text{in} \quad \Omega \times (0, T], \\ U(x, 0) = U_0(x), \quad P(x, 0) = P_0(x), \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinetic viscosity, $U_0(x)$ and $P_0(x)$ are the initial values. Assume that all functions in (1.1) have the period 2π for x_2. We also suppose that U satisfies the homogeneous boundary conditions in the x_1-direction

$$U(-1, x_2, t) = U(1, x_2, t) = 0, \quad \forall x_2 \in I_2.$$

Besides, to fix $P(x,t)$, we require

$$\mu(P) \equiv \int_\Omega P(x,t) dx = 0, \quad \forall t \in [0, T].$$

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We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the usual inner product and norm of $L^2(\Omega)$, etc.. Let $C^\infty_{0,p}(\Omega)$ be the subset of $C^\infty(\Omega)$, whose elements vanish at $x_1=\pm1$ and have the period 2π for $x_2\in I_2$. $H^1_{0,p}(\Omega)$ is the closure of $C^\infty_{0,p}(\Omega)$ in $H^1(\Omega)$.

2. The Scheme

Let M and N be positive integers. Assume that there exist positive constants c_1 and c_2 such that

$$c_1N\le M\le c_2N.$$

We denote by $\mathcal{P}_M$ the space of all polynomials with degree $\le M$, defined on I_1. Let

$$V_M=\{v(x_1)\in\mathcal{P}_M/v(-1)=v(1)=0\}.$$

Set l be integer, and

$$\tilde V_N=\mathrm{Span}\{e^{ilx_2}/\,|l|\le N\}.$$

Let V_N be the subset of $\tilde V_N$, containing all real-valued functions. Define

$$V_{M,N}=(V_M\times V_N)^2,\quad S_{M-1,N}=\{v\in\mathcal{P}_{M-1}\times V_N/\mu(v)=0\}.$$

Let $P^1_{M,N}:(H^1_{0,p}(\Omega))^2\longrightarrow V_{M,N}$ be the projection operator such that for any $u\in(H^1_{0,p}(\Omega))^2$,

$$(\nabla(u-P^1_{M,N}u),\nabla v)=0,\quad\forall v\in V_{M,N}.$$

While $P_{M-1,N}:L^2(\Omega)\longrightarrow\mathcal{P}_{M-1}(I_1)\times V_N$ is the orthogonal projection such that for any $u\in L^2(\Omega)$,

$$(u-P_{M-1,N}u,v)=0,\quad\forall v\in\mathcal{P}_{M-1}\times V_N.$$

Obviously, if $u\in L^2(\Omega)$ and $\mu(u)=0$, then $\mu(P_{M-1,N}u)=0$.

Now, let $\{x_1^{(j)},\omega^{(j)}\}$ be the nodes and weights of Gauss-Lobatto integration, i.e.,

$$x_1^{(0)}=-1,x_1^{(M)}=1,x_1^{(j)}(j=1,\cdots,M-1)\text{ zeroes of }L'_M,$$
$$\omega^{(j)}=\frac{2}{M(M+1)(L_M(x_1^{(j)}))^2},j=0,\cdots,M,$$

where L_M is the Legendre polynomial of degree M. Let $h=\frac{2\pi}{2N+1}$ be the mesh size for x_2. Define

$$\Omega_{M,N}=\{(x_1^{(j)},lh)/1\le j\le M-1,-N\le l\le N\},$$
$$\bar\Omega_{M,N}=\{(x_1^{(j)},lh)/0\le j\le M,-N\le l\le N\}.$$

The discrete inner products and norms are defined as follows

$$<u,v>_M=\sum_{j=0}^{M}u(x_1^{(j)})v(x_1^{(j)})\omega^{(j)},$$

$$(u,v)_{M,N}=\frac{1}{2N+1}\sum_{j=0}^{M}\sum_{l=-N}^{N}u(x_1^{(j)},lh)\bar v(x_1^{(j)},lh)\omega^{(j)},$$

$$\|u\|_{M,N}=(u,u)^{\frac12}_{M,N},\quad|u|_{1,M,N}=(\sum_{j=1}^{2}\|\partial_ju\|^2_{M,N})^{\frac12}.$$

It is known that if $uv \in \mathcal{P}_{2M-1} \times V_{2N}$, then

$$(u,v)_{M,N} = (u,v). \tag{2.1}$$

On the other hand, let $\{X_1^{(j)}, \bar{\omega}^{(j)}\}$ be the nodes and weights of Gauss integration, i.e.,

$$X_1^{(j)} (j = 0, \cdots, M-1) \text{ zeroes of } L_M,$$
$$\bar{\omega}^{(j)} = \frac{2}{(1-(X_1^{(j)})^2)(L_M'(X_1^{(j)}))^2}, j = 0, \cdots, M-1.$$

Similarly,

$$\Omega_{M,N}^* = \{(X_1^{(j)}, lh)/0 \le j \le M-1, -N \le l \le N\},$$
$$(u,v)_{M,N,*} = \frac{1}{2N+1} \sum_{\Omega_{M,N}^*} u\bar{v}\bar{\omega} = \frac{1}{2N+1} \sum_{j=0}^{M-1} \sum_{l=-N}^{N} u(X_1^{(j)}, lh)\bar{v}(X_1^{(j)}, lh)\bar{\omega}^{(j)},$$
$$(u,v)_{M,N,*} = (u,v), \quad \text{for } uv \in \mathcal{P}_{2M-1} \times V_{2N}. \tag{2.2}$$

Let $\tilde{\Omega}_{M,N}^*$ be the subset of $\Omega_{M,N}^*$, which we suppress an arbitrary point x^* from $\Omega_{M,N}^*$. Let P_C be the interpolation from $C(\bar{\Omega})$ to $\mathcal{P}_M \times V_N$ such that

$$P_C u(x) = u(x), \quad x \in \bar{\Omega}_{M,N}.$$

While P_C^* is the interpolation from $C(\bar{\Omega})$ to $\mathcal{P}_{M-1} \times V_N$ such that

$$P_C^* u(x) = u(x), \quad x \in \Omega_{M,N}^*.$$

We use small parameter technique for continuity equation (see [7]). Then the incompressible condition is approximated by

$$\beta \partial_t P + \nabla \cdot U = 0, \quad \beta > 0.$$

To approximate the nonlinear term, we define

$$d_c(u,v) = \sum_{j=1}^{2} \partial_j P_C(v^{(j)} u),$$

where $v^{(j)}$ is the component of v.

Let τ be the step of the time, and

$$R_\tau = \{t/t = k\tau, 1 \le k \le \left[\frac{T}{\tau}\right]\}.$$

We denote $u(x,t)$ by $u(t)$ or u sometimes. Let

$$u_{\hat{t}}(t) = \frac{1}{2\tau}(u(t+\tau) - u(t-\tau)), \quad \hat{u}(t) = \frac{1}{2}(u(t+\tau) + u(t-\tau)).$$

The Fourier-Legendre pseudospectral scheme for solving (1.1) is to find $u(t) \in V_{M,N}$, $p(t) \in S_{M-1,N}$ for $t \in R_\tau$, such that

$$\begin{cases} u_{\hat{t}} + d_c(u,u) - \nu \nabla^2 \hat{u} + \nabla \hat{p} = P_C f, \qquad \text{in } \Omega_{M,N}, \\ \beta p_{\hat{t}} + \nabla \cdot \hat{u} = 0, \qquad \text{in } \tilde{\Omega}_{M,N}^*, \\ u(0) = P_{M,N}^1 U_0, \quad u(\tau) = P_{M,N}^1 (U_0 + \tau \partial_t U(0)), \\ p(0) = P_{M-1,N} P_0, \quad p(\tau) = P_{M-1,N}(P_0 + \tau \partial_t P(0)), \end{cases} \tag{2.3}$$

where

$$\partial_t U(0) = f(0) - (U_0 \cdot \nabla)U_0 + \nu\nabla^2 U_0 - \nabla P_0,$$
$$\beta\partial_t P(0) + \nabla \cdot U_0 = 0.$$

The number of equations in (2.3) is $(2M-1)(2N+1)-1$. The boundary conditions for u and $\mu(p)=0$ provide other $2(2N+1)+1$ equations. So the number of unknown coefficients of the expansions of u and p equals the number of equations.

3. Some Lemmas

We first introduce some notations. For any integer $r \ge 0$, let $H^r(I_1)$ be the Hilbertian Sobolev space with the usual norm $\|\cdot\|_{r,I_1}$ and semi-norm $|\cdot|_{r,I_1}$, and $H_0^r(I_1)$ be the closure of $C_0^\infty(I_1)$ in $H^r(I_1)$. For any real $r > 0$, $H^r(I_1)$ is defined by the interpolation between the spaces $H^{[r]}(I_1)$ and $H^{[r+1]}(I_1)$, etc..

Let B be a Banach space with the norm $\|\cdot\|_B$, and Λ be an interval in R. Define

$$L^2(\Lambda, B) = \{v(z) : \Lambda \longrightarrow B/v \text{ is strongly measurable and } \|v\|_{L^2(\Lambda,B)} < \infty\},$$
$$C(\Lambda, B) = \{v(z) : \Lambda \longrightarrow B/v \text{ is strongly measurable and } \|v\|_{C(\Lambda,B)} < \infty\}$$

where

$$\|v\|_{L^2(\Lambda,B)} = (\int_\Lambda \|v(z)\|_B^2\, dz)^{\frac{1}{2}}, \quad \|v\|_{C(\Lambda,B)} = \max_{z\in\Lambda} \|v(z)\|_B .$$

For any non-negative integer α, let

$$H^\alpha(\Lambda, B) = \{v(z) \in L^2(\Lambda, B)/\, \|v\|_{H^\alpha(\Lambda,B)} < \infty\}$$

equipped with

$$\|v\|_{H^\alpha(\Lambda,B)} = (\sum_{k=0}^{\alpha} \left\|\frac{\partial^k v}{\partial z^k}\right\|_{L^2(\Lambda,B)}^2)^{\frac{1}{2}}.$$

For real $\alpha > 0$, $H^\alpha(\Lambda, B)$ is defined by the interpolation between $H^{[\alpha]}(\Lambda, B)$ and $H^{[\alpha+1]}(\Lambda, B)$. Let

$$H^{r,s}(\Omega) = L^2(I_2, H^r(I_1)) \bigcap H^s(I_2, L^2(I_1)), \quad r, s \ge 0,$$
$$M^{r,s}(\Omega) = H^{r,s}(\Omega) \bigcap H^1(I_2, H^{r-1}(I_1)) \bigcap H^{s-1}(I_2, H^1(I_1)), \quad r, s \ge 1,$$

with the following norms

$$\|v\|_{H^{r,s}(\Omega)} = (\|v\|^2_{L^2(I_2,H^r(I_1))} + \|v\|^2_{H^s(I_2,L^2(I_1))})^{\frac{1}{2}},$$
$$\|v\|_{M^{r,s}(\Omega)} = (\|v\|^2_{H^{r,s}(\Omega)} + \|v\|^2_{H^1(I_2,H^{r-1}(I_1))} + \|v\|^2_{H^{s-1}(I_2,H^1(I_1))})^{\frac{1}{2}}.$$

We denote by $H_{0,p}^{r,s}(\Omega)$ and $M_{0,p}^{r,s}(\Omega)$ the closures of $C_{0,p}^\infty(\Omega)$ in $H^{r,s}(\Omega)$ and $M^{r,s}(\Omega)$, etc.. Let $|||v|||_{q,\infty} = \max_{t\in R_\tau} \|v(t)\|_{q,\infty}$.

Next, we list some lemmas. Throughout the paper, c will be a positive constant which may be different in different cases.

Lemma 1. *If $u \in C(\bar\Omega)$ and $v \in \mathcal{P}_M \times V_N$, then*

$$\|v\| \le \|v\|_{M,N} \le \sqrt{3}\,\|v\|,$$
$$|(u,v)_{M,N} - (u,v)| \le c(\|u - P_{M-1,N}u\| + \|u - P_C u\|)\,\|v\| .$$

Furthermore if also $u \in \mathcal{P}_M \times V_N$, *then*

$$|(u,v)_{M,N} - (u,v)| \le cM^{-r} \|u\|_{H^{r,0}(\Omega)} \|v\|.$$

Proof. Let

$$v = \sum_{|l| \le N} v_l(x_1) e^{ilx_2}.$$

Clearly,

$$\|v\|_{M,N}^2 = \sum_{|l| \le N} < v_l, v_l >_M .$$

By (9.3.2) of [8], we get the first conclusion. Since (2.1), we have

$$\begin{aligned}
|(u,v)_{M,N} - (u,v)| &= |(u,v)_{M,N} - (P_{M-1,N}u, v) + (P_{M-1,N}u, v) - (u,v)| \\
&\le |(P_C u - P_{M-1,N}u, v)_{M,N}| + |(P_{M-1,N}u - u, v)| \\
&\le \|P_C u - P_{M-1,N}u\|_{M,N} \|v\|_{M,N} + \|u - P_{M-1,N}u\| \|v\| \\
&\le c(\|u - P_{M-1,N}u\| + \|u - P_C u\|) \|v\|.
\end{aligned}$$

Finally, we obtain from (9.3.5) and (9.4.6) of [8] that for $u = \sum_{|l| \le N} u_l e^{ilx_2} \in \mathcal{P}_M \times V_N$,

$$\begin{aligned}
|(u,v)_{M,N} - (u,v)| &= \left| \sum_{|l| \le N} [< u_l, v_l >_M - (u_l, v_l)_{L^2(I_1)}] \right| \\
&\le cM^{-r} \|u\|_{H^{r,0}(\Omega)} \|v\|.
\end{aligned}$$

Lemma 2 (Theorem A.1 of [2]). *If* $v \in H_p^{r,s}(\Omega)$ *and* $r, s \ge 0$, *then*

$$\|v - P_{M-1,N}v\| \le c(M^{-r} + N^{-s}) \|v\|_{H^{r,s}(\Omega)}.$$

Lemma 3. *If* $v \in H^1_{0,p}(\Omega) \bigcap H^\gamma(I_2, H^r(I_1)) \bigcap H^s(I_2, H^\alpha(I_1))$, $\quad \alpha = 0$ *or* $1, 0 \le \gamma \le s$ *and* $r, s \ge 1$, *then*

$$\left\| v - P^1_{M,N} v \right\|_{H^\gamma(I_2, H^\alpha(I_1))} \le c(M^{\alpha - r} + N^{\gamma - s}) \|v\|_{H^\gamma(I_2, H^r(I_1)) \bigcap H^s(I_2, H^\alpha(I_1))}.$$

Proof. Let $P^1_M : H^1_0(I_1) \longrightarrow V_M$ be the projection operator such that for $u \in H^1_0(I_1)$,

$$(\partial_1(u - P^1_M u), \partial_1 w)_{L^2(I_1)} = 0, \quad \forall w \in V_M.$$

By (9.4.22) of [8],

$$\left\| u - P^1_M u \right\|_{\alpha, I_1} \le cM^{\alpha - r} \|u\|_{r, I_1}, \quad \alpha = 0, 1. \tag{3.1}$$

Let

$$\begin{aligned}
v_l(x_1) &= \frac{1}{2\pi} \int_{I_2} v(x_1, x_2) e^{-ilx_2} dx_2, \\
P^1_{M,N} v &= \sum_{|l| \le N} v^*_l(x_1) e^{ilx_2}, \\
a_l(u, w) &= (\partial_1 u, \partial_1 w)_{L^2(I_1)} + |l|^2 (u, w)_{L^2(I_1)}, \quad |l| \le N.
\end{aligned}$$

Then $v_l^* \in V_M$ and $a_l(v_l - v_l^*, w) = 0$ for all $w \in V_M$. We have

$$a_l(u,u) = |u|_{1,I_1}^2 + |l|^2 \|u\|_{0,I_1}^2 ,$$
$$|a_l(u,w)| \le c(|u|_{1,I_1} + |l| \|u\|_{0,I_1})(|w|_{1,I_1} + |l| \|w\|_{0,I_1}).$$

Then we get from (3.1) that

$$\begin{aligned}|v_l - v_l^*|_{1,I_1}^2 + |l|^2 \|v_l - v_l^*\|_{0,I_1}^2 * &= a_l(v_l - v_l^*, v_l - v_l^*)\\ &\le c \inf_{w\in V_M} (|v_l - w|_{1,I_1}^2 + |l|^2 \|v_l - w\|_{0,I_1}^2)\\ &\le c(\left|v_l - P_M^1 v_l\right|_{1,I_1}^2 + |l|^2 \left\|v_l - P_M^1 v_l\right\|_{0,I_1}^2)\\ &\le cM^{2-2r} \|v_l\|_{r,I_1}^2 .\end{aligned}$$

Moreover by mean of the duality,

$$\|v_l - v_l^*\|_{\alpha,I_1} \le cM^{\alpha - r} \|v_l\|_{r,I_1}, \quad \alpha = 0,1.$$

Therefore, we have

$$\begin{aligned}\left\|v - P_{M,N}^1 v\right\|_{H^\gamma(I_2,H^\alpha(I_1))}^2 &\le c \sum_{|l|\le N} |l|^{2\gamma} \|v_l - v_l^*\|_{\alpha,I_1}^2 + c \sum_{|l|>N} |l|^{2\gamma} \|v_l\|_{\alpha,I_1}^2\\ &\le cM^{2(\alpha-r)} \sum_{|l|\le N} |l|^{2\gamma} \|v_l\|_{r,I_1}^2 + cN^{2(\gamma-s)} \sum_{|l|>N} |l|^{2s} \|v_l\|_{\alpha,I_1}^2\\ &\le c(M^{2(\alpha-r)} + N^{2(\gamma-s)}) \|v\|_{H^\gamma(I_2,H^r(I_1))\bigcap H^s(I_2,H^\alpha(I_1))}^2 .\end{aligned}$$

Lemma 4 (Theorem A.3 of [2]). *If* $v \in H^{r,s}(\Omega)\bigcap H^{s'}(I_2,H^{r'}(I_1))$ *with* $r,r' > \frac{1}{2}$ *and* $s,s' > \frac{1}{2}$, *then*

$$\|v - P_C v\| \le c(M^{\frac{1}{2}-r} \|v\|_{L^2(I_2,H^r(I_1))} + N^{-s} \|v\|_{H^s(I_2,L^2(I_1))} + M^{\frac{1}{2}-r'} N^{-s'} \|v\|_{H^{s'}(I_2,H^{r'}(I_1))}).$$

Lemma 5. *If* $v \in H_{0,p}^1(\Omega)\bigcap H^{s+m}(I_2,H^{r+k}(I_1))$ *with* $r > \frac{3}{2}$, $s > \frac{1}{2}$, *then*

$$\left\|\partial_1^k \partial_2^m P_{M,N}^1 v\right\|_\infty \le c \|v\|_{H^{s+m}(I_2,H^{r+k}(I_1))}, \quad k = 0,1.$$

Proof. Noting that

$$\left\|\partial_1^k \partial_2^m P_{M,N}^1 v\right\|_\infty \le \sum_{|l|\le N} |l|^m |v_l^*|_{k,\infty,I_1},$$

we can prove the conclusion by the same way in Lemma 5 of [6].

Lemma 6. *If* $v \in \mathcal{P}_M \times V_N$, *then*

$$\|v\|_\infty \le cMN^{\frac{1}{2}} \|v\| .$$

Proof. Let

$$v = \sum_{|l|\le N} v_l(x_1) e^{ilx_2}.$$

Then $v_l(x_1) \in \mathcal{P}_M$. By (9.4.3) of [8],

$$\|v\|_\infty \le \sum_{|l|\le N} \|v_l\|_{\infty,I_1} \le cM \sum_{|l|\le N} \|v_l\|_{L^2(I_1)} \le cMN^{\frac{1}{2}} \|v\| .$$

Lemma 7 (Lemma 4.16 of [9]). *Assume that the following conditions are fulfilled*
(i) $E(t)$ *is a non-negative function defined on* R_τ*;*
(ii) ρ, M_1 *and* M_2 *are non-negative constants;*
(iii) *for all* $t \in R_\tau$,

$$E(t) \leq \rho + M_1\tau \sum_{\zeta\in R_\tau, \zeta<t} [E(\zeta) + M_2E^2(\zeta)];$$

(iv) $E(0) \leq \rho$, *and for some* $t_1 \in R_\tau$,

$$\rho e^{2M_1t_1} \leq \frac{1}{M_2}.$$

Then for all $t \in R_\tau, \quad t \leq t_1$, *we have*

$$E(t) \leq \rho e^{2M_1t}.$$

4. Error Estimation

First, we consider the generalized stability of scheme (2.3). If the initial values and the right terms of scheme (2.3) have the errors $\tilde{u}(0), \tilde{u}(\tau), \tilde{p}(0), \tilde{p}(\tau), \tilde{f}(t)$ and $\tilde{g}(t)$, then $u(t)$ and $p(t)$ have the errors $\tilde{u}(t)$ and $\tilde{p}(t)$. They satisfy

$$\begin{cases} \tilde{u}_{\hat{t}} + d_c(u,\tilde{u}) + d_c(\tilde{u}, u+\tilde{u}) - \nu\nabla^2\hat{\tilde{u}} + \nabla\hat{\tilde{p}} = P_C\tilde{f}, \quad \text{in } \Omega_{M,N}, \\ \beta\tilde{p}_{\hat{t}} + \nabla\cdot\hat{\tilde{u}} = P_C^*\tilde{g}, \quad \text{in } \tilde{\Omega}^*_{M,N}. \end{cases} \tag{4.1}$$

By the boundary conditions of $\tilde{u}$, (2.2) and $\mu(\tilde{p}) = 0$, we get

$$0 = \int_\Omega (\beta\tilde{p}_{\hat{t}} + \nabla\cdot\hat{\tilde{u}})dx = (\beta\tilde{p}_{\hat{t}} + \nabla\cdot\hat{\tilde{u}}, 1)_{M,N,*} = \frac{1}{2N+1}\sum_{\Omega^*_{M,N}} (\beta\tilde{p}_{\hat{t}} + \nabla\cdot\hat{\tilde{u}})\bar{\omega}.$$

Since the second equation of (4.1),

$$\beta\tilde{p}_{\hat{t}}(x^*) + \nabla\cdot\hat{\tilde{u}}(x^*) = -\frac{1}{\bar{\omega}(x^*)}\sum_{\tilde{\Omega}^*_{M,N}} P_C^*\tilde{g}\bar{\omega}.$$

Let $\tilde{g}(x^*) = -\frac{1}{\bar{\omega}(x^*)}\sum_{\tilde{\Omega}^*_{M,N}} P_C^*\tilde{g}\bar{\omega}$, and so

$$\beta\tilde{p}_{\hat{t}} + \nabla\cdot\hat{\tilde{u}} = P_C^*\tilde{g}, \quad \text{in } \Omega^*_{M,N}.$$

Hence, we have

$$\begin{cases} (\tilde{u}_{\hat{t}} + d_c(u,\tilde{u}) + d_c(\tilde{u}, u+\tilde{u}) - \nu\nabla^2\hat{\tilde{u}} + \nabla\hat{\tilde{p}}, v)_{M,N} = (P_C\tilde{f}, v)_{M,N}, \quad \forall v \in V_{M,N}, \\ (\beta\tilde{p}_{\hat{t}} + \nabla\cdot\hat{\tilde{u}}, \varphi)_{M,N,*} = (P_C^*\tilde{g}, \varphi)_{M,N,*}, \quad \forall\varphi \in S_{M-1,N}. \end{cases}$$

Noting that $\tilde{p}$ and φ belong to $S_{M-1,N}$, we obtain from (2.1) and (2.2) that

$$\begin{cases} (\tilde{u}_{\hat{t}} + d_c(u,\tilde{u}) + d_c(\tilde{u}, u+\tilde{u}) - \nu\nabla^2\hat{\tilde{u}}, v)_{M,N} + (\nabla\hat{\tilde{p}}, v) = (P_C\tilde{f}, v)_{M,N}, \quad \forall v \in V_{M,N}, \\ (\beta\tilde{p}_{\hat{t}} + \nabla\cdot\hat{\tilde{u}}, \varphi) = (P_C^*\tilde{g}, \varphi), \quad \forall\varphi \in S_{M-1,N}. \end{cases} \tag{4.2}$$

By taking $v = 2\hat{\tilde{u}}$ in (4.2), we have from (2.1) and integrating by parts that

$$(\|\tilde{u}\|_{M,N}^2)_{\hat{t}} + 2\nu \left|\hat{\tilde{u}}\right|_{1,M,N}^2 + \sum_{j=1}^{2} F_j + 2(\nabla\hat{\tilde{p}}, \hat{\tilde{u}}) \le \frac{1}{2}\left\|\hat{\tilde{u}}\right\|_{M,N}^2 + 2\left\|P_C\tilde{f}\right\|_{M,N}^2, \tag{4.3}$$

where

$$F_1 = 2(d_c(u,\tilde{u}) + d_c(\tilde{u},u), \hat{\tilde{u}})_{M,N}, \quad F_2 = 2(d_c(\tilde{u},\tilde{u}), \hat{\tilde{u}})_{M,N}.$$

We choose $\varphi = 2\hat{\tilde{p}}$ in (4.2),

$$\beta(\|\tilde{p}\|^2)_{\hat{t}} + 2(\nabla\cdot\hat{\tilde{u}}, \hat{\tilde{p}}) = 2(P_C^*\tilde{g}, \hat{\tilde{p}}) \le \frac{\beta}{2}\left\|\hat{\tilde{p}}\right\|^2 + \frac{2}{\beta}\|P_C^*\tilde{g}\|^2. \tag{4.4}$$

Combining (4.3) with (4.4), we integrate by parts,

$$\begin{aligned}(\|\tilde{u}\|_{M,N}^2 + \beta\|\tilde{p}\|^2)_{\hat{t}} + 2\nu\left|\hat{\tilde{u}}\right|_{1,M,N}^2 + \sum_{j=1}^{2}F_j \le& \frac{1}{2}\left\|\hat{\tilde{u}}\right\|_{M,N}^2 + \frac{\beta}{2}\left\|\hat{\tilde{p}}\right\|^2 \\ &+ 2\left\|P_C\tilde{f}\right\|_{M,N}^2 + \frac{2}{\beta}\|P_C^*\tilde{g}\|^2.\end{aligned} \tag{4.5}$$

We estimate $|F_j|$. By (2.1), integrating by parts, Lemma 1 and Lemma 6, we have

$$\begin{aligned}|F_1| &= 2\left|(P_C(\tilde{u}^{(1)}u) + P_C(u^{(1)}\tilde{u}), \partial_1\hat{\tilde{u}}) + (P_C(\tilde{u}^{(2)}u) + P_C(u^{(2)}\tilde{u}), \partial_2\hat{\tilde{u}})_{M,N}\right| \\ &\le c\|u\|_\infty\|\tilde{u}\|_{M,N}\left|\hat{\tilde{u}}\right|_{1,M,N} \le \frac{\nu}{2}\left|\hat{\tilde{u}}\right|_{1,M,N}^2 + \frac{c}{\nu}\|u\|_\infty^2\|\tilde{u}\|^2, \\ |F_2| &\le c\|\tilde{u}\|_\infty\|\tilde{u}\|_{M,N}\left|\hat{\tilde{u}}\right|_{1,M,N} \le \frac{\nu}{2}\left|\hat{\tilde{u}}\right|_{1,M,N}^2 + \frac{cM^2N}{\nu}\|\tilde{u}\|^4.\end{aligned}$$

By substituting the above estimations into (4.5), we get

$$\begin{aligned}(\|\tilde{u}\|_{M,N}^2 + \beta\|\tilde{p}\|^2)_{\hat{t}} + \nu\left|\hat{\tilde{u}}\right|_{1,M,N}^2 \le& \frac{1}{2}\left\|\hat{\tilde{u}}\right\|_{M,N}^2 + \frac{\beta}{2}\left\|\hat{\tilde{p}}\right\|^2 + \frac{c}{\nu}\|u\|_\infty^2\|\tilde{u}\|^2 + \frac{cM^2N}{\nu}\|\tilde{u}\|^4 \\ &+ 2\left\|P_C\tilde{f}\right\|_{M,N}^2 + \frac{2}{\beta}\|P_C^*\tilde{g}\|^2.\end{aligned} \tag{4.6}$$

In fact,

$$\begin{aligned}\left\|\hat{\tilde{u}}(t)\right\|_{M,N}^2 &\le \frac{1}{2}(\|\tilde{u}(t+\tau)\|_{M,N}^2 + \|\tilde{u}(t-\tau)\|_{M,N}^2), \\ \left\|\hat{\tilde{p}}(t)\right\|^2 &\le \frac{1}{2}(\|\tilde{p}(t+\tau)\|^2 + \|\tilde{p}(t-\tau)\|^2).\end{aligned}$$

Let

$$\begin{aligned}M_1 =& 6 + \frac{c}{\nu}\||u\||_\infty^2, \quad M_2 = \frac{cM^2N}{\nu}, \\ E(t) =& \|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2 + 4\nu\tau\sum_{\zeta\in R_\tau,\zeta<t}\left|\hat{\tilde{u}}(\zeta)\right|_1^2 \\ \rho(t) =& 3(3\|\tilde{u}(0)\|^2 + \beta\|\tilde{p}(0)\|^2) + 2(3\|\tilde{u}(\tau)\|^2 + \beta\|\tilde{p}(\tau)\|^2) \\ &+ 8\tau\sum_{\zeta\in R_\tau,\zeta<t}(3\left\|P_C\tilde{f}(\zeta)\right\|^2 + \frac{1}{\beta}\|P_C^*\tilde{g}(\zeta)\|^2).\end{aligned}$$

Then by letting $\tau \leq 1$ and summing (4.6) for $t \in R_\tau$, we have from Lemma 1 that

$$E(t) \leq \rho(t) + \tau \sum_{\zeta \in R_\tau, \zeta < t} [M_1 E(\zeta) + M_2 E^2(\zeta)].$$

We use Lemma 7 to obtain the following result.

Theorem 1. *There exist positive constants M_1 and M_3 depending only on $|||u|||_\infty$ and ν, such that if for some $t_1 \in R_\tau$,*

$$\rho(t_1) e^{2M_1 t_1} \leq \frac{M_3}{M^2 N},$$

then for all $t \in R_\tau, \quad t \leq t_1$,

$$E(t) \leq \rho(t) e^{2M_1 t}.$$

Next, we derive the convergence of scheme (2.3). By analysis as before, we know that if u and p are the solutions of (2.3), then

$$\begin{cases} (u_{\hat{t}} + d_c(u,u) - \nu \nabla^2 \hat{u}, v)_{M,N} + (\nabla \hat{p}, v) = (P_C f, v)_{M,N}, \quad \forall v \in V_{M,N}, \\ (\beta p_{\hat{t}} + \nabla \cdot \hat{u}, \varphi) = 0, \quad \forall \varphi \in S_{M-1,N}. \end{cases} \tag{4.7}$$

Let

$$U^*(t) = P^1_{M,N} U(t), \quad P^*(t) = P_{M-1,N} P(t), \quad e(t) = u(t) - U^*(t), \quad \phi(t) = p(t) - P^*(t).$$

By (1.1) and (4.7), we get

$$\begin{cases} (e_{\hat{t}} + d_c(U^*, e) + d_c(e, U^* + e) - \nu \nabla^2 \hat{e}, v)_{M,N} + (\nabla \hat{\phi}, v) = \sum\limits_{j=1}^{6} A_j(v,t), \quad \forall v \in V_{M,N}, \\ (\beta \phi_{\hat{t}} + \nabla \cdot \hat{e}, \varphi) = \sum\limits_{j=7}^{8} A_j(\varphi, t), \quad \forall \varphi \in S_{M-1,N}, \\ e(0) = 0, \quad e(\tau) = P^1_{M,N}(U_0 + \tau \partial_t U(0) - U(\tau)), \\ \phi(0) = 0, \quad \phi(\tau) = P_{M-1,N}(P_0 + \tau \partial_t P(0) - P(\tau)), \end{cases} \tag{4.8}$$

where

$$\begin{aligned} A_1(v,t) &= (\partial_t U, v) - (U^*_{\hat{t}}, v)_{M,N}, \\ A_2(v,t) &= ((U \cdot \nabla) U, v) - (d_c(U^*, U^*), v)_{M,N}, \\ A_3(v,t) &= -\nu(\nabla^2 \hat{U}^*, v) + \nu(\nabla^2 \hat{U}^*, v)_{M,N}, \\ A_4(v,t) &= -\nu(\nabla^2 U, v) + \nu(\nabla^2 \hat{U}, v), \\ A_5(v,t) &= (\nabla(P - \hat{P}^*), v), \\ A_6(v,t) &= -(f, v) + (P_C f, v)_{M,N}, \\ A_7(\varphi,t) &= -\beta(P_{\hat{t}}, \varphi), \\ A_8(\varphi,t) &= (\nabla \cdot (U - \hat{U}^*), \varphi). \end{aligned}$$

We have to estimate the right terms in (4.8). Obviously,

$$|A_1(v,t)| \leq |(\partial_t U - U_{\hat{t}}, v)| + |(U_{\hat{t}}, v) - (U_{\hat{t}}, v)_{M,N}| + \left|(U_{\hat{t}} - U^*_{\hat{t}}, v)_{M,N}\right|.$$

Thanks to Lemma 1, we know

$$\left|(U_{\hat{t}} - U_{\hat{t}}^*, v)_{M,N}\right| = \left|(P_C U_{\hat{t}} - U_{\hat{t}}^*, v)_{M,N}\right| \le 3\,\|v\|\,\left\|P_C U_{\hat{t}} - U_{\hat{t}}^*\right\|$$
$$\le 3\,\|v\|\,(\|P_C U_{\hat{t}} - U_{\hat{t}}\| + \left\|U_{\hat{t}} - U_{\hat{t}}^*\right\|).$$

By Poincare inequality,

$$\|v\| \le c\,|v|_1\,.$$

From Lemma 1-Lemma 4, we have that for $r > 1$ and $s \ge 1$,

$$|A_1(v,t)| \le c\,\|v\|\,(\tau^{\frac{3}{2}}\,\|U\|_{H^3(t-\tau,t+\tau;L^2(\Omega))} + \|U_{\hat{t}} - P_{M-1,N}U_{\hat{t}}\| + \|U_{\hat{t}} - P_C U_{\hat{t}}\|$$
$$+ \left\|U_{\hat{t}} - U_{\hat{t}}^*\right\|) \le \frac{\nu}{32}\,|v|_1^2 + c\tau^3\,\|U\|^2_{H^3(t-\tau,t+\tau;L^2(\Omega))} + c\tau^{-1}(M^{-2r} + N^{-2s})$$
$$\|U\|^2_{H^1(t-\tau,t+\tau;H^{r+\frac{1}{2},s}(\Omega)\bigcap H^1(I_2,H^{r-\frac{1}{2}}(I_1))}\,.$$

Let $A_2(v,t) = B_1 + B_2$ where

$$B_1 = (d_c(U^*,U^*),v) - (d_c(U^*,U^*),v)_{M,N},$$
$$B_2(t) = ((U\cdot\nabla)U - d_c(U^*,U^*),v).$$

Since $\partial_1 P_C((U^*)^{(1)}U^*) \in \mathcal{P}_{M-1}\times V_N$,

$$B_1 = (\partial_2 P_C((U^*)^{(2)}U^*),v) - (\partial_2 P_C((U^*)^{(2)}U^*),v)_{M,N}$$
$$= -(P_C((U^*)^{(2)}U^*),\partial_2 v) + (P_C((U^*)^{(2)}U^*),\partial_2 v)_{M,N}.$$

Let I be the identity operator. By Lemma 1, we have

$$|B_1| \le c\,|v|_1\,\left\|(I - P_{M-1,N})P_C((U^*)^{(2)}U^*)\right\|$$
$$\le c\,|v|_1\,(\left\|(I - P_{M-1,N})(I - P_C)((U^*)^{(2)}U^*)\right\| + \left\|(I - P_{M-1,N})((U^*)^{(2)}U^*)\right\|).$$

Using Lemma 2-Lemma 5, we get that for $r > 1, s \ge 1, \alpha > \frac{3}{2}$ and $\gamma > \frac{1}{2}$,

$$\left\|(I - P_{M-1,N})(I - P_C)((U^*)^{(2)}U^*)\right\| \le c\left\|(I - P_C)((U^*)^{(2)}U^*)\right\|$$
$$\le c\left\|(I-P_C)(((U^*)^{(2)} - U^{(2)})U^*)\right\| + c\left\|(I-P_C)(U^{(2)}(U^*-U))\right\| + c\left\|(I-P_C)(U^{(2)}U)\right\|$$
$$\le cM^{-\frac{1}{2}}\left\|((U^*)^{(2)} - U^{(2)})U^*\right\|_{L^2(I_2,H^1(I_1))} + cN^{-1}\left\|((U^*)^{(2)} - U^{(2)})U^*\right\|_{H^1(I_2,L^2(I_1))}$$
$$+ cM^{-\frac{1}{2}}N^{-1}\left\|((U^*)^{(2)} - U^{(2)})U^*\right\|_{H^1(I_2,H^1(I_1))} + cM^{-\frac{1}{2}}\left\|U^{(2)}(U^* - U)\right\|_{L^2(I_2,H^1(I_1))}$$
$$+ cN^{-1}\left\|U^{(2)}(U^* - U)\right\|_{H^1(I_2,L^2(I_1))} + cM^{-\frac{1}{2}}N^{-1}\left\|U^{(2)}(U^* - U)\right\|_{H^1(I_2,H^1(I_1))}$$
$$+ c(M^{-r} + N^{-s})\left\|U^{(2)}U\right\|_{H^{r+\frac{1}{2},s}(\Omega)\bigcap H^1(I_2,H^{r-\frac{1}{2}}(I_1))}$$
$$\le c(M^{-r} + N^{-s})\,\|U\|_{H^{\gamma+1}(I_2,H^{\alpha+1}(I_1))\bigcap W^{2,\infty}(\Omega)}$$
$$\cdot\,\|U\|_{H^{r+\frac{1}{2},s}(\Omega)\bigcap H^1(I_2,H^{r-\frac{1}{2}}(I_1))\bigcap H^{s-\frac{1}{2}}(I_2,H^1(I_1))}$$
$$+ c(M^{-r} + N^{-s})\left\|U^{(2)}U\right\|_{H^{r+\frac{1}{2},s}(\Omega)\bigcap H^1(I_2,H^{r-\frac{1}{2}}(I_1))}\,.$$

By the embedding theorem, we have

$$\|uv\|_{H^{r,s}(\Omega)} \le c\,\|u\|_{M^{r+1,s+1}(\Omega)}\,\|v\|_{M^{r+1,s+1}(\Omega)}\,,$$
$$\|uv\|_{H^1(I_2,H^r(I_1))} \le c\,\|u\|_{H^2(I_2,H^r(I_1))\bigcap H^1(I_2,H^{r+1}(I_1))}\,\|v\|_{H^2(I_2,H^r(I_1))\bigcap H^1(I_2,H^{r+1}(I_1))}\,.$$

Clearly,

$$\begin{aligned}\left\|(I-P_{M-1,N})((U^*)^{(2)}U^*)\right\| \le{}& \left\|(I-P_{M-1,N})((U^*)^{(2)}-U^{(2)})U^*\right\| \\ &+ \left\|(I-P_{M-1,N})U^{(2)}(U^*-U)\right\| + \left\|(I-P_{M-1,N})U^{(2)}U\right\|.\end{aligned}$$

Then we obtain

$$\begin{aligned}|B_1| \le{}& \frac{\nu}{32}\,|v|_1^2 + c(M^{-2r}+N^{-2s})\{\|U\|^2_{H^{\gamma+1}(I_2,H^{\alpha+1}(I_1))\bigcap W^{2,\infty}(\Omega)} \\ &\cdot \|U\|^2_{H^{r+\frac12,s}(\Omega)\bigcap H^1(I_2,H^{r-\frac12}(I_1))\bigcap H^{s-\frac12}(I_2,H^1(I_1))} \\ &+ \|U\|^4_{M^{r+\frac32,s+1}(\Omega)\bigcap H^2(I_2,H^{r-\frac12}(I_1))}\}.\end{aligned}$$

We now estimate $|B_2|$. Moreover,

$$\begin{aligned}|B_2| &\le c\,|v|_1 \sum_{j=1}^{2} \left\|U^{(j)}U - P_C((U^*)^{(j)}U^*)\right\| \\ &\le c\,|v|_1 \sum_{j=1}^{2} (\left\|U^{(j)}U-(U^*)^{(j)}U^*\right\| + \left\|(I-P_C)((U^*)^{(j)}U^*)\right\|).\end{aligned}$$

Similarly,

$$\begin{aligned}|B_2| \le{}& \frac{\nu}{32}\,|v|_1^2 + c(M^{-2r}+N^{-2s})\{\|U\|^2_{H^{\gamma+1}(I_2,H^{\alpha+1}(I_1))\bigcap W^{2,\infty}(\Omega)} \\ &\cdot \|U\|^2_{H^{r+\frac12,s}(\Omega)\bigcap H^1(I_2,H^{r-\frac12}(I_1))\bigcap H^{s-\frac12}(I_2,H^1(I_1))} \\ &+ \|U\|^4_{M^{r+\frac32,s+1}(\Omega)\bigcap H^2(I_2,H^{r-\frac12}(I_1))}\}.\end{aligned}$$

It can be verified that

$$\begin{aligned}|A_3(v,t)| &= \left|\nu(\partial_2\hat U^*,\partial_2 v) - \nu(\partial_2\hat U^*,\partial_2 v)_{M,N}\right| \\ &\le c\nu\,|v|_1\left\|(I-P_{M-1,N})\partial_2\hat U^*\right\| \\ &\le c\nu\,|v|_1\left\|(I-P_{M-1,N})\partial_2(\hat U^*-\hat U)\right\| + c\nu\,|v|_1\left\|(I-P_{M-1,N})\partial_2\hat U\right\| \\ &\le c\nu(M^{-r}+N^{-s})\,|v|_1\,\|U\|_{C(t-\tau,t+\tau;H^1(I_2,H^r(I_1))\bigcap H^{s+1}(I_2,L^2(I_1)))} \\ &\le \frac{\nu}{32}\,|v|_1^2 + c(M^{-2r}+N^{-2s})\,\|U\|^2_{C(t-\tau,t+\tau;H^1(I_2,H^r(I_1))\bigcap H^{s+1}(I_2,L^2(I_1)))}\,, \\ |A_4(v,t)| &\le \frac{\nu}{32}\,|v|_1^2 + c\tau^3\,\|U\|^2_{H^2(t-\tau,t+\tau;H^1(\Omega))}\,, \\ |A_5(v,t)| &\le |v|_1\left\|P-\hat P^*\right\| \\ &\le \frac{\nu}{32}\,|v|_1^2 + c\tau^3\,\|P\|^2_{H^2(t-\tau,t+\tau;L^2(\Omega))} + c(M^{-2r}+N^{-2s})\,\|P\|^2_{C(t-\tau,t+\tau;H^{r,s}(\Omega))}\,, \\ |A_6(v,t)| &\le \frac{\nu}{32}\,|v|_1^2 + c(M^{-2r}+N^{-2s})\,\|f\|^2_{H^{r+\frac12,s}(\Omega)\bigcap H^1(I_2,H^{r-\frac12}(I_1))}\,.\end{aligned}$$

Moreover,

$$
\begin{aligned}
|A_7(\varphi,t)| \leq& \frac{\beta}{16}\|\varphi\|^2 + c\beta\tau^{-1}\|P\|^2_{H^1(t-\tau,t+\tau;L^2(\Omega))},\\
|A_8(\varphi,t)| \leq& \frac{\beta}{16}\|\varphi\|^2 + c\beta^{-1}(M^{-2r}+N^{-2s})\|U\|^2_{C(t-\tau,t+\tau;M^{r+1,s+1}(\Omega))}\\
&+ c\beta^{-1}\tau^3\|U\|^2_{H^2(t-\tau,t+\tau;H^1(\Omega))},\\
\|e(\tau)\|^2 \leq& c\tau^4\|U\|^2_{H^2(0,T;H^1(\Omega))}, \quad \|\phi(\tau)\|^2 \leq c\tau^4\|P\|^2_{H^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

Suppose that

$$\tau = O((M^2N)^{-\frac{1}{2}}), \qquad c_3\tau^2 \leq \beta \leq c_4\tau^2,$$

where c_3 and c_4 are the positive constants. If $r,s \geq 3$, then

$$\beta^{-1}(\tau^4 + M^{-2r} + N^{-2s}) + \beta \leq \frac{C}{M^2N}.$$

By an argument as in Theorem 1, we get the following result.

Theorem 2. *Assume that*

(i) $\tau = O((M^2N)^{-\frac{1}{2}}), \quad c_3\tau^2 \leq \beta \leq c_4\tau^2;$

(ii) *for* $r,s \geq 3, \alpha > \frac{3}{2}$ *and* $\gamma > \frac{1}{2}$,

$$
\begin{aligned}
U \in& C(0,T;M_{0,p}^{r+\frac{3}{2},s+1}(\Omega)) \bigcap H^{\gamma+1}(I_2,H^{\alpha+1}(I_1)) \bigcap W^{2,\infty}(\Omega) \bigcap H^2(I_2,H^{r-\frac{1}{2}}(I_1)))\\
&\bigcap H^1(0,T;H^{r+\frac{1}{2},s}(\Omega) \bigcap H^1(I_2,H^{r-\frac{1}{2}}(I_1))) \bigcap H^2(0,T;H^1(\Omega)) \bigcap H^3(0,T;L^2(\Omega)),\\
P \in& C(0,T;H_p^{r,s}(\Omega)) \bigcap H^2(0,T;L^2(\Omega)), \quad f \in C(0,T;H_p^{r+\frac{1}{2},s}(\Omega) \bigcap H^1(I_2,H^{r-\frac{1}{2}}(I_1))).
\end{aligned}
$$

Then for all $t \leq T$,

$$\|U(t)-u(t)\|^2 \leq M_4(\beta^{-1}(\tau^4 + M^{-2r} + N^{-2s}) + \beta),$$

where M_4 *is a positive constant depending only on* ν *and the norms of* U *and* P *in the spaces mentioned in the above.*

5. Numerical Results

In this section, we present the numerical results to confirm our method. The exact solutions of (1.1) are given by

$$
\begin{aligned}
U_1 &= 0.1e^{At}(1-x_1^2)^2\cos 2x_2,\\
U_2 &= 0.2e^{At}(x_1-x_1^3)\sin 2x_2,\\
P &= 0.1e^{Bt}(x_1-x_1^3)\sin 2x_2.
\end{aligned}
$$

The errors for the speed and the pressure are defined by

$$E(U_i(t)) = \left(\frac{\sum\limits_{x\in\Omega_{M,N}} |U_i(t) - u_i(t)|^2}{\sum\limits_{x\in\Omega_{M,N}} |U_i(t)|^2} \right)^{\frac{1}{2}}, \qquad i = 1, 2,$$

$$E(P(t)) = \left(\frac{\sum\limits_{x\in\Omega_{M,N}} |P(t) - p(t)|^2}{\sum\limits_{x\in\Omega_{M,N}} |P(t)|^2} \right)^{\frac{1}{2}},$$

where u_i and p are the solutions of (2.3).

We solve (1.1) by the scheme (2.3). The calculation is carried out with $M = N = 4$ and $A = B = 0.1$. The numerical results are tabulated in Table 1,2,3 and 4. The results are very accurate if β is chosen suitably, see Table 1 and 2. Table 3 and 4 show that β becomes less, the accuracy decreases. It agrees with our theoretical analysis very well. Since the nonlinear terms are computed on the collocation points, this method is performed very simply.

Table 1. The errors of (2.3), $\tau = 0.01, \nu = 0.001$ and $\beta = 0.05$.

t	$E(U_1(t))$	$E(U_2(t))$	$E(P(t))$
1	0.1203E-2	0.1512E-2	0.3942E-2
2	0.1083E-2	0.1456E-2	0.9666E-3
3	0.5059E-3	0.7982E-3	0.9696E-2
4	0.2427E-3	0.8051E-3	0.9040E-2
5	0.1348E-2	0.2895E-2	0.9248E-4

Table 2. The errors of (2.3), $\tau = 0.005, \nu = 0.0001$ and $\beta = 0.05$.

t	$E(U_1(t))$	$E(U_2(t))$	$E(P(t))$
1	0.1180E-2	0.1479E-2	0.3508E-2
2	0.9922E-3	0.1333E-2	0.3060E-3
3	0.5792E-3	0.8790E-3	0.7297E-2
4	0.3678E-3	0.8014E-3	0.6611E-2
5	0.1071E-2	0.1780E-2	0.1066E-2

Table 3. The errors of (2.3), $\tau = 0.005, \nu = 0.001$ and $\beta = 0.01$.

t	$E(U_1(t))$	$E(U_2(t))$	$E(P(t))$
1	0.1285E-3	0.1738E-3	0.2403E-2
2	0.1019E-3	0.2548E-3	0.6076E-2
3	0.5564E-3	0.1058E-2	0.5218E-2
4	0.6615E-3	0.1938E-2	0.9113E-2
5	0.1743E-3	0.5391E-1	0.2549E-1

Table 4. The errors of (2.3), $\tau = 0.005, \nu = 0.001$ and $\beta = 0.005$.

t	$E(U_1(t))$	$E(U_2(t))$	$E(P(t))$
1	0.8480E-4	0.1214E-3	0.4712E-3
2	0.3105E-3	0.3628E-2	0.4710E-2
3	0.3088E-2	0.1705	0.4598E-1
4	0.1230	0.7085E+1	0.2690E+1
5	0.3955E+2	0.2445E+3	0.5070E+3

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