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A FAMILY OF HIGH-ODER PARALLEL ROOTFINDERS FOR POLYNOMIALS*1)

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Abstract

In this paper we present a family of parallel iterations of order $m+2$ with parameter $m=0,1\cdots$ for simultaneous finding all zeros of a polynomial without evaluation of derivatives, which includes the well known Weierstrass-Durand-Dochev-Kerner and Börsch-Supan-Nourein iterations as the special cases for $m=0$ and $m=1$, respectively. Some numerical examples are given.

Key words: Parallel iteration, zeros of polynomial, order of convergence

1. Introduction

Let

$$f(t)=\sum_{i=0}^{n}a_it^{n-i}=\prod_{j=1}^{n}(t-\xi_i),\quad a_0=1 \tag{1}$$

be a monic complex polynomial of degree n with zeros $\xi_1,\cdots,\xi_n$. Some authors have studied the parallel iterations without evaluation of derivatives for simultaneous finding all zeros of $f(t)$ (see [1]-[10]). The famous one is Weierstrass-Durand-Dochev-Kerner iteration

$$x_i^{k+1}=x_i^k-u_i^k\quad i=1,2,\cdots,n,\quad k=0,1,\cdots, \tag{2}$$

where x_i^k is the k-th approximation of $\xi_i(1\le i\le n)$ and

$$u_i^k=\frac{f(x_i^k)}{\prod\limits_{j\neq i}(x_i^k-x_j^k)},\quad i=1,\cdots,n,\quad k=0,1,\cdots, \tag{3}$$

which does not require any information of derivatives and was presented independently by Weierstrass[7], Durand[2], Dochev[3] and Kerner[4]. It is well known that the convergence of (2) is quadratic if $\xi_i\neq\xi_j$ for $i\neq j$. Another one is

$$x_i^{k+1}=x_i^k-\frac{u_i^k}{1+\sum\limits_{j\neq i}\dfrac{u_j^k}{x_i^k-x_j^k}},\quad i=1,2,\cdots,n,\quad k=0,1,\cdots, \tag{4}$$

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which was derived by Börsch-Supan[1], later, by Nourein[5], and the convergence is cubic if $\xi_i \neq \xi_j$ for $i \neq j$.

In this paper we present a family of parallel iterations of order $m+2$ with parameter $m = 0, 1 \cdots$, which includes Weierstrass-Durand-Dochev-Kerner iteration (2) and Börsch-Supan-Nourein iteration (4) as the special cases for $m = 0$ and $m = 1$, respectively. Some numerical examples are given in section 4.

2. Construction of the Iterations

For purposes of brevity, all formulas, sums and products (such as in (2), (3) and (4) above) involving indices i, j and ν will assume the range $1, 2, \cdots, n$ and the iterative index $k = 0, 1, \cdots$, unless explicit stated otherwise. Naturally, we always regard $\sum_{l=\nu}^{\mu}(\cdots) = 0$ for $\mu < \nu$. Moreover, we simply write $x_i, u_i, \cdots$ for $x_i^k, u_i^k, \cdots$ and x_i^+ for x_i^{k+1}.

To construct the family of the iterations we first give the following

Proposition. *Let $x_1, x_2, \cdots, x_n \notin \{\xi_1, \xi_2, \cdots, \xi_n\}$ be distinct. Define*

$$u_j = \frac{f(x_j)}{\prod_{\nu \neq j}(x_j - x_\nu)}. \tag{5}$$

$$\begin{cases} \delta_i = & x_i - \xi_i \\ S_{il} = & \sum_{j \neq i} \frac{u_j}{(x_i - x_j)^l}, \ l = 1, 2, \cdots, \\ T_{im} = & \sum_{l=1}^{m} S_{il}\delta_i^{l-1}, \ m = 0, 1, \cdots, \\ R_{im} = & \delta_i^m \sum_{j \neq i} \frac{u_j}{(x_i - x_j)^m(\xi_i - x_j)}, \ m = 0, 1, \cdots. \end{cases} \tag{6}$$

Then for all $m = 0, 1, \cdots$ the fixed point relation

$$\xi_i = x_i - \delta_i = x_i - \frac{u_i}{1 + T_{im} + R_{im}}, \ m = 0, 1, \cdots \tag{7}$$

holds.

proof. Using Lagrange interpolation, we have

$$f(t) = (\sum \frac{u_j}{t - x_j} + 1) \prod (t - x_j). \tag{8}$$

Substituting $t = \xi_i \notin \{x_1, \cdots, x_n\}$ into (8) and observing $f(\xi_i) = 0$, we obtain

$$\frac{u_i}{\xi_i - x_i} + 1 + \sum_{j \neq i} \frac{u_j}{\xi_i - x_j} = 0, \tag{9}$$

$$\delta_i = x_i - \xi_i = \frac{u_i}{1 + \sum_{j \neq i} \frac{u_j}{\xi_i - x_j}}. \tag{10}$$

It means that the relation

$$\delta_i = \frac{u_i}{1 + T_{im} + R_{im}} \tag{11}$$

holds for $m = 0$. Clearly,

$$R_{im} = S_{i(m+1)}\delta_i^m + R_{i(m+1)},\ m = 0, 1, \cdots,$$

$$T_{im} + R_{im} = T_{i(m+1)} + R_{i(m+1)}.$$

So (11) is true for all $m = 0, 1, \cdots$ and (7) is proved. □

Suppose that $x_i (1 \le i \le n)$ are reasonably close approximations of the zeros ξ_i of $f(t)$. Ignoring the term R_{im} and replacing T_{im} by some approximation t_{im} in the fixed point equation (7), then for fixed m we can obtain from (6)–(7) the further approximation x_i^+ of the zero ξ_i. And we can construct the following

Algorithm.

$$\begin{cases} x_i^+ = & x_i - \delta_{im}, \\ \delta_{im} = & \dfrac{u_i}{1 + t_{im}},\ m = 0, 1, \cdots, \\ t_{im} = & \displaystyle\sum_{l=1}^{m} S_{il}\delta_{i(m-l)}^{l-1},\ m = 0, 1, \cdots, \\ S_{il} = & \displaystyle\sum_{j \ne i} \frac{u_j}{(x_i - x_j)^l},\ l = 1, 2, \cdots. \end{cases} \tag{12}$$

(12) constitute a family of parallel iterations with parameter $m = 0, 1, \cdots$ for finding all zeros of $f(t)$.

As the examples, let $m = 0, 1, 2, 3, 4$, the corresponding corrections are

$$\delta_{i0} = u_i,$$

$$\delta_{i1} = \frac{u_i}{1 + S_{i1}},$$

$$\delta_{i2} = \frac{u_i}{1 + S_{i1} + S_{i2}u_i},$$

$$\delta_{i3} = \frac{u_i}{1 + S_{i1} + \dfrac{S_{i2}}{1 + S_{i1}}u_i + S_{i3}u_i^2},$$

$$\delta_{i4} = \frac{u_i}{1 + S_{i1} + \dfrac{S_{i2}}{1 + S_{i1} + S_{i2}u_i}u_i + \dfrac{S_{i3}}{(1 + S_{i1})^2}u_i^2 + S_{i4}u_i^3},$$

respectively. Clearly, δ_{i0} and δ_{i1} are just Weierstrass-Durand-Dochev-Kerner correction (3) and Börsch-Supan-Nourein correction in (4), respectively.

3. The Convergence of the Iterations

About the convergence of the iterations constructed in previous section we give the following

Theorem. *Suppose that $f(t)$ is a monic complex polynomial of degree n with simple zeros $\xi_1, \cdots, \xi_n$ and $x_1, \cdots, x_n$ are distinct, reasonably close approximations of the zeros. Then the iterations* (12) *is convergent of order $m+2$.*

Proof. Denote

$$\epsilon = \max_{1 \le i \le n} |x_i - \xi_i|.$$

To complete the proof of Theorem it is enough to prove that

$$\delta_{im} - \delta_i = \xi_i - x_i^+ = O(\epsilon^{m+2}),\ m = 0, 1, \cdots. \tag{13}$$

It is easy to see from (5), (6) and (12) that

$$\begin{cases} u_i = & O(\epsilon),\ S_{il} = O(\epsilon),\ l = 1, 2, \cdots, \\ T_{im} = & O(\epsilon),\ t_{im} = O(\epsilon),\ \delta_{im} = O(\epsilon),\ m = 0, 1, \cdots, \\ R_{im} = & O(\epsilon^{m+1}),\ m = 0, 1, \cdots. \end{cases} \tag{14}$$

From (11) and (12) we see that

$$\delta_{im} - \delta_i = \delta_i\left(\frac{1 + T_{im} + R_{im}}{1 + t_{im}} - 1\right) = \frac{\delta_i}{1 + t_{im}}(T_{im} - t_{im} + R_{im}). \tag{15}$$

Observing (6), (12) and (14), we have

$$T_{i0} = t_{i0} = 0,\ T_{i1} = t_{i1} = S_{i1},$$

$$\delta_{i0} - \delta_i = O(\epsilon^2),\ \delta_{i1} - \delta_i = O(\epsilon^3).$$

Therefore (13) holds for $m = 0, 1$. If $\delta_{i\nu} - \delta_i = O(\epsilon^{\nu+2})$ for $\nu < m$, then by (6), (12) and (14) we obtain

$$\begin{aligned} T_{im} - t_{im} &= \sum_{l=2}^{m} S_{il}(\delta_i^{l-1} - \delta_{i(m-l)}^{l-1}) = \sum_{l=2}^{m} S_{il}(\delta_i - \delta_{i(m-l)}) \sum_{\nu=0}^{l-2} \delta_i^{\nu} \delta_{i(m-l)}^{l-2-\nu} \\ &= O(\epsilon^{1+(m-l+2)+(l-2)}) = O(\epsilon^{m+1}). \end{aligned} \tag{16}$$

Substituting (14) and (16) into (15), we conclude that (13) is also true for $m \ge 2$. The theorem is proved. □

4. Numerical Examples

In this section we give four numerical examples. In the following we denote a complex $a + \sqrt{-1}b$ by (a, b). And the iteration errors $\max_{1 \le i \le n} |x_i^k - \xi_i|$ are listed in the tables, where $\{x_i^k\}$ are produced by iterations (12) with some m. In Example 1 a trivial quadratic polynomial is given to verify the conclusion of Theorem. Example 2 is chosen from p.68 of [6]. Example 3 is a polynomial of degree 9 with coefficients $a_i = i,\ i = 1, \cdots, 9$. Moreover, Example 4 shows that the iterations are also available for the polynomials with multiple zeros. Of course, the convergence is linear in this case as the other iterations.

The initials for all examples are chosen such that they are equidistantly located on a circular with center $\frac{-a_1}{n} = \frac{1}{n}\sum \xi_j$ and all zeros are within it. The numerical experiment shows that the iterations are always convergent in this way even when the zeros are close together or multiple.

Example 1. Let $f_1(t) = t^2 - 3t + 2 = (t-1)(t-2)$ be the quadratic polynomial with zeros 1,2 and take the initials $(2.20711, .707107)$, $(.792893, -.707107)$.

Example 2. Let $f_2(t) = t^9 + 3t^8 - 3t^7 - 9t^6 + 3t^5 + 9t^4 + 99t^3 + 297t^2 - 100t - 300$ be the polynomial of degree 9 with zeros -3, ± 1, $\pm 2i$, $\pm 2 \pm i$. The iterations begin with the initials $(3.6059, .6946)$, $(2.2378, 3.0642)$, $(-.3333, 4.0000)$, $(-2.9045, 3.0642)$, $(-4.2726, .6946)$, $(-3.7974, -2.0000)$, $(-1.7014, -3.7588)$, $(1.0347, -3.7588)$, $(3.1308, -2.0000)$.

Example 3. Let $f_3(t) = t^9 + t^8 + 2t^7 + 3t^6 + 4t^5 + 5t^4 + 6t^3 + 7t^2 + 8t + 9$ be the polynomial of degree 9 with coefficients $a_i = i, i = 1, \cdots, 9$. Its zeros are $(.9719, .8546)$, $(.4385, 1.2796)$, $(-.3326, 1.2244)$, $(-.9708, .7485)$, $(-1.2141, .0000)$, $(-.9708, -.7485)$, $(-.3326, -1.2244)$, $(.4385, -1.2796)$, $(.9719, -.8546)$. The initials are given by $(2.8433, .5209)$, $(1.8173, 2.2981)$, $(-.1111, 3.0000)$, $(-2.0395, 2.2981)$, $(-3.0655, .5209)$, $(-2.7092, -1.5000)$, $(-1.1372, -2.8191)$, $(.9149, -2.8191)$, $(2.4870, -1.5000)$.

Example 4. Let $f_4(t) = t^4 - 4t^3 + 6t^2 - 4t + 1 = (t-1)^4$ be the polynomial of degree 4 with quadruple zero 1 and take the initials $(1.9239, .3827)$, $(.6173, .9239)$, $(.0761, -.3827)$, $(1.3827, -.9239)$.

Table 1. Iteration Errors for Example 1

k/m	0	1	2	3	4	5	6
0	.74	.74	.74	.74	.74	.74	.74
1	.27	.13	$.89E-01$	$.62E-01$	$.44E-01$	$.32E-01$	$.24E-01$
2	$.71E-01$	$.31E-02$	$.21E-03$	$.60E-05$	$.87E-07$	$.69E-09$	$.32E-11$
3	$.60E-02$	$.30E-07$	$.35E-14$	$.23E-24$			
4	$.35E-04$						
5	$.12E-08$						

Table 2. Iteration Errors for Example 2

k/m	0	1	2
0	3.8	3.8	3.8
1	3.2	2.9	2.7
2	2.7	2.1	1.8
3	2.4	1.3	1.2
4	2.0	.74	.22
5	1.6	$.54E-01$	$.27E-03$
6	1.1	$.24E-04$	$.44E-15$
7	.58	$.68E-15$	
8	.16		
9	$.12E-01$		
10	$.91E-04$		
11	$.45E-08$		
12	$.22E-15$		

Table 3. Iteration Errors for Example 3

k/m	0	1	2
0	1.6	1.6	1.6
1	1.3	1.1	.98
2	1.0	.67	.54
3	.82	.37	.25
4	.63	.20	.19
5	.47	$.83E-01$	$.38E-02$
6	.33	$.17E-02$	$.12E-08$
7	.24	$.17E-07$	$.17E-15$
8	.22	$.16E-15$	
9	.58		
10	.25		
11	$.63E-01$		
12	$.32E-02$		
13	$.11E-04$		
14	$.15E-09$		
15	$.17E-15$		

Table 4.1 Iteration Errors for Example 4

k/m	0	1	2
0	1.00000	1.00000	1.00000
1	.750000	.600000	.542857
2	.562500	.360000	.294694
3	.421875	.216000	.159977
4	.316406	.129600	$.868445E{-}01$
5	.237305	$.777600E{-}01$	$.471441E{-}01$
6	.177979	$.466560E{-}01$	$.255925E{-}01$
7	.133484	$.279936E{-}01$	$.138931E{-}01$
8	.100113	$.167962E{-}01$	$.754196E{-}02$
9	$.750847E{-}01$	$.100777E{-}01$	$.409421E{-}02$
10	$.563135E{-}01$	$.604662E{-}02$	$.222257E{-}02$
11	$.422351E{-}01$	$.362797E{-}02$	$.120654E{-}02$
12	$.316764E{-}01$	$.217678E{-}02$	$.654993E{-}03$
13	$.237573E{-}01$	$.130607E{-}02$	
14	$.178179E{-}01$	$.783655E{-}03$	
15	$.133635E{-}01$		
16	$.100226E{-}01$		
17	$.751695E{-}02$		

Table 4.2 Iteration Errors for Example 4

k/m	3	4	5
0	1.00000	1.00000	1.00000
1	.490040	.452833	.421561
2	.240139	.205058	.177714
3	.117678	$.928568E{-}01$	$.749174E{-}01$
4	$.576668E{-}01$	$.420486E{-}01$	$.315823E{-}01$
5	$.282590E{-}01$	$.190410E{-}01$	$.133139E{-}01$
6	$.138480E{-}01$	$.862238E{-}02$	$.561261E{-}02$
7	$.678609E{-}02$	$.390450E{-}02$	$.236606E{-}02$
8	$.332546E{-}02$	$.176808E{-}02$	$.997444E{-}03$
9	$.162961E{-}02$	$.800649E{-}03$	
10	$.798579E{-}03$		

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