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ON THE CENTRAL RELAXING SCHEMES I: SINGLE CONSERVATION LAWS[*1)]

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Abstract

In this first paper we present a central relaxing scheme for scalar conservation laws, based on using the local relaxation approximation. Our scheme is obtained without using linear or nonlinear Riemann solvers. A cell entropy inequality is studied for the semidiscrete central relaxing scheme, and a second order MUSCL scheme is shown to be TVD in the zero relaxation limit. The next paper will extend the central relaxing scheme to multi–dimensional systems of conservation laws in curvilinear coordinates, including numerical experiments for 1D and 2D problems.

Key words: Hyperbolic conservation laws, the relaxing scheme, TVD, cell entropy inequality.

1. Introduction

In [6], Jin and Xin constructed a class of upwind relaxing schemes for nonlinear conservation laws

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{d}\frac{\partial f_i(u)}{\partial x_i}=0, \tag{1.1}$$

with initial data $u(0,x)=u_0(x),\ x=(x_1,...,x_d)$, by using the idea of the local relaxation approximation [2,3,6,10].

The relaxing scheme is obtained in the following way: A linear hyperbolic system with a stiff source term is first constructed to approximate the original equation (1.1) with a small dissipative correction. Then this linear hyperbolic system is solved easily by underresolved stable numerical discretizations. The main advantage of their schemes is to use neither nonlinear Riemann solvers spatially nor nonlinear system of algebraic equations solvers temporally. However, the numerical experiments have shown that

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implementation of the upwind relaxing schemes for general hyperbolic system seems to be inconvenient, because of using linear Riemann solvers of a linear hyperbolic system with a stiff source term spatially.

To overcome this drawback, we will construct a central relaxing scheme for systems of conservation laws in this series without using linear or nonlinear Riemann solvers. The schemes are shown to be TVD(total variation diminishing) and be of the similar relaxed form as in [6] in the zero relaxation limit for scalar case; a cell entropy inequality for semidiscrete schemes is also proved. Numerical experiments for 1D and 2D problems are presented in [14], which show that resolution of the central relaxing schemes is comparable to the upwind relaxing schemes presented in [6].

The paper is organized as follows. In section 2, we simply recall the relaxing system with a stiff source term, constructed by Jin and Xin to approximate Eq.(1.1). Section 3 is to construct a class of central difference approximations for the relaxing system. The schemes are also shown to have correct asymtotic limit as $\epsilon \to 0^+$, and be TVD(total variation diminishing [4]) in the zero relaxation limit in section 4. In section 5, we discuss the numerical entropy condition for the semidiscrete central relaxing scheme based on Osher–Tadmor numerical entropy flux. We conclude the paper with a few remarks in section 6.

2. Preliminaries

In this section we simply review the relaxing system with a stiff source, introduced Jin and Xin in [6] to approximate Eq.(1.1). For the sake of simplicity in the presentation, we will focus on the 1D single scalar conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.1}$$

with initial data

$$u(0,x) = u_0(x). \tag{2.2}$$

A linear system with a stiff source term (hereafter called the *relaxing system*) can be introduced as follows:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \\ &\frac{\partial v}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{1}{\epsilon}(v - f(u)), \end{aligned} \tag{2.3}$$

where the small positive parameter ϵ is the relaxation rate, and a is a positive constant satisfying

$$\mid f'(u) \mid \leq \sqrt{a}, \text{ for all } u \in \mathcal{R}. \tag{2.4}$$

Remark: (1) Here we can also use the more general $a(x,t)$ instead of the above constant a. The similar results can also be analyzed. (2) For scalar conservation laws in arbitrary space dimensions, a relaxing system with a stiff source term can also introduced as above, we refer the reader to [6,14] for more details.

In the small relaxation limit $\epsilon \to 0^+$, the relaxing system (2.3) can be approximated to leading order by the following *relaxed* equations

$$v = f(u), \tag{2.5a}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.5b}$$

The state satisfying (2.5a) is called the *local equilibrium*. Using the Chapman-Enskog expansion [1], we can derive the following first order approximation to Eq.(2.3)

$$v = f(u) - \epsilon\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}, \tag{2.6a}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon\frac{\partial}{\partial x}(\{a - [f'(u)]^2\}\frac{\partial u}{\partial x}), \tag{2.6b}$$

It is clear that the above second equation (2.6b) is dissipative under condition (2.4) (which is referred to as the *subcharacteristic condition* by Liu in [10]). Here, we will choose the special initial condition for the relaxing system (2.3) as follows:

$$\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x) \equiv f(u_0(x)). \end{aligned} \tag{2.7}$$

The aim is to avoid the initial layer introduced by the relaxing system (2.3). In doing so the state is already in equilibrium initially. On the other hand, to avoid any new boundary layers in solving boundary value problems, we can also impose the boundary conditions for v to be consistent to the local equilibrium.

The relaxation limit for systems of conservation laws with a stiff source term was first studied by Liu in [10]. Convergence of solutions of the general relaxing systems are considered later in [2,3]. In this paper we are concerned with construction of a class of the central relaxing approximations for conservation laws based on the relaxing system (2.3).

3. The Central Relaxing Schemes for Conservation Laws

Based on the relaxing system (2.3) for conservation laws, we can consider construction of the relaxing schemes for conservation laws. Moreover, we will restrict our attention to one–dimensional scalar case in the following.

Introduce the spatial grid points x_j, $j \in \mathcal{Z}$ with the uniform mesh width $\Delta x = x_{j+1} - x_j$, i.e. Δx is a constant, and denote by $w_j(t)$ the approximate point value of $w(x,t)$ at $x = x_j$. The discrete time level are spaced uniformly with the step

$\Delta t = t^{n+1} - t^n$ for $n \in \mathcal{Z}^+ \cup \{0\}$. In the following, $\lambda = \frac{\Delta t}{\Delta x}$ is assumed a constant. The relaxing schemes are obtained by discreting the system (2.3), for which it is convenient to treat the spatial and time discretization separately .

I. The spatial discretizations

A spatial discretization to Eq.(2.3) in conservation form can be written as

$$\frac{\partial}{\partial t} u_j + \frac{1}{\Delta x}(v_{j+1/2} - v_{j-1/2}) = 0, \tag{3.1a}$$

$$\frac{\partial}{\partial t} v_j + \frac{a}{\Delta x}(u_{j+1/2} - u_{j-1/2}) = -\frac{1}{\epsilon}(v_j - f(u_j)), \tag{3.1b}$$

where the numerical flux $u_{j+1/2}$ and $v_{j+1/2}$ will be defined in two ways specified below.

Algorithm I: (First order central scheme) A 1st order numerical flux in a central form is defined as:

$$\begin{aligned} v_{j+1/2} &= \frac{1}{2}(v_{j+1} + v_j) - \frac{1}{2\lambda}(u_{j+1} - u_j), \\ u_{j+1/2} &= \frac{1}{2}(u_{j+1} + u_j) - \frac{1}{2a\lambda}(v_{j+1} - v_j). \end{aligned} \tag{3.2}$$

Algorithm II: (Second order MUSCL scheme) A 2nd order numerical flux can be defined as:

$$\begin{aligned} v_{j+1/2} &= \frac{1}{2}(v^R + v^L) - \frac{1}{2\lambda}(u^R - u^L), \\ u_{j+1/2} &= \frac{1}{2}(u^R + u^L) - \frac{1}{2a\lambda}(v^R - v^L), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} v^L &= v_j + \frac{1}{2}\phi(r_j)\Delta v_{j+1/2}, \quad & v^R &= v_{j+1} - \frac{1}{2}\phi(\frac{1}{r_{j+1}})\Delta v_{j+1/2}, \\ u^L &= u_j + \frac{1}{2}\phi(s_j)\Delta u_{j+1/2}, \quad & u^R &= u_{j+1} - \frac{1}{2}\phi(\frac{1}{s_{j+1}})\Delta u_{j+1/2}, \\ r_j &= \frac{v_j - v_{j-1}}{v_{j+1} - v_j}, & s_j &= \frac{u_j - u_{j-1}}{u_{j+1} - u_j}, \end{aligned} \tag{3.4}$$

and $\phi(r)$ is some symmetric limiters [12].

Remark: (1) One simple choice of limiters is the so-called minmod limiters

$$\phi(r) = \max(0, \min(1, r)).$$

A sharper limiter was introduced by van Leer[15] as

$$\phi(r) = (\mid r \mid + r)/(1 + \mid r \mid).$$

(2) Because the current spatial discretizations are using the Lax-Friedrichs type central difference, linear or nonlinear Riemann solvers are not used. (3) The schemes (3.1)–(3.4) can be extended easily to several space variables.

II. The time discretizations

Numerical schemes for stiff relaxing systems such as Eq.(2.3) were studied in [7]. Proper implicit time discretizations should be taken to overcome the stability constraints brought by the stiff source. A simple way is to keep the convection terms explicit and the stiff source terms implicit. Since the source terms in Eq.(2.3) is linear in the variable v, we can avoid to solve nonlinear systems of algebraic equation. As in [6,7], a general second order Runge-Kutta splitting scheme to Eq.(3.1) can be given

$$\begin{aligned}
&\bar{u}_j = u_j^n, && \bar{v}_j = v_j^n - \frac{\Delta t}{\epsilon}(\bar{v}_j - f(\bar{u}_j)),\\
&u_j^{(1)} = \bar{u}_j - \lambda\Delta_+\bar{v}_{j-1/2}, && v_j^{(1)} = \bar{v}_j - a\lambda\Delta_+\bar{u}_{j-1/2},\\
&\bar{\bar{u}}_j = u_j^{(1)}, && \bar{\bar{v}}_j = v_j^{(1)} + \alpha\frac{\Delta t}{\epsilon}(\bar{\bar{v}}_j - f(\bar{\bar{u}}_j)) + \beta\frac{\Delta t}{\epsilon}(\bar{v}_j - f(\bar{u}_j)),\\
&u_j^{(2)} = \bar{\bar{u}}_j - \lambda\Delta_+\bar{\bar{v}}_{j-1/2}, && v_j^{(2)} = \bar{\bar{v}}_j - a\lambda\Delta_+\bar{\bar{u}}_{j-1/2},\\
&u_j^{n+1} = \frac{1}{2}(u_j^n + u_j^{(2)}), && v_j^{n+1} = \frac{1}{2}(v_j^n + v_j^{(2)}).
\end{aligned} \tag{3.5}$$

where two parameters α and β should satisfy the consistancy condition: $\alpha + \beta = -1$. For example, $\alpha = +1,\ \beta = -2$.

4. The TVD Properties of the Relaxed Schemes

It is known that not all stable schemes for the relaxing system (2.3) yield the correct solutions when the small relaxation rate is not well resolved [6,7]. Thus we expect a good numerical scheme should possess the correct relaxation limit in the sense that the zero relaxation limit ($\epsilon \to 0^+$) be a consistent and stable discretization of Eq.(2.1). As was demonstrated in [7], assume $0 < \epsilon,\ \frac{\epsilon}{\Delta x},\ \frac{\epsilon}{\Delta t} << 1$. If $v(x,0) = f(u(x,0))$, then we have

$$\bar{v} = f(\bar{u}) + O(\epsilon),\ \bar{\bar{v}} = f(\bar{\bar{u}}) + O(\epsilon), \tag{4.1}$$

which imply that the solutions are local equilibria at the two intermediate time step in Eq.(3.5). Applying (4.1) in second order Runge-Kutta splitting scheme (3.5), one arrives at the *relaxed schemes* after ignoring the $O(\epsilon)$ terms

$$\begin{aligned}
u_j^{(1)} &= u_j - \lambda\Delta_+ v_{j-1/2}\ |_{v^n = f(u^n)},\\
u_j^{(2)} &= u^{(1)} - \lambda\Delta_+ v_{j-1/2}^{(1)}\ |_{v^{(1)} = f(u^{(1)})},\\
u_j^{n+1} &= \frac{1}{2}(u_j^n + u_j^{(2)}),
\end{aligned} \tag{4.2}$$

which is the second order TVD-type Runge-Kutta time discretization for Eq.(2.1) or (2.5) (See [11]).

The relaxed spatial discretization can also be done easily. For example, the relaxed

1st order central scheme is

$$v_{j+1/2}\mid_{v=f(u)}=\frac{1}{2}(f(u_j)+f(u_{j+1}))-\frac{1}{2\lambda}(u_{j+1}-u_j), \tag{4.3}$$

which is numerical flux of the Lax-Friedrichs type scheme for Eq.(2.1).

Similary, the relaxed MUSCL-type central scheme can be obtained

$$\begin{aligned} v_{j+1/2}\mid_{v=f(u)}=&\frac{1}{2}[f(u_j)+f(u_{j+1})+\frac{1}{2}(\phi(r_j)-\phi(\frac{1}{r_{j+1}}))(f(u_{j+1})-f(u_j))]\\ &-\frac{1}{2\lambda}[u_{j+1}-u_j-\frac{1}{2}(\phi(\frac{1}{s_{j+1}})+\phi(s_j))(u_{j+1}-u_j)]. \end{aligned} \tag{4.4}$$

Clearly this is a second order consistent spatial discretization to (2.1).

In conclusion, the relaxing schemes we discussed here have the correct relaxation limit. Moreover, we have:

Theorem 1. *The relaxed first order central scheme* ((3.1a), (4.2), *and* (4.3)) *is a monotone scheme for Eq.*(2.1) *under the CFL condition*

$$\lambda \sup_{j\in\mathcal{Z}} \mid f'(u)\mid \le 1. \tag{4.5}$$

The above conclusion are obvious. We omit its proof.

Theorem 2. *The relaxed second order MUSCL scheme ((3.1a),(4.2),and (4.4)) is a TVD scheme for Eq.(2.1) under conditions*

$$0\le \phi(r), \phi(\frac{1}{r})\le 2, \tag{4.6a}$$

for all symmetric limiter ϕ, and the CFL condition

$$\lambda \sup_{j\in\mathcal{Z}} \mid f'(u)\mid \le \min\left\{\frac{2-\phi(\frac{1}{s_{j+1}})}{2-\phi(\frac{1}{r_{j+1}})+\phi(r_j)}, \frac{2-\phi(s_j)}{2+\phi(\frac{1}{r_{j+1}})-\phi(r_j)}\right\}. \tag{4.6b}$$

Proof. Due to the special structures of second order TVD-type Runge-Kutta time discretization (4.2), we can only consider the simple one-step conservative scheme for the relaxing system (2.3) with uniform grids

$$\begin{aligned} &u_j^{n+1}-u_j^n+\lambda(v_{j+1/2}^n-v_{j-1/2}^n)=0,\\ &v_j^{n+1}-v_j^n+a\lambda(u_{j+1/2}^n-u_{j-1/2}^n)=-\frac{\Delta t}{\epsilon}(v_j^{n+1}-f(u_j^{n+1})), \end{aligned} \tag{4.7}$$

The relaxed MUSCL scheme takes the following form (see Eq.(4.4))

$$\begin{aligned} &v_j=f(u_j),\\ &u_j^{n+1}-u_j^n+\lambda(\bar f_{j+1/2}-\bar f_{j-1/2})=0, \end{aligned} \tag{4.8}$$

with

$$
\begin{aligned}
\bar{f}_{j+1/2} = \frac{1}{2}\left[f(u_j) + f(u_{j+1}) + \frac{1}{2}(\phi(r_j) - \phi(\frac{1}{r_{j+1}}))(f(u_{j+1}) - f(u_j))\right] \\
-\frac{1}{2\lambda}\left[u_{j+1} - u_j - \frac{1}{2}(\phi(\frac{1}{s_{j+1}}) + \phi(s_j))(u_{j+1} - u_j)\right].
\end{aligned}
\tag{4.9}
$$

for all symmetric limiter ϕ.

One can rewrite the relaxed MUSCL scheme (4.8) in the incremental form

$$
u_j^{n+1} = u_j^n + C_{j+1/2}^n \Delta_{j+1/2} u - D_{j-1/2}^n \Delta_{j-1/2} u,
$$

where

$$
\begin{aligned}
C_{j+1/2}^n &= \frac{1}{4}\left[\Big(2 - \phi(\frac{1}{s_{j+1}})\Big) - \Big(2 - \phi(\frac{1}{r_{j+1}}) + \phi(r_j)\Big)\lambda \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j}\right], \\
D_{j+1/2}^n &= \frac{1}{4}\left[\Big(2 - \phi(s_j)\Big) + \Big(2 + \phi(\frac{1}{r_{j+1}}) - \phi(r_j)\Big)\lambda \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j}\right].
\end{aligned}
\tag{4.10}
$$

Under the conditions (4.6), the coefficients $C_{j+1/2}^n$ and $D_{j+1/2}^n$ in (4.10) satisfy

$$
\begin{aligned}
&C_{j+1/2}^n \geq 0,\ \ D_{j+1/2}^n \geq 0, \\
&C_{j+1/2}^n + D_{j+1/2}^n \leq 1.
\end{aligned}
\tag{4.11}
$$

thus the relaxed MUSCL-type central scheme is TVD.

Remark: Condition (4.6a) can guarantee scheme to be of second order accuracy and very high resolution, because it includes the critical point $\phi(1) = 1$ [12,16].

5. The Cell Entropy Inequality for the Semidiscrete Central Relaxing Schemes

In this section we discuss numerical entropy condition for the semidiscrete first order central relaxing scheme and second order MUSCL-type relaxing scheme. In the following we will always assume $(\eta,\ q)$ to represent the entropy pairs for models (2.3)[2,3], where the convex function $\eta(u,v) \in C^2(\mathcal{R}^2)$, then they must satisfy the consistency condition:

$$
(\eta_u, \eta_v)\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} = (q_u, q_v),
\tag{5.1}
$$

where η_u represents the partial differentiation with respect to u. Furthermore, we have

$$
\begin{aligned}
\eta_{uu} - a\eta_{vv} = 0, \\
q_{uu} - a q_{vv} = 0.
\end{aligned}
\tag{5.2}
$$

Thus the general representation of the entropy pairs for (2.3) is

$$\begin{aligned} \eta &= G(v+\sqrt{a}u) + H(v-\sqrt{a}u), \\ q &= \sqrt{a}(G(v+\sqrt{a}u) - H(v-\sqrt{a}u)), \end{aligned} \tag{5.3}$$

for any functions G and H in $C^2(\mathcal{R})$.

It is easy to verify that η is a convex function if and only if $H''G'' \geq 0$. At the equilibrium state $v = f(u)$, we have

$$\begin{aligned} \eta \mid_{v=f(u)} &= G(f(u)+\sqrt{a}u) + H(f(u)-\sqrt{a}u) \equiv \bar{\eta}(u), \\ q \mid_{v=f(u)} &= \sqrt{a}(G(f(u)+\sqrt{a}u) - H(f(u)+\sqrt{a}u)) \equiv \bar{q}(u), \end{aligned} \tag{5.4}$$

and expect to have

$$\eta_v \mid_{v=f(u)} = 0. \tag{5.5}$$

Thus the pair $(\bar{\eta}(u), \bar{q}(u))$ forms an entropy pair for scalar conservation laws (2.1), that is

$$\bar{\eta}'(u) f'(u) = \bar{q}'(u), \ \text{ if } H'' \geq 0 \text{ and } G'' \geq 0,$$

At the equilibrium state $v = f(u)$, we expect to have the following entropy condition for equation (2.1)

$$\frac{\partial \bar{\eta}}{\partial t} + \frac{\partial \bar{q}}{\partial x} \leq 0. \tag{5.6}$$

and the numerical entropy condition (5.14) for the numerical method for solving scalar conservation laws (2.1).

For the sake of simplicity in the presentation, define $w^+ = v + \sqrt{a}u$ and $w^- = v - \sqrt{a}u$, which imply $v = \frac{1}{2}(w^+ + w^-)$ and $u = \frac{1}{2\sqrt{a}}(w^+ - w^-)$.

For the relaxing schemes (3.1), the following numerical entropy inequality

$$\frac{\partial \eta}{\partial t} + \frac{1}{\Delta x}(Q_{j+1/2} - Q_{j-1/2}) + \frac{\eta_v}{\epsilon}(v_j - f(u_j)) \leq 0, \tag{5.7}$$

is needed to guarantee convergence of numerical solution to the entropy solution, where $Q_{j+1/2}$ is some numerical entropy flux to be consistent with the entropy flux q.

Let $Q_{j+1/2}$ be Osher-Tadmor type numerical entropy flux

$$Q_{j+1/2} = q_{j+1} + \eta_u \mid_{j+1} (v_{j+1/2} - v_{j+1}) + a\eta_v \mid_{j+1} (u_{j+1/2} - u_{j+1}). \tag{5.8}$$

and multipy both side of equation (4.1) by $(\eta_u, \eta_v)_j$, then we have

$$\frac{\partial \eta}{\partial t} + \frac{1}{\Delta x}\eta_u \mid_j (v_{j+1/2} - v_{j-1/2}) + \frac{a}{\Delta x}\eta_v \mid_j (u_{j+1/2} - u_{j-1/2}) = -\frac{\eta_v}{\epsilon}(v_j - f(u_j)). \tag{5.9}$$

Thus we have

$$\begin{aligned} LHS = \frac{1}{\Delta x}\{&[q_{j+1} + \eta_u \mid_{j+1} (v_{j+1/2} - v_{j+1}) + a\eta_v \mid_{j+1} (u_{j+1/2} - u_{j+1})] \\ &-[q_j + \eta_u \mid_j (v_{j+1/2} - v_j) + a\eta_v \mid_j (u_{j+1/2} - u_j)]\}, \end{aligned} \tag{5.10}$$

where LHS denote the left hand side of inequality (5.7).

Substitute (5.3) in (5.10), then

$$LHS = \frac{\sqrt{a}}{\Delta x}\left\{\int_{w_j^-}^{w_{j+1}^-}(s - w_{j+1/2}^-)H''(s)ds + \int_{w_j^+}^{w_{j+1}^+}(w_{j+1/2}^+ - t)G''(t)dt\right\}. \tag{5.11}$$

Therefore, to guarantee the numerical entropy inequality (5.7) to be satisfied, we have

Theorem 3. *A sufficient condition for the inequality (5.7) with numerical entropy flux given in Eq.(5.8) to be satisfied is, for all $j \in \mathcal{Z}$,*

$$\begin{aligned} &sign(w_{j+1}^+ - w_j^+)(w_{j+1/2}^+ - t) \le 0, \text{ for every } t \text{ between } w_{j+1}^+ \text{ and } w_j^+, \\ &sign(w_{j+1}^- - w_j^-)(s - w_{j+1/2}^-) \le 0, \text{ for every } s \text{ between } w_{j+1}^- \text{ and } w_j^-, \end{aligned} \tag{5.12}$$

if H and G are two convex functions.

Theorem 4. *Assume the subcharacteristic condition (2.4) and CFL-type condition $\lambda\sqrt{a} \le 1$ are satisfied . For the first order central scheme (3.1)–(3.2), the entropy inequality (5.7) with numerical entropy flux given in Eq.(5.8) is valid, if H and G are two convex function.*

Proof. From Eqs.(3.1)–(3.2), $(w_{j+1/2}^+ - t)$ and $(s - w_{j+1/2}^-)$ can be rewritten in the form

$$\begin{aligned} &(-\frac{1}{2\lambda\sqrt{a}} + \frac{1}{2})(w_{j+1}^+ - t) + (\frac{1}{2\lambda\sqrt{a}} + \frac{1}{2})(w_j^+ - t), \\ &(\frac{1}{2\lambda\sqrt{a}} + \frac{1}{2})(s - w_{j+1}^-) + (-\frac{1}{2\lambda\sqrt{a}} + \frac{1}{2})(s - w_j^-). \end{aligned}$$

It is easy to verify that inequalities are satisfied under the above assumption.

Theorem 5. *Assume that the hypotheses of* **Theorem 4** *are satisfied . For the second order MUSCL-type scheme (3.1),(3.3),and* (3.4′) *the entropy inequality (5.7) numerical entropy flux defined in Eq.(5.8) is valid, if the following conditions hold,*

$$\begin{aligned} 0 \le \phi(r), \phi(\frac{1}{r}) \le 1, \\ \left[2 \pm \phi(\frac{1}{r}) \mp \phi(r)\right] \lambda\sqrt{a} \le 2 - \phi(\frac{1}{r}) - \phi(r), \end{aligned} \tag{5.13}$$

for all symmetric limiter ϕ.

Proof. From Eq.(3.4), we have

$$w^+_{j+1/2} = \frac{1}{2}(v^R + v^L) - \frac{1}{2\lambda}(u^R - u^L) + \frac{\sqrt{a}}{2}(u^R + u^L) - \frac{1}{2\lambda\sqrt{a}}(v^R - v^L),$$
$$w^-_{j+1/2} = \frac{1}{2}(v^R + v^L) - \frac{1}{2\lambda}(u^R - u^L) - \frac{\sqrt{a}}{2}(u^R + u^L) + \frac{1}{2\lambda\sqrt{a}}(v^R - v^L),$$

then, for any t between w^+_{j+1} and w^+_j, for any s between w^-_{j+1} and w^-_j

$$\begin{aligned}(w^+_{j+1/2} - t) &= \left(-\frac{1}{2\lambda\sqrt{a}} + \frac{1}{2}\right)(w^{+,R} - t) + \left(\frac{1}{2\lambda\sqrt{a}} + \frac{1}{2}\right)(w^{+,L} - t)\\
&= \frac{1}{4\lambda\sqrt{a}}\left[(\lambda\sqrt{a} - 1)(2 - \phi(\frac{1}{s})) + \phi(s)(\lambda\sqrt{a} + 1)\right](w^+_{j+1} - t)\\
&\quad + \frac{1}{4\lambda\sqrt{a}}\left[\phi(\frac{1}{s})(\lambda\sqrt{a} - 1) + (2 - \phi(s))(\lambda\sqrt{a} + 1)\right](w^+_j - t),\\
(s - w^-_{j+1/2}) &= \left(\frac{1}{2\lambda\sqrt{a}} + \frac{1}{2}\right)(s - w^{-,R}) + \left(-\frac{1}{2\lambda\sqrt{a}} + \frac{1}{2}\right)(s - w^{-,L})\\
&= \frac{1}{4\lambda\sqrt{a}}\left[(\lambda\sqrt{a} + 1)(2 - \phi(\frac{1}{s})) + \phi(s)(\lambda\sqrt{a} - 1)\right](s - w^-_{j+1})\\
&\quad + \frac{1}{4\lambda\sqrt{a}}\left[\phi(\frac{1}{s})(\lambda\sqrt{a} + 1) + (2 - \phi(s))(\lambda\sqrt{a} - 1)\right](s - w^-_j).\end{aligned}$$

where without ambiguity we have omitted the subscript j in ϕ. It is not difficult to verify that numerical entropy inequalities (5.7) are satisfied with Osher-Tadmor type numerical entropy flux, under the condition (5.13). This completes the proof.

Remark: At the equilibrium state $v = f(u)$, the entropy inequlity (5.7) becomes

$$\frac{\partial\bar{\eta}}{\partial t} + \frac{1}{\Delta x}(\bar{Q}_{j+1/2} - \bar{Q}_{j-1/2}) \le 0, \tag{5.14}$$

where $\bar{\eta}$ and $\bar{q}$ defind in (5.4), and

$$\bar{Q}_{j+1/2} \equiv Q_{j+1/2}\,|_{v=f(u)}\ .$$

Inequality (5.14) forms a cell entropy inequality for the relaxed scheme

$$\begin{aligned}&v_j = f(u_j),\\ &\frac{\partial}{\partial t}u_j + \frac{1}{\Delta x}(v_{j+1/2} - v_{j-1/2}) = 0.\end{aligned} \tag{5.15}$$

6. Conclusions

In this first paper we have presented a central relaxing scheme for scalar conservation laws, based on using the local relaxation approximation. Our scheme is obtained without using linear or nonlinear Riemann solvers. The schemes are shown to be TVD and be of the similar relaxed form as in [6] in the zero relaxation limit. For scalar equations, a cell entropy inequality for semidiscrete central relaxing schemes is also studied.

The implementation of the current schemes will presented in [14]. Moreover, the theoretical studies, such as the entropy condition for the fully discrete relaxing schemes and the convergence of the relaxing schemes , need to be considered in future.

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