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LAYER STRIPPING METHOD FOR POTENTIAL INVERSION[*1)]

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Abstract

To solve the potential inversion problem of the coupled system for one-way wave equations, the absorbing boundary conditions in the lateral direction are derived. The difference schemes are constructed and a layer stripping method is proposed. Some numerical experiments are presented.

Key words: 2-D potential inversion, Layer stripping method, Absorbing boundary condition

1. Introduction

The potential inversion problem of the following Plasma wave equation is discussed in this paper:

$$\Big[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} + v(x,z)\Big]p(x,z,t) = 0, \ \ x \in R, z > 0, t > 0, \tag{1.1}$$

$$p(x,z,0) = \frac{\partial p}{\partial t}(x,z,0) = 0, \tag{1.2}$$

$$p(x,0,t) = \delta(t), \tag{1.3}$$

$$\frac{\partial}{\partial z}p(x,0,t) = h(x,t). \tag{1.4}$$

That is, giving an impulse at the surface $z = 0$, to determine the wavefield p and potential v from the impulse response h.

There are three kinds of inverse problems of this Plasma wave equation:

(1) To determine the differential equation from its spectral function[1];

(2) To determine the potential from the wave function form at large distance. It is the so-called inverse scattering problem[2,3];

(3) To determine the potential from the response on the boundary to a unit impulse at some reference time $t = 0$[4,5].

Our problem belongs to the third kind.

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In the one dimensional case, there are comprehensive results for this problem. But for the multi-dimensional case, there is not any satisfactory result, whether theoretical or numerical. Because in this case the problem is non-linear and ill-possed.

We have done some theoretical analysis about this potential inversion problem and split the original full-way wave equation into the system of one-way wave equations by using the wave splitting method based on the theory of pseudo-differential operator[6]. In order to make the problem closed, we also transformed the impulse condition (1.3) into the conditions on the characteristic surface by singularity analysis. We also proved the stability of the direct problem for the system of equations, treated as Cauchy problems in the direction of depth.

As the results of wave splitting and singularity analysis the potential inversion problem of the one-way wave equations is [6]:

$$\Big(\frac{\partial}{\partial t}-\frac{\partial}{\partial z}\Big)U_1-\frac{\partial^2}{\partial t^2}\Big[\sum_{m=1}^{n}a_m q_U(s_m)\Big]+\frac{vp}{2}=0, \tag{1.5}$$

$$\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\Big)D_1-\frac{\partial^2}{\partial t^2}\Big[\sum_{m=1}^{n}a_m q_D(s_m)\Big]+\frac{vp}{2}=0, \tag{1.6}$$

$$\frac{\partial p}{\partial z}=U_1-D_1, \tag{1.7}$$

$$U_1=\frac{\partial U}{\partial t}, \tag{1.8}$$

$$D_1=\frac{\partial D}{\partial t}, \tag{1.9}$$

$$\Big(\frac{\partial^2}{\partial t^2}-s_m^2\frac{\partial^2}{\partial x^2}\Big)q_U(s_m,x,z,t)=\frac{\partial^2}{\partial x^2}U(x,z,t), \tag{1.10}$$

$$\Big(\frac{\partial^2}{\partial t^2}-s_m^2\frac{\partial^2}{\partial x^2}\Big)q_D(s_m,x,z,t)=\frac{\partial^2}{\partial x^2}D(x,z,t). \tag{1.11}$$

The initial conditions on the surface $z=0$ are

$$U_1(x,0,t)=-D_1(x,0,t)=\frac{1}{2}h(x,t), \tag{1.12}$$

$$p(x,0,t)=0. \tag{1.13}$$

The conditions for sufficient large value T are:

$$U(x,z,T)=U_1(x,z,T)=q_U(x,z,T)=\frac{\partial q_U}{\partial t}(x,z,T)=0. \tag{1.14}$$

As the results of singularity analysis we get the conditions on the characteristic surface $t=z+0$

$$D(x,z,t=z+0)=-\int_0^z\frac{v(x,y)}{2}dy=g(x,z), \tag{1.15}$$

$$D_1(x,z,t=z+0)=\frac{1}{2}\int_0^z\Big[\frac{\partial^2}{\partial x^2}-v(x,y)\Big]g(x,y)dy-\frac{1}{2}h(x,0), \tag{1.16}$$

$$q_D(x,z,t=z+0)=\frac{\partial q_D}{\partial t}(x,z,t=z+0)=0, \tag{1.17}$$

$$v(x,z) = -4U_1(x,z,t=z+0). \tag{1.18}$$

where

$$a_m = \frac{1}{n+1}\sin^2\Big(\frac{m\pi}{n+1}\Big), \quad s_m = \cos\Big(\frac{m\pi}{n+1}\Big). \tag{1.19}$$

If taking different values of n, we can get different orders of approximate one-way wave equations. For example, if taking $n = 1$ we get the so-called 15° approximate equations. And if taking $n = 2$ we get the so-called 45° approximate equations.

It is important to point out that the order of all above equations is no more than two for all values of n. So it is very simple to discrete the equations and to do some theoretical analysis for the corresponding difference equations. On the other hand, the forms of all equations are the same for all values of n, so it is possible for us to handle them in a uniform manner. All of these features are valuable for practical computations.

In this paper we construct the absorbing boundary conditions and presented a layer stripping numerical method to solve the potential inversion problem. Some numerical experiments performed by this method are illustrated.

2. Absorbing Boundary Conditions

One of the persistent problems in the numerical simulation of wave phenomena is the artificial reflections that are introduced by the boundaries of the computational domain. This problem is common to all calculations where artificial boundaries are introduced to limit the computational size. To remove spurious reflections in a direct and efficient manner, one would like to specify boundary conditions which absorb energy incident on the boundaries. According to the above request, we construct the so-called absorbing boundary conditions.

The construction of the absorbing boundary conditions should obey the following rule: On the left boundary there is no reflection for the left-going wave, therefore the right-going wave is zero. And it is the same on the right boundary. But this rule is difficult to be satisfied strictly, we should only obey it as well as possible.

Now we introduce the construction of absorbing boundary conditions for the inverse problem.

The original full-way wave equation is:

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}\Big)p + vp = 0.$$

If we delete the term of vp which is negligible near the boundaries of computational domain, the dispersion relation is: $\omega^2 - k_x^2 - k_z^2 = 0$, that is $k_z = \pm\omega\sqrt{1-\big(\frac{k_x}{\omega}\big)^2}$. It represents the upcoming wave if taking the sign as "+". And it represents the downgoing wave if taking the sign as "−".

Firstly we consider the upcoming wave. The upcoming wave will be full transparent without reflection if the following dispersion relation for the upcoming wave is satisfied on the boundaries:

$$k_z = \omega\sqrt{1-\Big(\frac{k_x}{\omega}\Big)^2}.$$

But there is a term of k_x^2 corresponding to the two-order differential of x in this equation, the discretization of which on the boundaries will use not only the wavefield inside the boundaries but also the wavefield outside the boundaries. It is impossible in practical computation. So we had only to get the boundary conditions approximately, that is, we should express the square-root under the request that there are only one-order terms of k_x/ω. To do this we use the Pade weighted approximation[8], that is, we use the following rational fraction

$$\frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x}{b_0 + b_1 x}$$

to approximate the square-root $\sqrt{1-x^2}$, where a_i, b_i are constant values. Now the dispersion of the upcoming wave can be written as:

$$k_z \approx \omega \frac{a_0 + a_1 \left|\frac{k_x}{\omega}\right|}{b_0 + b_1 \left|\frac{k_x}{\omega}\right|} = \omega \frac{a_0 \pm a_1 \frac{k_x}{\omega}}{b_0 \pm b_1 \frac{k_x}{\omega}}$$

If $\frac{k_x}{\omega} > 0$, the sign is taken as "+", it represents the left-going wave;
If $\frac{k_x}{\omega} < 0$, the sign is taken as "−", it represents the right-going wave.

If we write the above relation into differential equations, we get the absorbing boundary conditions for the upcoming wave:

$$a_0 \frac{\partial^2 U}{\partial t^2} + a_1 \frac{\partial^2 U}{\partial t \partial x} - b_0 \frac{\partial^2 U}{\partial t \partial z} - b_1 \frac{\partial^2 U}{\partial x \partial z} = 0, \quad x = x_{\min}, \tag{2.1}$$

$$a_0 \frac{\partial^2 U}{\partial t^2} - a_1 \frac{\partial^2 U}{\partial t \partial x} - b_0 \frac{\partial^2 U}{\partial t \partial z} + b_1 \frac{\partial^2 U}{\partial x \partial z} = 0, \quad x = x_{\max}. \tag{2.2}$$

We can get the absorbing boundary conditions for the downgoing wave in the same manner.

$$a_0 \frac{\partial^2 D}{\partial t^2} - a_1 \frac{\partial^2 D}{\partial t \partial x} + b_0 \frac{\partial^2 D}{\partial t \partial z} - b_1 \frac{\partial^2 D}{\partial x \partial z} = 0, \quad x = x_{\min}, \tag{2.3}$$

$$a_0 \frac{\partial^2 D}{\partial t^2} + a_1 \frac{\partial^2 D}{\partial t \partial x} + b_0 \frac{\partial^2 D}{\partial t \partial z} + b_1 \frac{\partial^2 D}{\partial x \partial z} = 0, \quad x = x_{\max}. \tag{2.4}$$

3. Difference Schemes

Difference schemes for integration U_1, D_1, U, D and p are constructed by the trapezoidal formula:

$$\frac{U_{1ij+1}^{k+1} - U_{1ij}^{k}}{\Delta t} + \frac{v_{ij-\frac{1}{2}} p_{ij-\frac{1}{2}}^{k+\frac{1}{2}}}{2} - \frac{\Delta_x^2}{\Delta x^2} \Big[\frac{U_{ij-1}^{k+1} + U_{ij}^{k}}{4} + \sum_{m=1}^{n} a_m s_m^2 q_U(s_m)_{ij-\frac{1}{2}}^{k+\frac{1}{2}} \Big] = 0, \tag{3.1}$$

$$\frac{D_{1ij}^k - D_{1ij-1}^{k-1}}{\Delta t} + \frac{v_{ij-\frac{1}{2}} p_{ij-\frac{1}{2}}^{k-\frac{1}{2}}}{2} - \frac{\Delta_x^2}{\Delta x^2}\Big[\frac{D_{ij}^k + D_{ij-1}^{k-1}}{4} + \sum_{m=1}^{n} a_m s_m^2 q_D(s_m)_{ij-\frac{1}{2}}^{k-\frac{1}{2}}\Big] = 0, \tag{3.2}$$

$$\frac{p_{ij-\frac{1}{2}}^{k+\frac{1}{2}} - p_{ij-\frac{3}{2}}^{k+\frac{1}{2}}}{\Delta z} = \frac{1}{2}[(U_{1ij-1}^k + U_{1ij-1}^{k+1}) - (D_{1ij-1}^k + D_{1ij-1}^{k+1})] \tag{3.3}$$

$$\frac{U_{ij}^{k+1} - U_{ij}^k}{\Delta t} = \frac{1}{2}(U_{1ij}^{k+1} + U_{1ij}^k) \tag{3.4}$$

$$\frac{D_{ij}^k - D_{ij}^{k-1}}{\Delta t} = \frac{1}{2}(D_{1ij}^k + D_{1ij}^{k-1}) \tag{3.5}$$

where F_{ij}^k is the approximate value of $F(i\Delta x, j\Delta z, k\Delta t)$, $\Delta_x^2 = F_{i+1} - 2F_i + F_{i-1}$, $\Delta t = \Delta z$.

Difference schemes for integrating q_U and q_D are explicit, commonly used for solving 1-D wave equations.

$$\begin{aligned} & q_U(s_m)_{ij-\frac{1}{2}}^{k+\frac{1}{2}} - 2q_U(s_m)_{ij-\frac{1}{2}}^{k+\frac{3}{2}} + q_U(s_m)_{ij-\frac{1}{2}}^{k+\frac{5}{2}} - s_m^2 \frac{\Delta t^2}{\Delta x^2}\Delta_x^2 q_U(s_m)_{ij-\frac{1}{2}}^{k+\frac{3}{2}} \\ & \qquad = \frac{\Delta t^2}{4\Delta x^2}\Delta_x^2(U_{ij-1}^{k+1} + U_{ij-1}^{k+2} + U_{ij}^{k+1} + U_{ij}^{k+2}), \end{aligned} \tag{3.6}$$

$$\begin{aligned} & q_D(s_m)_{ij-\frac{1}{2}}^{k-\frac{1}{2}} - 2q_D(s_m)_{ij-\frac{1}{2}}^{k-\frac{3}{2}} + q_D(s_m)_{ij-\frac{1}{2}}^{k-\frac{5}{2}} - s_m^2 \frac{\Delta t^2}{\Delta x^2}\Delta_x^2 q_D(s_m)_{ij-\frac{1}{2}}^{k-\frac{3}{2}} \\ & \qquad = \frac{\Delta t^2}{4\Delta x^2}\Delta_x^2(D_{ij-1}^{k-1} + D_{ij-1}^{k-2} + D_{ij}^{k-1} + D_{ij}^{k-2}). \end{aligned} \tag{3.7}$$

These difference schemes are similar to that constructed for the one-way wave equations[9], its stability condition for the direct initial value problem is $s_m\Delta t/\Delta x < 1$.

The truncation error of these difference schemes is $O(\Delta x^2 + \Delta z^2 + \Delta t^2)$.

Now we construct the difference schemes for absorbing boundary conditions. For convenience, we take the upcoming wave U_1 at the left boundary as an example. The condition (2.1) can be written as

$$a_0\frac{\partial U_1}{\partial t} + a_1\frac{\partial U_1}{\partial x} - b_0\frac{\partial U_1}{\partial z} - b_1\frac{\partial^2 U}{\partial x\partial z} = 0,$$

the difference scheme is

$$\begin{aligned} & \frac{a_0}{2\Delta t}(U_{10j}^{k+1} + U_{10j-1}^{k+1} - U_{10j}^k - U_{10j-1}^k) - \frac{b_0}{2\Delta z}(U_{10j}^k + U_{10j}^{k+1} - U_{10j-1}^k - U_{10j-1}^{k+1}) \\ & \quad + \frac{a_1}{4\Delta x}(U_{11j-1}^k + U_{11j}^k + U_{11j-1}^{k+1} + U_{11j}^{k+1} - U_{10j-1}^k - U_{10j}^k - U_{10j-1}^{k+1} - U_{10j}^{k+1}) \\ & \quad - \frac{b_1}{2\Delta x\Delta z}(U_{1j}^k - U_{0j}^k + U_{1j}^{k+1} - U_{0j}^{k+1} - U_{1j-1}^k + U_{0j-1}^k - U_{1j-1}^{k+1} + U_{0j-1}^{k+1}) = 0. \end{aligned}$$

The potential inversion problem formulated in terms of coupled system is numerically solved by finite difference method constructed above in layer stripping fashion.

The numerical procedure for solving the potential inversion problem is as follows:

1. Suppose we have $U^k_{1ij-1}, D^k_{1ij-1}, U^k_{ij-1}, D^k_{ij-1}$ and $p^{k+\frac{1}{2}}_{ij-\frac{3}{2}}$ for all i and k.

2. From (3.3) we can obtain $p^{k+\frac{1}{2}}_{ij-\frac{1}{2}}$. For $j = 1$ we use the first order approximation

$$\frac{p^{k+\frac{1}{2}}_{i\frac{1}{2}} - p^{k+\frac{1}{2}}_{i0}}{\Delta z/2} = \frac{1}{2}[(U^k_{1i0} + U^{k+1}_{1i0}) - (D^k_{1i0} + D^{k+1}_{1i0})]$$

3. In order to obtain U^k_{1ij}, U^k_{ij} from (3.1), (3.4) and D^k_{1ij}, D^k_{ij} from (3.2), (3.5), we use the predictor-corrector method in which one iteration is needed. $v_{ij-\frac{1}{2}}$ is taken to be $-4U^{j-1}_{1ij-1}$ in accordance with condition (1.18) in the predictor step, and $-2(U^{j-1}_{1ij-1} + \widetilde{U}^j_{1ij})$ in corrector step where $\widetilde{U}^j_{1ij}$ is the value of U^j_{1ij} obtained in the predictor step.

4. D^k_{1ij} and $q_D(s_m)^{k+\frac{1}{2}}_{ij-\frac{1}{2}}$ are obtained by solving (3.2) and (3.5) successively from $k = j$ to $k = J = T/\Delta t$. U^k_{1ij} and $q_U(s_m)^{k-\frac{1}{2}}_{ij-\frac{1}{2}}$ are obtained by solving (3.1) and (3.4) successively from $k = J - 1$ to $k = j$.

5. The potential inversion problem is numerically solved layer by layer for $j = 1, 2, \cdots$.

4. Numerical Experiments

We give some numerical results below. It should be noted that the following is only a representative and illustrative sample of our results. The impulse response $h(x,t)$ needed in the inverse problem are generated by solving numerically the forward problem (1.1–1.3) with the known potential $v(x,z)$. It is worthwhile to point out that the computation in the inversion is not the reverse of that in the direct problem, because the equations involved in the direct and in the inverse problems are quite different.

All examples take $n = 1$ in (1.19) and use $\Delta x = 0.1, \Delta t = \Delta z = 1/30$. The 3-D plots depict 2-D function $v(x,z)$. The grid number are 30×30. In each Figure, (a) is the model of potential function, (b) is the numerical results of reconstruction.

In Fig. 1, $v(x,z) = \cos \pi x \cos 5\pi z$. The absorbing boundary conditions are imposed on the x boundaries.

In Figs. 2 and 3, $v(x,z)$ is a mountain like function.

In Fig. 4, $v(x,z)$ is also a mountain like function, but it is discontinuous in the direction of x.

From the numerical experiments we draw the following conclusions:

1. The proposed method is numerically stable and the numerical results show that it can reproduce the potentials with satisfactory accuracy.

2. The absorbing boundary conditions in x direction are effective. That is, when the waves propagate to the boundaries there isn't notable reflections. The waves transmit along the previous directions with the previous velocities, there is no notable reflections.

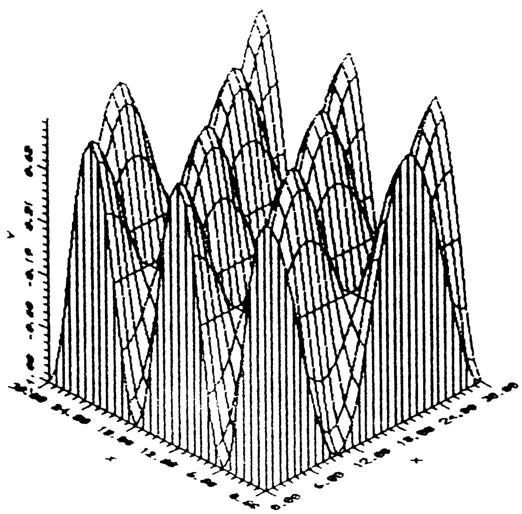
Fig. 1(a)

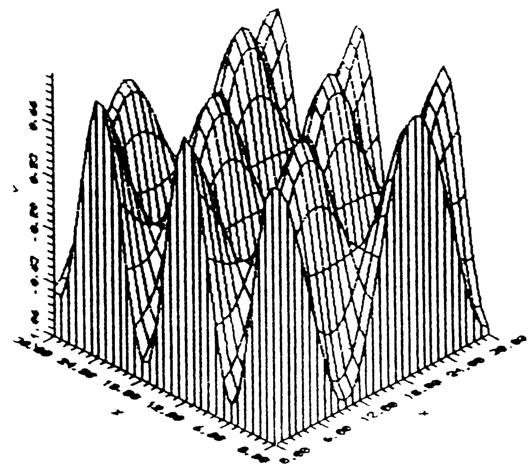
Fig. 1(b)

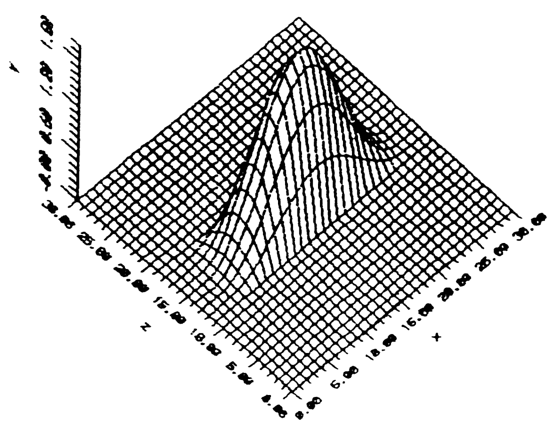
Fig. 2(a)

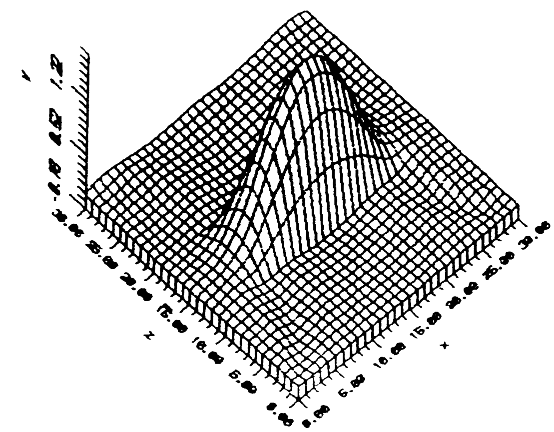
Fig. 2(b)

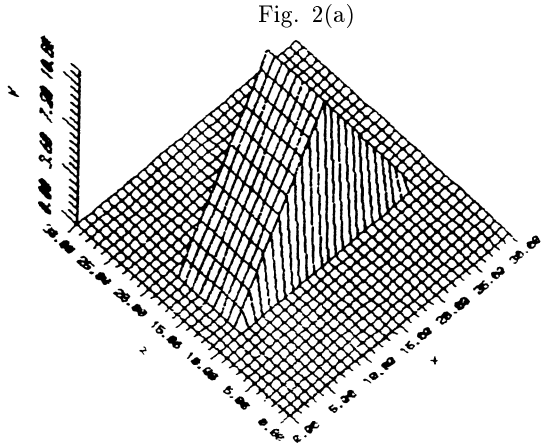
Fig. 3(a)

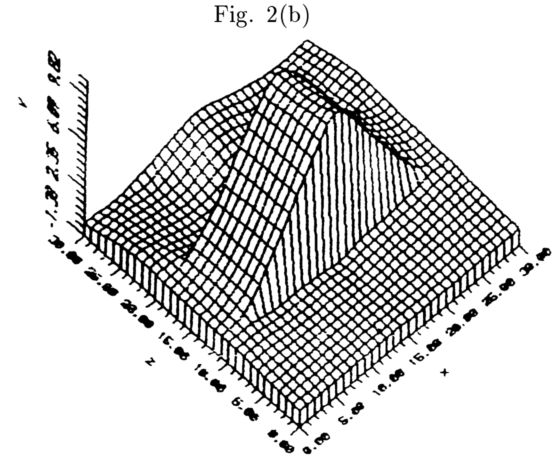
Fig. 3(b)

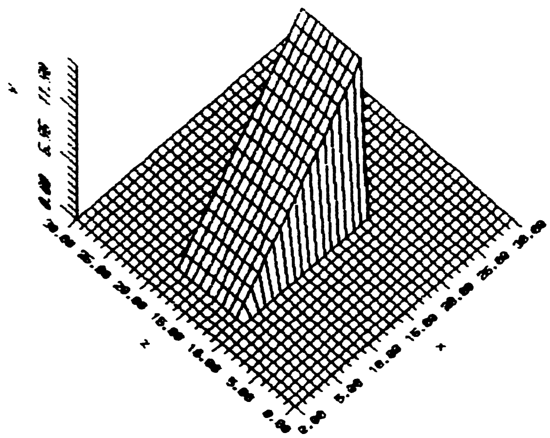

Fig. 4(a)

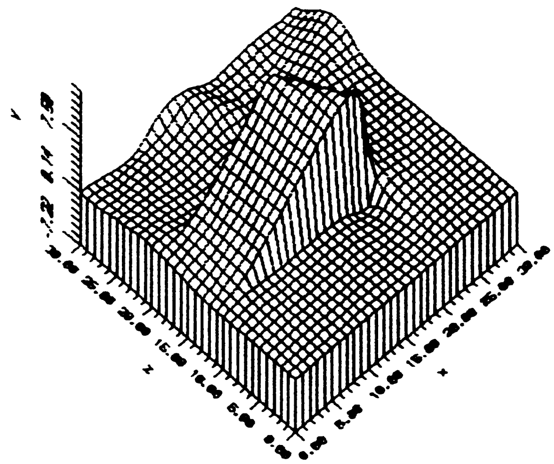

Fig. 4(b)

3. Almost for all inverted potentials the errors between the jumps of the function values are very small. The reason for this is that although we do approximate splitting to the wave equation, the conditions on the characteristic surface are precise, which are obtained by singularity analysis.

4. The reconstructed potentials are well when the lateral variation is slow. The reasons for this are as follows. On one side, the discretization of the equations is very crude because the two order derivative of x is simply replaced by its two order difference quotient. On the other side, we use the 15° approximate equations which requires relatively little lateral change.

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