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THE FULL DISCRETE DISCONTINUOUS FINITE ELEMENT ANALYSIS FOR FIRST–ORDER LINEAR HYPERBOLIC EQUATION*1)

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Abstract

In this paper, the full discrete discontinuous Galerkin finite element method to solove 2–dimensional first–order linear hyperbolic problem is considered. Two practical schemes, Euler scheme and Crank–Nicolson scheme, are constructed. For each of them, the stability and error estimation with optimal order approximation is established in the norm stronger than L^2–norm.

Key words: Hyperbolic equation, Discontinous F.E.M., Euler scheme

1. Introduction

Let $\bar{\Omega}$ be a bounded domain in $\mathbf{R}^2$ with piecewise smooth boundary $\partial\Omega$, $[0,T]$ be a time interval. Consider the first-order hyperbolic problem as following

$$\frac{\partial u}{\partial t} + \beta(x,t)\cdot\nabla u + \sigma(x,t)u = f(x,t), \quad t \in (0,T], x \in \tilde{\Omega}(t), \tag{1.0a}$$

$$u(x,t) = g(x,t), \quad t \in [0,T], x \in \partial\Omega_-(t), \tag{1.0b}$$

$$u(x,t) = u_0(x), \quad x \in \Omega. \tag{1.0c}$$

where $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$, $\beta(x,t) = (\beta_1(x,t), \beta_2(x,t))$, $\partial\Omega_-(t) = \{x \in \partial\Omega : \beta(x,t)\cdot\gamma < 0\}$, $\gamma(x)$ is the outward unit normal to $\partial\Omega$; $\tilde{\Omega}(t) = \bar{\Omega}\backslash\partial\Omega_-(t)$. As usual, $\partial\Omega_-(t)$ is refered to as inflow boundary at time t, and $\partial\Omega_+(t) = \partial\Omega\backslash\partial\Omega_-(t)$ is called outflow boundary at time t.

For simplicity in finite element analysis, suppose that boundary $\partial\Omega_-(t)$ is independent of t. Thus for all $t \in (0,T]$ we can write

$$\partial\Omega_-(t) \equiv \Gamma_-, \partial\Omega_+(t) \equiv \Gamma_+, \tilde{\Omega}(t) = \bar{\Omega}\backslash\Gamma_- \equiv \Omega^\star$$

and problem (1.0) can be written as

$$\frac{\partial u}{\partial t} + \beta(x,t)\cdot\nabla u + \sigma(x,t)u = f(x,t), \quad (x,t) \in \Omega^\star \times (0,T], \tag{1.1a}$$

$$u(x,t) = g(x,t), \quad (x,t) \in \Gamma_- \times [0,T], \tag{1.1b}$$

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$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.1c}$$

We shall consider the full discrete discontinuous Galerkin method for problem (1.1).

Set $D = \Omega \times (0,T]$, $L^p(0,T;X) \equiv L^p(X)$, $p = 2,+\infty$, where X is a Banach space. Assume that $\beta_i \in L^\infty(C^1(\bar{\Omega}))$, $i = 1,2$; $\sigma \in L^\infty(L^\infty(\Omega))$, $f \in L^2(L^2(\Omega))$, $g \in L^2(L^2(\Gamma_-))$; $u_0 \in L^2(\Omega)$.

Discontinuous Galerkin (DG) method is an explict method with good stability and satisfactory accuracy, thus it has become to be an effictive procedure to solve first-order hyperbolic problems. DG method was proposed by P. Lesaint and P.R. Raviart in 1978 ([1]), then it was developed by C. Johnson, G.R. Richart et al.[2–4]. In principle, we can use the DG method based on space-time finite element discretization for domain $\bar{\Omega}\times[0,T]$ to solve Problem (1.1), but in this case, we must solve a series of discretization problems defined on 3–dimensional subdomain $\bar{\Omega}\times[t_{n-1},t_n]$, $n = 1,2,\cdots$; As compared with full discrete Galerkin method, the computational scale of DG method is larger and the computing program is more complex.

In order to overcome the weakness of DG method, we now present a simplified DG method for time-dependent Problem (1.1), full discrete discontinuous Galerkin (FDDG) method, that is, using DG discretization only in space variables and using finite difference discretization in time variable t.

One can imagine that FDDG scheme possesses similar stability and convergence resultes with the DG scheme (based on space-time finite element). In fact, the theoritical analysis for FDDG scheme is more complex than that of DG scheme because of the non-uniform processing in time and space variables. It seems to us so far that there has been no paper to establish complete analysis for FDDG scheme of Problem (1.1).

In section 2 two practical FDDG schemes, Euler scheme and Crank–Nicolson (C—N) scheme, are constructed; In section 3 the stability and error estimate for Euler scheme are derived; In section 4 the theoretical results for Crank–Nicolson scheme are given briefly; Finally, a numerical example is given in sectiom 5.

Throughout context, we shall use letters C, C_i, ε, ε_i to denote some positive constants independent of time-step $\triangle t$ and finite element mesh parameter h, which have different values in different inequalities.

2. Full Discrete Discontinuous Galerkin Schemes

For convenience, let $\bar{\Omega}$ be a polygonal domain, $\mathcal{T}_h = \{k\}$ is a quasi–uniform triangular partition of $\bar{\Omega}$ with mesh parameter $h(0 < h \le h_0 < 1)$, k is an element in $\mathcal{T}_h$. Let $\triangle t = \tau$ be time–step, $t^n = n\tau$, $n = 0,1,\cdots,N = [T/\triangle t]$. Suppose that on all time levels $t = t^n (n = 0,1,\cdots,N)$, the same finite element mesh $\mathcal{T}_h$ for space domain $\bar{\Omega}$ is adopted. Denote

$$V_h = \{v \in L^2(\Omega) : v|_k \in P_r(k), \forall k \in \mathcal{T}_h\}, \tag{2.1}$$

where $P_r(k)$ is a set of polynomials with degree $\le r$ on k.

I. **Euler FDDG Scheme**

Set $\beta^n(x) = \beta(x,t^n)$. For $\forall k \in \mathcal{T}_h$, let ∂k be the boundary of k which consist of straight line sides l_j $(j = 1,2,3)$ and $\gamma(x)$ be the outward unit vector normal to ∂k.

Define for $\forall k \in \mathcal{T}_h$ on time level $t = t^n$,

$$\begin{aligned}
&\bar{\beta}_j^n = \frac{1}{|l_j|}\int_{l_j} \beta^n(x)\,ds, \quad j = 1,2,3, (|l_j| \text{ is length of } l_j),\\
&\bar{\beta}^n(x) = \bar{\beta}_j^n, \text{ for } x \in l_j, j = 1,2,3,\\
&\partial k_-^n = \{x \in \partial k, \bar{\beta}^n(x)\cdot\gamma(x) < 0\}, \partial k_+^n = \partial k \backslash \partial k_-^n.
\end{aligned} \tag{2.2}$$

∂k_-^n and ∂k_+^n are called inflow and outflow boundary of element k Respectively.

Obviously, if $l_j \subset \Gamma_-$ then $l_j \subset \partial k_-^n$ from the definition (2.2).

Note that $v|_{\partial k}$ may be discontinuous for $v \in V_h$. Define for $v, w \in V_h$ and $x \in \partial k$ on $t = t^n$,

$$\begin{aligned}
&v_+^n(x) = \lim_{s\to 0^+} v(x + s\bar{\beta}^n(x)), \quad v_-^n(x) = \lim_{s\to 0^-} v(x + s\bar{\beta}^n(x)),\\
&[v^n(x)] = v_+^n(x) - v_-^n(x),\\
&\langle v, w\rangle_{\partial k_-^n} = \int_{\partial k_-^n} vw|\bar{\beta}^n\cdot\gamma|\,ds, |v|^2_{\partial k_-^n} = \langle v, v\rangle_{\partial k_-^n},\\
&\langle v, w\rangle_{\Gamma_-^n} = \sum_{\partial k_-^n \subset \Gamma_-} \langle v, w\rangle_{\partial k_-^n}, |v|^2_{\Gamma_-^n} = \langle v, v\rangle_{\Gamma_-^n}.
\end{aligned} \tag{2.3}$$

Likewise, $\langle v, w\rangle_{\partial k_+^n}$, $|v|_{\partial k_+^n}$, $\langle v, w\rangle_{\Gamma_+^n}$, $|v|_{\Gamma_+^n}$ can be defined.

And also, denote

$$\begin{aligned}
&(v, w)_k = \int_k vw\,dx, ||v||_k^2 = (v, v)_k,\\
&(v, w) = \int_\Omega vw\,dx, ||v||^2 = (v, v),\\
&(v, w)_{H^1(\Omega)} = \sum_{k\in\mathcal{T}_h}(v, w)_{H^1(k)}, ||v||_1^2 = \sum_{k\in\mathcal{T}_h}(v, v)_{H^1(k)}.
\end{aligned}$$

Denote $q^n(x) = q(x, t^n)$ and $\triangle_t q^n = \frac{q^n - q^{n-1}}{\tau}$. Problem (1.1) on time $t = t^n$ can be written as

$$\triangle_t u^n + \beta^n\cdot\nabla u^n + \sigma^n u^n = f^n + E_1^n, \quad n = 1, 2, \cdots, N, \tag{2.4a}$$

$$u_-^n|_{\Gamma_-} = g^n, \tag{2.4b}$$

$$u^0 = u_0, x \in \Omega. \tag{2.4c}$$

where E_1^n is truncation error

$$E_1^n = \triangle_t u^n - \Big(\frac{\partial u}{\partial t}\Big)^n. \tag{2.5}$$

Omitting E_1^n from (2.4a) and consulting the definition of DG scheme[2], the Euler FDDG scheme of Problem (1.1) is defined as: Find $U^n \in V_h$, $n = 0, 1, \cdots, N$ such that, for each $k \in \mathcal{T}_h$,

$$(\triangle_t U^n + \beta^n\cdot\nabla U^n + \sigma^n U^n, v)_k + \langle [U^n], v_+\rangle_{\partial k_-^n} = (f^n, v)_k, \quad \forall v \in P_r(k), \tag{2.6a}$$

$$U_-^n|_{\partial k_-^n} = g^n, \text{ on } \partial k_-^n \subset \Gamma_-, \tag{2.6b}$$

$$(U^0 - u_0, v)_k = 0, \quad \forall v \in P_r(k). \tag{2.6c}$$

Initial-value function $U^0 \in V_h$ is determined by (2.6c). We can use (2.6a) and (2.6b) to compute U^n element by element starting from those elements where $\partial k_-^n \subset \Gamma_-$, when U^{n-1} has been solved.

Summing (2.6) for $k \in \mathcal{T}_h$, we have

$$(\triangle_t U^n + \beta^n \cdot \nabla U^n + \sigma^n U^n, v) + \sum_{k\in\mathcal{T}_h} \langle [U^n], v_+\rangle_{\partial k_-^n} = (f^n, v), \quad \forall v \in V_h, \tag{2.7a}$$

$$U_-^n|_{\Gamma_-} = g^n, \tag{2.7b}$$

$$(U^0 - u_0, v) = 0, \quad \forall v \in V_h. \tag{2.7c}$$

It is easy to see that Promble (2.6) is equivalent to Problem (2.7).

II. **Crank–Nicolson FDDG Scheme**

Set $t_n = (t^{n-1} + t^n)/2$, $q_n(x) = q(x, t_n)$, $\tilde{q}^n(x) = (q^{n-1} + q^n)/2$. Then on level $t = t_n$, (1.1a) can be written as

$$\triangle_t u^n + \beta_n \cdot \nabla \tilde{u}^n + \sigma_n \tilde{u}^n = f_n + E_2^n, \quad n = 1, 2 \cdots, N, \tag{2.8}$$

where the truncation error

$$E_2^n = \triangle_t u^n - \left(\frac{\partial u}{\partial t}\right)_n - \beta_n \cdot \nabla(u_n - \tilde{u}_n) - \sigma_n(u_n - \tilde{u}^n). \tag{2.9}$$

As the definition (2.2) introduced in Euler scheme (2.6), define for $\forall k \in \mathcal{T}_h$ with boundary $l_j (j = 1, 2, 3)$ on $t = t_n$,

$$\begin{aligned} &\bar{\beta}_n^{(j)} = \frac{1}{|l_j|}\int_{l_j} \beta_n(x)\, ds, \quad j = 1, 2, 3, \\ &\bar{\beta}_n(x) = \bar{\beta}_n^{(j)}, \quad \forall x \in l_j, \ j = 1, 2, 3, \\ &\partial k_-^n = \{x \in \partial k; \bar{\beta}_n(x) \cdot \gamma(x) < 0\}, \partial k_+^n = \partial k \backslash \partial k_-^n. \end{aligned} \tag{2.10}$$

And also, define for $v \in V_h$ and $x \in \partial k$ on $t = t^n$,

$$v_\pm^n(x) = \lim_{s\to 0^\pm} v(x + s\bar{\beta}_n(x)), [v^n(x)] = v_+^n - v_-^n.$$

Omitting E_2^n from (2.8), the Crank–Nicolson FDDG scheme of Promble (1.1) is defined as: Find $U^n \in V_h$ $(n = 0, 1, \cdots, N)$ such that, for $\forall k \in \mathcal{T}_h$,

$$(\triangle_t U^n + \beta_n \cdot \nabla \tilde{U}^n + \sigma_n \tilde{U}^n, v)_k + \langle [\tilde{U}^n], v_+\rangle_{\partial k_-^n} = (f_n, v)_k, \quad \forall v \in P_r(k), \tag{2.11a}$$

$$U_-^n|_{\partial k_-^n} = g^n, \text{ on } \partial k_-^n \subset \Gamma_-, \tag{2.11b}$$

$$(U^0 - u_0, v)_k = 0, \quad \forall v \in P_r(k). \tag{2.11c}$$

3. The Analysis for Euler FDDG Scheme

For simplicity in notations, set $\sum \triangleq \sum\limits_{k\in\mathcal{T}_h}$ and $\bigcup \triangleq \bigcup\limits_{k\in\mathcal{T}_h}$. On level $t=t^n$, set $Q_-^n=\bigcup\partial k_-^n$, $Q_+^n=\bigcup\partial k_+^n$ and denote

$$\begin{aligned}&\langle v,w\rangle_{Q_-^n}=\sum\langle v,w\rangle_{\partial k_-^n},\ \langle v,w\rangle_{Q_+^n}=\sum\langle v,w\rangle_{\partial k_+^n},\\&B(w^n,v;w^{n-1})\triangleq\sum(\triangle_t w^n+\beta^n\cdot\nabla w^n+\sigma^n w^n,v)_k+\langle[w^n],v_+\rangle_{Q_-^n}.\end{aligned}\tag{3.1}$$

3.1 Stability

Lemma 3.1. *There exist constants $C^\star$ and $C^{\star\star}>0$ independent of k,h,n such that, for $\forall v\in P_r(k)$,*

$$||v||_{L^2(\partial k)}\le C^\star h^{-\frac{1}{2}}||v||_k,\quad \forall k\in\mathcal{T}_h,\tag{3.2}$$

$$\Big|\int_{\partial k}v^2(\beta^n-\bar\beta^n)\cdot\gamma\,ds\Big|\le C^{\star\star}||v||_k^2,\quad\forall k\in\mathcal{T}_h.\tag{3.3}$$

Proof. Estimate (3.2) can be derived from the quasi–uniformity of $\mathcal{T}_h$ and the inverse estimation $||v||_{L^\infty(k)}\le M_0h^{-1}||v||_k$ for $P_r(k)$. The inequality (3.3) follows from (3.2) and the fact $||\beta^n-\bar\beta^n||_{L^\infty(\partial k)}\le M_1h||\beta||_{L^\infty(C^1(\bar\Omega))}$.

Lemma 3.2. *There exists constant $C_0>0$ independent of τ,h,n such that, for $\forall w^n$, $w^{n-1}\in V_h$ and $\forall w_-^n|_{\Gamma_-}\in L^2(\Gamma_-)$,*

$$\begin{aligned}B(w^n,w^n;w^{n-1})&+C_0||w^n||^2+\frac{1}{2}|w_-^n|^2_{\Gamma_-^n}\\&\ge\frac{1}{2}\left[\triangle_t||w^n||^2+\tau||\triangle_tw^n||^2+|[w^n]|^2_{Q_-^n}+|w_-^n|^2_{\Gamma_+^n}\right]\end{aligned}\tag{3.4}$$

where $\triangle_t||w^n||^2\triangleq(||w^n||^2-||w^{n-1}||^2)/\tau$.

Proof. By definition (3.1),

$$B(w^n,w^n;w^{n-1})\triangleq\sum(\triangle_tw^n+\beta^n\cdot\nabla w^n+\sigma^nw^n,w^n)_k+\langle[w^n],w_+^n\rangle_{Q_-^n}.\tag{3.5}$$

It's easy to see that

($\star$1) $(\triangle_tw^n,w^n)=\frac{1}{2}(\triangle_t||w^n||^2+\tau||\triangle_tw^n||^2)$,

($\star$2) $(\beta^n\cdot\nabla w^n+\sigma^nw^n,w^n)_k=((\sigma^n-\frac{1}{2}\mathrm{div}\,\beta^n)w^n,w^n)_k+\frac{1}{2}\int_{\partial k}(w^n)^2\beta^n\cdot\gamma\,ds$,

($\star$3) $\int_{\partial k}(w^n)^2\beta^n\cdot\gamma\,ds=\int_{\partial k}(w^n)^2\bar\beta^n\cdot\gamma\,ds+\int_{\partial k}(w^n)^2(\beta^n-\bar\beta^n)\cdot\gamma\,ds$,

($\star$4) $\int_{\partial k}(w^n)^2\bar\beta^n\cdot\gamma\,ds=\int_{\partial k_+^n}(w_-^n)^2\bar\beta^n\cdot\gamma\,ds-\int_{\partial k_-^n}(w_+^n)^2|\bar\beta^n\cdot\gamma|\,ds$.

Substituting ($\star$1)—($\star$4) into (3.5) we have

($\star$5)
$$\begin{aligned}B(w^n,w^n;w^{n-1})\ge&\ \frac{1}{2}(\triangle_t||w^n||^2+\tau||\triangle_tw^n||^2)-||\sigma-\frac{1}{2}\mathrm{div}\,\beta||_{L^\infty(L^\infty(\Omega))}\cdot||w^n||^2\\&+\frac{1}{2}\langle w_-^n,w_-^n\rangle_{Q_+^n}-\frac{1}{2}\langle w_+^n,w_+^n\rangle_{Q_-^n}+\langle[w^n],w_+^n\rangle_{Q_-^n}\\&-\frac{1}{2}\sum\Big|\int_{\partial k}(w^n)^2(\beta^n-\bar\beta^n)\cdot\gamma\,ds\Big|.\end{aligned}$$

Noting that

$$(w_-^n,w_-^n)_{Q_+^n}=\langle w_-^n,w_-^n\rangle_{Q_-^n}-\langle w_-^n,w_-^n\rangle_{\Gamma_-^n}+\langle w_-^n,w_-^n\rangle_{\Gamma_+^n}.$$

Applying (3.3) to term $\int_{\partial k}(w^n)^2(\beta^n-\bar{\beta}^n)\cdot\gamma\, ds$ and setting

$$C_0=\left\|\sigma-\frac{1}{2}\mathrm{div}\,\beta\right\|_{L^\infty(L^\infty(\Omega))}+\frac{1}{2}C^{\star\star},$$

then estimate (3.4) is obtained immediately from inequality ($\star$ 5).

Theorem 3.1. *For $\triangle t(=\tau)$ small enough, Euler FDDG scheme* (2.6) *has a unique solution $\{U^n\}$ and the following estimate is true:*

$$\max_{0\le n\le N}||U^n||^2+\sum_{n=1}^{N}(|[U^n]|^2_{Q^n_-}+|U^n_-|^2_{\Gamma^n_+})\tau\le C\{||f||^2_{L^2(L^2(\Omega))}+||g||^2_{L^2(L^2(\Gamma_-))}+||u_0||^2\}, \tag{3.6}$$

where constant C is independent of τ,h.

Proof. It is sufficient to show the estimation (3.6). In fact, from (2.7) we have

$$B(U^n,U^n;U^{n-1})=(f^n,U^n),\quad n=1,2,\cdots,N.$$

Applying Lemma 3.2 and (2.7b) to the right–side of above equality then using Gronwall inequality and noting that $||U^0||\le||u_0||$, we can see that provided $\triangle t(=\tau)$ is sufficiently small such that $1-(C_0+1)\tau\ge\mu_0>0$ and

$$\sum_{n=1}^{N}||f^n||^2\tau\le 2||f||^2_{L^2(L^2(\Omega))},\ \sum_{n=1}^{N}||g^n||^2_{L^2(\Gamma_-)}\tau\le 2||g||^2_{L^2(L^2(\Gamma_-))}$$

then

$$(\star 6)\qquad \begin{aligned}&||U^n||^2+\textstyle\sum_{l=1}^{n}(|[U^l]|^2_{Q^l_-}+|U^l_-|^2_{\Gamma^l_+})\tau\\&\le C\{||f||^2_{L^2(L^2(\Omega))}+||g||^2_{L^2(L^2(\Gamma_-))}+||u_0||^2\},\ n=1,2,\cdots,N,\end{aligned}$$

from which the conclusion (3.6) is proved.

If specifying $U^n_+|_{\Gamma_+}=0$ and $u^n_+|_{\Gamma_+}=0$, setting $Q\triangleq\bigcup\partial k=Q^n_-\bigcup\Gamma_+$ and denoting

$$||U||^2_\triangle\triangleq\max_{0\le n\le N}||U^n||^2+\sum_{n=1}^{N}|[U^n]|^2_Q\tau \tag{3.7}$$

then estimation (3.6) can be written as

$$||U||^2_\triangle\le C\{||f||^2_{L^2(L^2(\Omega))}+||g||^2_{L^2(L^2(\Gamma_-))}+||u_0||^2\}. \tag{3.8}$$

In order to establish the stability of scheme (3.6) in that norm which is stronger than $||\cdot||_\triangle$, it is necessary to make a more fine analysis for scheme (2.6).

Let O_k be the geometry centre of element $k\in\mathcal{T}_h$ and $w^n,w^{n-1}\in V_h$. Define piecewise functions on $\mathcal{T}_h$

$$w^n_\star=\triangle_t w^n+\beta^n\cdot\nabla w^n,\quad\forall k\in\mathcal{T}_h, \tag{3.9}$$

$$\underline{w}^n=\triangle_t w^n+\beta^n(O_k)\cdot\nabla w^n,\quad\forall k\in\mathcal{T}_h, \tag{3.10}$$

then $\underline{w}^n\in V_h$ and it is easy to show that

$$||\underline{w}^n-w^n_\star||_k\le C||w^n||_k, \tag{3.11}$$

$$||\underline{w}^n||_k \le C(||w^n||_k + ||w^n_\star||_k), \quad \forall k \in \mathcal{T}_h. \tag{3.12}$$

Lemma 3.3. *There exists a constant* $\tilde{C}_0 > 0$ *such that, for* $\forall w^n,\ w^{n-1} \in V_h$ *and* $\forall w^n_-|_{\Gamma_-} \in L^2(\Gamma_-)$,

$$B(w^n, \underline{w}^n; w^{n-1}) \ge \frac{3}{4}||w^n_\star||^2 - \tilde{C}_0(||w^n||^2 + h^{-1}|[w^n]|^2_{Q^n_-}). \tag{3.13}$$

Proof. By the definition (3.1) we have

$$\begin{aligned} B(w^n, \underline{w}^n; w^{n-1}) = &\sum\{(w^n_\star, w^n_\star)_k + (\sigma^n w^n, w^n_\star)_k\} + \langle [w^n], \underline{w}^n_+\rangle_{Q^n_-} \\ &+ \sum\{(w^n_\star, \underline{w}^n - w^n_\star)_k + (\sigma^n w^n, \underline{w}^n - w^n_\star)_k\}. \end{aligned} \tag{3.14}$$

Set $\sigma_1 = ||\sigma||_{L^\infty(L^\infty(\Omega))}$ then

$$(w^n_\star, w^n_\star)_k + (\sigma^n w^n, w^n_\star)_k \ge (1-\varepsilon)||w^n_\star||^2_k - \frac{\sigma_1^2}{4\varepsilon}||w^n||^2_k, \quad (0 < \varepsilon < 1).$$

It follows from (3.11), (3.12) and Lemma 3.1 that

$$\begin{aligned} (w^n_\star, \underline{w}^n - w^n_\star)_k \le &\varepsilon||w^n_\star||^2_k + \frac{C_1}{4\varepsilon}||w^n||^2_k, \\ (\sigma^n w^n, \underline{w}^n - w^n_\star)_k \le &C_2||w^n||^2_k, \\ \langle [w^n], \underline{w}^n_+\rangle_{\partial k^n_-} \le &C_3 h^{-\frac{1}{2}}|[w^n]|_{\partial k^n_-}||\underline{w}^n||_k \\ \le &\varepsilon(||w^n_\star||^2_k + ||w^n||^2_k) + \frac{C_4}{4\varepsilon}h^{-1}|[w^n]|^2_{\partial k^n_-}. \end{aligned}$$

Substituting inequalities above into (3.14) and taking $\varepsilon = \frac{1}{12}$ then (3.13) is derived.

Lemma 3.4. *There exists a constant* $C_0^\star > 0$ *such that*

$$||U^n_\star||^2 \le C_0^\star\{||U^n||^2 + h^{-1}|[U^n]|^2_{Q^n_-} + ||f^n||^2\}, \quad n = 1, 2, \cdots, N \tag{3.15}$$

Proof. Since $\underline{U}^n \in V_h$ we have

$$B(U^n, \underline{U}^n; U^{n-1}) = (f^n, \underline{U}^n), \quad n = 1, 2, \cdots, N.$$

The estimation (3.15) can be obtained by Lemma 3.3 and applying $\varepsilon - ab$ inequality and (3.12) to term $(f^n, \underline{U}^n)$.

Now define

$$||U||^2_{\beta,h} \triangleq ||U||^2_\triangle + h\sum_{n=1}^N ||\triangle_t U^n + \beta^n \cdot \nabla U^n||^2\tau, \tag{3.16}$$

where still specifying $U^n_+|_{\Gamma_+} = 0$, $n = 1, 2, \cdots, N$.

Theorem 3.2. *Euler FDDG scheme* (2.6) *is stable in norm* $||\cdot||_{\beta,h}$*, that is, for* τ *small enough, the solution* $\{U^n\}$ *of scheme* (2.6) *satisfies the following estimate:*

$$||U||^2_{\beta,h} \le C\{||f||^2_{L^2(L^2(\Omega))} + ||g||^2_{L^2(L^2(\Gamma_-))} + ||u_0||^2\}, \tag{3.17}$$

where C is independent of h, τ.

Proof. It follows from (2.7) and Lemma 3.2 that

$$\frac{1}{2}[\triangle_t||U^n||^2 + |[U^n]|^2_{Q^n_-} + |U^n_-|^2_{\Gamma^n_+}] \le ||f^n||^2 + C_1||U^n||^2 + ||g^n||^2_{L^2(\Gamma_-)}. \tag{3.18}$$

Multiplying (3.15) by $\alpha h(\alpha > 0)$ where α is small so that $\alpha C_0^\star \le \frac{1}{4}$ and adding the obtained inequality to (3.18) we have

$$\begin{aligned}\frac{1}{2\tau}(||U^n||^2 - ||U^{n-1}||^2) + \frac{1}{4}|[U^n]|^2_{Q^n_-} + \frac{1}{2}|U^n_-|^2_{\Gamma_+} + \alpha h||U^n_\star||^2 \\ \le C_2(||U^n||^2 + ||f^n||^2 + ||g^n||^2_{L^2(\Gamma_-)}), \quad n = 1, 2, \cdots, N.\end{aligned}$$

Summing above inequalities up for n and using the regularity treatment, then the conclusion (3.17) can be proved.

Comparing Theorem 3.2 with Theorem 3.1 we see that Theorem 3.2 delineates the stability of scheme (2.6) more deeply since estimate (3.17) shows that the change of U^n along the flow field direction $(\beta_1^n, \beta_2^n, 1)$ is also stable. By the way, we point out that it seems impossible to establish the same stability results as (3.17) for full discrete Galerkin scheme of Problem (1.1).

Remark 1 If the term $||\triangle_t U^n||^2$ is retained by applying Lemma 3.2 to derive (3.18), then the estimate (3.17) can be improved as

$$||U||^2_{\beta,h} + \tau\sum_{n=1}^N ||\triangle_t U^n||^2\tau \le C(||f||^2_{L^2(L^2(\Omega))} + ||g||^2_{L^2(L^2(\Gamma_-))} + ||u_0||^2). \tag{3.19}$$

If taking $\tau = \mu h$ ($\mu = const. > 0$) and noting that

$$||\beta^n \cdot \nabla U^n||^2 \le 2(||\triangle_t U^n||^2 + ||\triangle_t U^n + \beta^n \cdot \nabla U^n||^2)$$

then we can get

$$\begin{aligned}||U||^2_\triangle + \sum_{n=1}^N \{\tau||\triangle_t U^n||^2 + h||\beta^n \cdot \nabla U^n||^2\}\tau \\ \le C\{||f||^2_{L^2(L^2(\Omega))} + ||g||^2_{L^2(L^2(\Gamma_-))} + ||u_0||^2\}.\end{aligned} \tag{3.20}$$

Remark 2. In the analysis of DG method based on space-time finite element discrete to solve Problem (1.1), the condition

$$\sigma - \frac{1}{2}\mathrm{div}\,\beta \ge \alpha_0 > 0, (x, t) \in D$$

is assumed[2]. But as we have seen above that the assumption is not necessary for Euler FDDG scheme (2.6).

3.2 Convergence-order estimation

Let u be the solution of (1.1). Assume that $u \in L^\infty(H^{r+1}(\Omega)) \cap C(\bar{D})$, $\frac{\partial u}{\partial t} \in L^2(H^{r+1}(\Omega))$ and $\frac{\partial^2 u}{\partial t^2} \in L^2(L^2(\Omega))$. The truncation error E_1^n in following equation

$$B(u^n, v; u^{n-1}) = (f^n, v) + (E_1^n, v), \quad \forall v \in V_h,\ n = 1, 2, \cdots, N \tag{3.21}$$

can be bounded by

$$||E_1^n||^2 = \Big\| \triangle_t u^n - \Big(\frac{\partial u}{\partial t}\Big)^n \Big\|^2 = \frac{1}{\tau^2} \Big\| \int_{t^{n-1}}^{t^n} (t' - t^{n-1}) \frac{\partial^2 u}{\partial t^2} dt' \Big|^2$$
$$\leq C_1 \tau \Big\| \frac{\partial^2 u}{\partial t^2} \Big\|_{L^2(J_n, L^2(\Omega))}, \tag{3.22}$$

where $J_n = (t^{n-1}, t^n)$.

It follows from (3.2) and (2.7) that

$$B(u^n - U^n, v; u^{n-1} - U^{n-1}) = (E_1^n, v), \quad \forall v \in V_h,\ n = 1, 2, \cdots, N, \tag{3.23a}$$
$$(u^n - U^n)_-|_{\Gamma_-} = 0, \tag{3.23b}$$
$$(u^0 - U^0, v) = 0, \quad \forall v \in V_h. \tag{3.23c}$$

Define $\tilde{u}(t) : [0, T] \to V_h$ such that, for $\forall k \in V_h$,

$$(\tilde{u}(t) - u(t), v)_k = 0, \forall v \in P_r(k), t \in [0, T]. \tag{3.24}$$

Set $\xi^n = U^n - \tilde{u}^n$, $\eta^n = u^n - \tilde{u}^n$, $e^n = u^n - U^n = \eta^n - \xi^n$ and take $\tilde{u}_-^n|_{\Gamma_-} = g^n$ then

$$B(\xi^n, v; \xi^{n-1}) = B(\eta^n, v; \eta^{n-1}) - (E_1^n, v), \quad \forall v \in V_h,\ n = 1, 2, \cdots, N, \tag{3.25a}$$
$$\xi_-^n|_{\Gamma_-} = 0,\ \eta_-^n|_{\Gamma_-} = 0, \tag{3.25b}$$
$$\xi^0 = 0. \tag{3.25c}$$

Taking $v = \xi^n$ in (3.25a) we have

$$B(\xi^n, \xi^n; \xi^{n-1}) = B(\eta^n, \xi^n; \eta^{n-1}) - (E_1^n, \xi^n).$$

Using Lemma 3.2 and the boundary condition (3.25b) we can get

$$\frac{1}{2}[\triangle_t ||\xi^n||^2 + ||[\xi^n]||^2_{Q_-^n} + |\xi_-^n|^2_{\Gamma_+^n} + \tau ||\triangle_t \xi^n||^2]$$
$$\leq B(\xi^n, \xi^n; \xi^{n-1}) + C_0 ||\xi^n||^2 \leq B(\eta^n, \xi^n; \eta^{n-1})$$
$$+ C_1 (||\xi^n||^2 + ||E_1^n||^2). \tag{3.26}$$

Lemma 3.5. *There exists a constant $C > 0$ such that*

$$\triangle_t ||\xi^n||^2 + ||[\xi^n]||^2_{Q_-^n} + |\xi_-^n|^2_{\Gamma_+^n} + \tau ||\triangle_t \xi^n||^2$$
$$\leq C\{||\xi^n||^2 + ||\eta^n||^2 + h||\eta^n||_1^2 + |\eta_-^n|^2_{Q_-^n} + |\eta_-^n|^2_{\Gamma_+^n} + ||E_1^n||^2\},\ n = 1, 2, \cdots, N. \tag{3.27}$$

Proof. In fact,

$$B(\eta^n, \xi^n; \eta^{n-1}) = \sum (\triangle_t \eta^n + \beta^n \cdot \nabla \eta^n + \sigma^n \eta^n, \xi^n)_k + \langle [\eta^n], \xi_+^n \rangle_{Q_-^n}. \tag{3.28}$$

Since $\xi^n|_k \in P_r(k)$, from the definition of $\tilde{u}$ we have
$(\star\star 1)$ $\quad (\triangle_t \eta^n, \xi^n)_k = \frac{1}{\tau}(\eta^n - \eta^{n-1}, \xi^n)_k = 0,\ \forall k \in \mathcal{T}_h.$
Integrating by parts yields
$(\star\star 2)$ $\quad (\beta^n \cdot \nabla \eta^n + \sigma^n \eta^n, \xi^n)_k = -(\eta^n, \beta^n \cdot \nabla \xi^n)_k$
$+((\sigma^n - \operatorname{div} \beta^n)\eta^n, \xi^n)_k + \int_{\partial k} \eta^n \xi^n \beta^n \cdot \gamma\, ds.$
Noting that $(\beta^n(O_k) \cdot \nabla \xi^n)|_k \in P_r(k)$ and using the inverse estimate of $P_r(k)$ we can get
$(\star\star 3)$ $\quad (\eta^n, \beta^n \cdot \nabla \xi^n)_k = (\eta^n, (\beta^n(x) - \beta^n(O_k)) \cdot \nabla \xi^n)_k \le C_1||\eta^n||_k \cdot ||\xi^n||_k.$
And also
$(\star\star 4)$ $\quad \int_{\partial k} \eta^n \xi^n \beta^n \cdot \gamma\, ds$
$= \int_{\partial k_+^n} \eta_-^n \xi_-^n \bar{\beta}_n \cdot \gamma\, ds - \int_{\partial k_-^n} \eta_+^n \xi_+^n |\bar{\beta}_n \cdot \gamma|\, ds - \int_{\partial k} \eta^n \xi^n (\bar{\beta}^n - \beta^n) \cdot \gamma\, ds.$
Using Lemma 3.1 and the trace inequality we obtain
$(\star\star 5)$ $\quad \int_{\partial k} \eta^n \xi^n (\bar{\beta}^n - \beta^n) \cdot \gamma\, ds \le (\int_{\partial k} (\eta^n)^2 |\beta^n - \bar{\beta}^n|\, ds)^{\frac{1}{2}} \cdot (\int_{\partial k} (\xi^n)^2 |\beta^n - \bar{\beta}^n|\, ds)^{\frac{1}{2}}$
$\le Ch^{\frac{1}{2}}||\eta^n||_{L^2(\partial k)} \cdot ||\xi^n||_k \le Ch^{\frac{1}{2}}||\eta^n||_{H^1(k)} \cdot ||\xi^n||_k$
$\le ||\xi^n||_k^2 + C^2 h ||\eta^n||_{1,k}^2.$
Combining $(\star\star 1)$—$(\star\star 5)$ with (3.28) and noting (3.25b) we have

$$\begin{aligned} B(\eta^n, \xi^n; \eta^{n-1}) \le & C_3(||\eta^n||^2 + ||\xi^n||^2) + C_2 h ||\eta^n||_1^2 + \langle \eta_-^n, \xi_-^n \rangle_{Q_+^n} \\ & - \langle \eta_+^n, \xi_+^n \rangle_{Q_-^n} + \langle [\eta^n], \xi_+^n \rangle_{Q_-^n} \\ \le & C_4(||\eta^n||^2 + ||\xi^n||^2 + h||\eta^n||_1^2) + \frac{1}{4}(|[\xi^n]|_{Q_-^n}^2 + ||\xi_-^n||_{\Gamma_+^n}^2) \\ & + 2(|\eta_-^n|_{Q_-^n}^2 + |\eta_-^n|_{\Gamma_+^n}^2). \end{aligned}$$

Substituting above inequality into (3.26), the desired estimate (3.27) is proved.

Define for $n = 1, 2, \cdots, N$,

$$\xi_\star^n = \triangle_t \xi^n + \beta^n \cdot \nabla \xi^n, \forall k \in \mathcal{T}_h, \tag{3.29a}$$
$$\underline{\xi}^n = \triangle_t \xi^n + \beta^n(O_k) \cdot \nabla \xi^n,\ \forall k \in \mathcal{T}_h. \tag{3.29b}$$

Lemma 3.6. *There exists a constant $\tilde{C} > 0$ such that*

$$\begin{aligned} ||\xi_\star^n||^2 \le & \tilde{C}_1[||\xi^n||^2 + h^{-1}|[\xi^n]|_{Q_-^n}^2 + ||\nabla \eta^n||^2 + ||\eta^n||^2 \\ & + h^{-1}|[\eta^n]|_{Q_-^n}^2 + ||E_1^n||^2],\ n = 1, 2, \cdots, N. \end{aligned} \tag{3.30}$$

Proof. Taking $v = \underline{\xi}^n$ in (3.25a) and using Lemma 3.3 we can get

$$\begin{aligned} \frac{3}{4}||\xi_\star^n||^2 \le & B(\xi^n, \underline{\xi}^n; \xi^{n-1}) + \tilde{C}_0(||\xi^n||^2 + h^{-1}|[\xi^n]|_{Q_-^n}^2) \\ = & B(\eta^n, \underline{\xi}^n; \eta^{n-1}) - (E_1^n, \underline{\xi}^n) + \tilde{C}_0(||\xi^n||^2 + h^{-1}|[\xi^n]|_{Q_-^n}^2). \end{aligned}$$

From (3.12) we have

$$(E_1^n, \underline{\xi}^n) \le \frac{1}{4}(||\xi_\star^n||^2 + ||\xi^n||^2) + C_1||E_1^n||^2.$$

Thus

$$\frac{1}{2}||\xi_\star^n||^2 \leq B(\eta^n,\underline{\xi}^n;\eta^{n-1}) + C_2(||\xi^n||^2 + h^{-1}|[\xi^n]|^2_{Q_-^n} + ||E_1^n||^2). \tag{3.31}$$

Note that

$$B(\eta^n,\underline{\xi}^n;\eta^{n-1}) = \sum(\triangle_t\eta^n + \beta^n\cdot\nabla\eta^n + \sigma^n\eta^n,\underline{\xi}^n)_k + \langle[\eta^n],\underline{\xi}_+^n\rangle_{Q_-^n}, \tag{3.32}$$

$$\begin{aligned}
&(\triangle 1) \quad (\triangle_t\eta^n,\underline{\xi}^n)_k = 0,\\
&(\triangle 2) \quad (\beta^n\cdot\nabla\eta^n + \sigma^n\eta^n,\underline{\xi}^n)_k \leq C_3(||\nabla\eta^n||_k + ||\eta^n||_k)\cdot(||\xi_\star^n||_k + ||\xi^n||_k),\\
&(\triangle 3) \quad \langle[\eta^n],\underline{\xi}_+^n\rangle_{Q_-^n} \leq C_4 h^{-\frac{1}{2}}\sum|[\eta^n]|_{\partial k_-^n}(||\xi_\star^n||_k + ||\xi^n||_k).
\end{aligned}$$

Combining (3.31) with (3.32), ($\triangle$1)—($\triangle$3) and using ε–ab inequality the estimate (3.30) is derived.

Theorem 3.3. *Let $u,\{U^n\}$ be the solutions of Problem* (1.1) *and Euler FDDG scheme* (2.6) *respectively. Assume that*

$$u \in L^\infty(H^{r+1}(\Omega))\cap C(\bar{D}),\ \frac{\partial u}{\partial t}\in L^2(H^{r+1}(\Omega)),\ \frac{\partial^2 u}{\partial t^2}\in L^2(L^2(\Omega)).$$

Then there exists a constant C independent of τ, h such that, for τ small enough,

$$\begin{aligned}
\max_{0\leq n\leq N}||e^n||^2 + &\sum_{n=1}^N|[e^n]|_Q^2\tau + \sum_{n=1}^N(\tau||\triangle_t e^n||^2 + h||\triangle_t e^n + \beta^n\cdot\nabla e^n||^2)\tau\\
&\leq C(h^{2r+1}+\tau^2),
\end{aligned} \tag{3.33}$$

where $U_+^n|_{\Gamma_+} = u_+^n|_{\Gamma_+} = \tilde{u}_+^n|_{\Gamma_+} = 0$ are specified.

Proof. Multiplying (3.30) by $\alpha h(\alpha>0)$ with α proper small and adding this new inequality to (3.27), we can get

$$\begin{aligned}
&\triangle_t||\xi^n||^2 + |[\xi^n]|_Q^2 + \tau||\triangle_t\xi^n||^2 + h||\xi_\star^n||^2\\
&\quad\leq C_1\{||\xi^n||^2 + ||\eta^n||^2 + |[\eta^n]|_Q^2 + h||\nabla\eta^n||^2 + |\eta_-^n|^2_{Q_-^n} + ||E_1^n||^2\},\ n=1,2,\cdots,N.
\end{aligned}$$

Multiplying above inequalities by τ then summing up for n and applying Gronwall ineqality, recalling $\xi^0=0$ we obtain for τ small enough,

$$\begin{aligned}
&||\xi^n||^2 + \sum_{j=1}^n|[\xi^j]|_Q^2\tau + \sum_{j=1}^n(\tau||\triangle_t\xi^j||^2 + h||\xi_\star^j||^2)\tau\\
&\quad\leq C_2\Big\{\sum_{j=1}^n[||\eta^j||^2 + |[\eta^j]|_Q^2 + h||\nabla\eta^j||^2 + |\eta_-^n|^2_{Q_-^n}]\tau + \sum_{j=1}^n||E_1^j||^2\tau\Big\}.
\end{aligned} \tag{3.34}$$

From (3.22) we have

$$\sum_{j=1}^n||E_1^j||^2\tau \leq C_3\tau^2\sum_{j=1}^n\Big\|\frac{\partial^2 u}{\partial t^2}\Big\|^2_{L^2(J_j,L^2(\Omega))} = C_3\tau^2\Big\|\frac{\partial^2 u}{\partial t^2}\Big\|^2_{L^2(L^2(\Omega))}$$

Therefore from (3.34),

$$\max_{0\le n\le N}||\xi^n||^2+\sum_{n=1}^N|[\xi^n]|_Q^2\tau+\sum_{n=1}^N(\tau||\triangle_t\xi^n||^2+h||\triangle_t\xi^n+\beta^n\cdot\nabla\xi^n||^2)\tau$$
$$\le C_4\Big\{\sum_{n=1}^N[||\eta^n||^2+|[\eta^n]|_Q^2+h||\nabla\eta^n||^2+|\eta_-^n|_{Q_-^n}^2]\tau+\tau^2\Big\}. \tag{3.35}$$

Moreover, we need to estimate terms $||\eta^n||$, $||\nabla\eta^n||$, $|[\eta^n]|_Q$, $|\eta_-^n|_{Q_-^n}$. To this end, let $\Pi_h u(t)$ be the interpolation of $u(t)$ in V_h. By the definition of $\tilde{u}$ and the finite element interpolation theorem we know that

$$||\eta^n||_k=||u^n-\tilde{u}^n||_k=\inf_{\varphi\in P_r(k)}||u^n-\varphi||_k\le||u^n-\Pi_h u^n||_k$$
$$\le Ch^{r+1}||u^n||_{r+1,k},\quad n=1,2,\cdots,N \tag{3.36}$$

where $||\cdot||_{r+1,k}\triangleq||\cdot||_{H^{r+1}(k)}$. Also

$$||\eta^n||_{\partial k}\triangleq||\eta^n||_{L^2(\partial k)}=||u^n-\tilde{u}^n||_{\partial k}\le||u^n-\Pi_h u^n||_{\partial k}+||\Pi_h u^n-\tilde{u}^n||_{\partial k}.$$

According to the result given by [3] we have

$$||u^n-\Pi_h u^n||_{\partial k}\le C_0h^{r+\frac{1}{2}}||u^n||_{r+1,k}.$$

Thus using Lemma 3.1 and triangle inequality to term $||\Pi_h u^n-\tilde{u}^n||_{\partial k}$ we can get

$$||\eta^n||_{\partial k}\le C_1h^{r+\frac{1}{2}}||u^n||_{r+1,k}. \tag{3.37}$$

Hence

$$|[\eta^n]|_Q\le C_2h^{r+\frac{1}{2}}||u^n||_{r+1,\Omega}. \tag{3.38}$$

Since

$$\nabla\eta^n=\nabla(u^n-\Pi_h u^n)+\nabla(\Pi_h u^n-\tilde{u}^n),$$

we can get from (3.36)

$$||\nabla\eta^n||_k\le C_3h^r||u^n||_{r+1,k}. \tag{3.39}$$

Substituting (3.36)—(3.39) into (3.35) to obtain

$$\max_{0\le n\le N}||\xi^n||^2+\sum_{n=1}^N|[\xi^n]|_Q^2\tau+\sum_{n=1}^N(\tau||\triangle_t\xi^n||^2+h||\triangle_t\xi^n+\beta^n\cdot\nabla\xi^n||^2)\tau$$
$$\le C_4(h^{2r+1}+\tau^2). \tag{3.40}$$

Noting that

$$||\triangle_t\eta^n||_k^2\le\frac{1}{\tau}\int_{J_n}\Big\|\frac{\partial u}{\partial t}\Big\|_k^2dt.$$

By the regularity analysis[5] we can get

$$\sum_{n=1}^{N} ||\triangle_t \eta^n||^2 \tau \leq C_5 h^{2r+2} ||\frac{\partial u}{\partial t}||^2_{L^2(H^{r+1}(\Omega))}. \tag{3.41}$$

Thus from (3.35) and applying triangle inequality, the convergence order estimate (3.33) is obtained.

The convergence order given by (3.33) is optimal since error $\beta^n \cdot \nabla e^n$ and $[e^n]_Q$ are considered.

4. Crank–Nicolson FDDG Scheme

Applying the treatment analogous to that used in section 3 for Euler scheme (2.6), we can establish the theoretical analysis for C—N FDDG scheme (2.11). Here we only give some concerned results on the stability and error estimation.

Define

$$G(w^n, v; w^{n-1}) = \sum (\triangle_t w^n + \beta_n \cdot \nabla \tilde{w}^n + \sigma_n \tilde{w}^n, v)_k + \langle [\tilde{w}^n], v_+ \rangle_{Q^n_-}, \tag{4.1}$$

where the definitions of notations $\tilde{w}^n, \beta_n, \sigma_n, \partial k^n_-, \cdots, Q^n_-$ have been given in §2.

Obviously, the C—N scheme (2.11) can be written as: find $U^n \in V_h$, $n = 0, 1, \cdots, N$, such that

$$G(U^n, v; U^{n-1}) = (f_n, v), \quad \forall v \in V_h, n = 1, 2, \cdots, N \tag{4.2a}$$

$$U^n_-|_{\Gamma_-} = g^n, \tag{4.2b}$$

$$(U^0 - u_0, v) = 0, \forall v \in V_h. \tag{4.2c}$$

Taking $v = \tilde{w}^n$ in $G(w^n, v; w^{n-1})$ and noting that $(\triangle_t w^n, \tilde{w}^n) = \frac{1}{2}\triangle_t ||w^n||^2$, we can get

Lemma 4.1. *There exists a constant $C_0 > 0$ such that, for arbitrary $w^n, w^{n-1} \in V_h$ and $w^n_-|_{\Gamma_-} \in L^2(\Gamma_-)$,*

$$G(w^n, \tilde{w}^n; w^{n-1}) + C_0 ||w^n||^2 + \frac{1}{2}|\tilde{w}^n_-|^2_{\Gamma_-} \geq \frac{1}{2}(\triangle_t ||w^n||^2 + |[\tilde{w}^n]|^2_{Q^n_-} + |\tilde{w}^n_-|^2_{\Gamma_+}). \tag{4.3}$$

Define piecewise functions on $\mathcal{T}_h$ for $n = 1, 2, \cdots, N$,

$$\tilde{U}^n_\star = \triangle_t U^n + \beta_n \cdot \nabla \tilde{U}^n, \ \forall k \in \mathcal{T}_h, \tag{4.4}$$

$$\underline{\tilde{U}}^n = \triangle_t U^n + \beta_n(O_k) \cdot \nabla \tilde{U}^n, \ \forall k \in \mathcal{T}_h, \tag{4.5}$$

where O_k is the geometry centre of element k.

Lemma 4.2. *Let $\{U^n\}$ be the solution of scheme (2.11) then there exists a constant $\tilde{C} > 0$ such that*

$$||\tilde{U}^n_\star||^2 \leq \tilde{C}\{||\tilde{U}^n||^2 + h^{-1}|[\tilde{U}^n]|^2_{Q^n_-} + ||f_n||^2\}, \ n = 1, 2, \cdots, N. \tag{4.6}$$

By Lemma 4.1, 4.2 and using the similar argument to prove Theorem 3.2 in §3 we can obtain

Theorem 4.1. *C—N FDDG scheme* (2.11) *has a unique solution* $\{U^n\}_0^N$ *which satisfies the following stability estimation: for* τ *small enough,*

$$\max_{0\le n\le N}||U^n||^2+\sum_{n=1}^{N}||[\tilde{U}^n]||_Q^2\tau+h\sum_{n=1}^{N}||\triangle_t U^n+\beta_n\cdot\nabla\tilde{U}^n||^2\tau \\ \le C\{||f||^2_{L^2(L^2(\Omega))}+||g||^2_{L^2(L^2(\Gamma_-))}+||u_0||^2\}, \tag{4.7}$$

where $U^n_+|_{\Gamma_+}=0$ $(n=1,2,\cdots,N)$ *are specified.*

Finally, applying the analogous approach used in §3 to establish Theorem 3.3 we can prove the following theorem.

Theorem 4.2. *Let* $u,\{U^n\}_0^N$ *be the solutions of Problem* (1.1) *and C—N scheme* (2.11) *respectively. Assume that*

$$u\in L^\infty(H^{r+1}(\Omega))\cap C(\bar{D}),\frac{\partial u}{\partial t}\in L^2(H^{r+1}(\Omega)),\frac{\partial^2 u}{\partial t^2}\in L^2(H^1(\Omega))$$

and $\frac{\partial^3 u}{\partial t^3}\in L^2(L^2(\Omega))$. *Then for* τ *small enough,*

$$\max_{0\le n\le N}||e^n||^2+\sum_{n=1}^{N}||[\tilde{e}^n]||_Q^2\tau+h\sum_{n=1}^{N}||\triangle_t e^n+\beta_n\cdot\nabla\tilde{e}^n||^2)\tau\le C(h^{2r+1}+\tau^4), \tag{4.8}$$

where constant C *is independent of* τ,h.

5. A Numerical Example

Consider the following hyperbolic problem

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=f(x,t),(x,t)\in(0,1]\times(1,2], \tag{5.1a}$$
$$u(x,1)=u_0(x),x\in[0,1], \tag{5.1b}$$
$$u(0,t)=g(t),t\in[1,2]. \tag{5.1c}$$

where

$$f(x,t)=\frac{1}{\varepsilon(1-e^{-1/\varepsilon})}(1-x-t)e^{-(1-x)t/\varepsilon},$$
$$u_0(x)=(1-e^{-(1-x)/\varepsilon})/(1-e^{-1/\varepsilon}),$$
$$g(t)=(1-e^{-t/\varepsilon})/(1-e^{-1/\varepsilon}).$$

The exact solution of (5.1) is

$$u(x,t)=(1-e^{-(1-x)t/\varepsilon})/(1-e^{-1/\varepsilon}). \tag{5.2}$$

Take $h=\frac{1}{M}$, $\triangle t=\tau=\frac{1}{N}$ where M,N are two positive integers. Let $x_i=ih$, $i=0,1,\cdots,M$, $I_i=[x_{i-1},x_i]$ and $t^n=1+n\tau$, $n=0,1,\cdots,N$. Denote

$$P_1(I_i)=\{v=ax+b,a,b\in R,x\in I_i\},\ i=1,2,\cdots,M,$$

$$V_h = \{v \in L^2(0,1), v|_{I_i} \in P_1(I_i),\ i = 1,2,\cdots,M\}.$$

The Euler FDDG scheme for Problem (1.1) is as follows: Find $U^n \in V_h$, $n = 0,1,\cdots,N$, such that, for element I_i,

$$\Big(\frac{U^n - U^{n-1}}{\tau}, v\Big)_{I_i} + (U^n_x, v)_{I_i} + [U^n(x_{i-1})]\cdot v_+(x_{i-1}) = (f^n, v)_{I_i},$$
$$\forall v \in P_1(I_i), n = 1,2,\cdots,N, \tag{5.3a}$$
$$(U^0, v)_{I_i} = (u_0, v)_{I_i}, \forall v \in P_1(I_i), \tag{5.3b}$$
$$U^n_-(0) = g^n, \tag{5.3c}$$

where $U^n_x = \frac{\partial U^n}{\partial x}$, $[U^n(x_{i-1})] \triangleq U^n_+(x_{i-1}) - U^n_-(x_{i-1})$.

Let $\varphi_{i-1}(x)$, $\varphi_i(x)$ be the basis functions of $P_1(I_i)$ satisfying

$$\varphi_l(x_s) = \delta_{ls}, \quad l, s = i-1, i$$

then $U^n(x)$ can be written as

$$U^n(x) = U^n_+(x_{i-1})\varphi_{i-1}(x) + U^n_-(x_i)\varphi_i(x), x \in I_i, i = 1,2,\cdots,N. \tag{5.4}$$

Clearly, $U^n(x)$ can be solved element by element in order $I_1, I_2, \cdots, I_N$ from (5.3).

Taking $\varepsilon = 10^{-3}$ and choosing $M = 10^3, N = 10$, we computed the solution $\{U^n\}_0^N$ of FDDG scheme (5.3).

To compare the numerical results with standard Galerkin method, we also solved Problem (5.1) by full discrete Euler Galerkin scheme.

Some numerical results at time $t = 2(n = 10)$ are given in following table. (For the values of solution $U^n(x)$ of Euler FDDG scheme at node $x = x_i$, we specify $U^N(x_i) \triangleq U^N_-(x_i)$.)

Table (at t=2.0)

x	exact solution	FD–DG solution	Galerkin solution
0	1	1	1
0.5	1	1	1
0.93	1	0.99997	0.99997
0.94	0.99995	0.99989	0.99975
0.95	0.99966	0.99950	0.99813
0.96	0.99752	0.99723	0.98621
0.97	0.98168	0.98201	0.89856
0.98	0.86464	0.87379	0.26408
1	0	0.08530	−0.7324

From the results above we see that FDDG scheme is better than standard Galerkin scheme. Specially, in the neighborhood $x = 1$ where the exact solution $u(x,2)$ presents rapid change from $1 \searrow 0$. The solution of FDDG scheme can approximate the exact solution still, but the solution of standard Galerkin method is of instable so that it can not simulate the exact solution of Problem (5.1).

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