Journal of Computational Mathematics, Vol.16, No.1, 1998, 15–26.

SPECTRAL-DIFFERENCE METHOD FOR TWO-DIMENSIONAL COMPRESSIBLE FLUID FLOW*[1]

Yue-shan Xiong
(*Department of Computer, National University of Defense Technology, Changsha, China*)

Abstract

We develop a combined Fourier spectral-finite difference method for solving 2-dimensional, semi-periodic compressible fluid flow problem. The error estimation, as well as the convergence rate, is presented.

Key Words: Spectral-difference method, compressible fluid flow, error estimation.

1. Introduction

We consider the following compressible flow equations:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + (u\cdot\nabla)u_i - \dfrac{1}{\rho}\dfrac{\partial}{\partial x_i}(\kappa\nabla\cdot u) - \dfrac{1}{\rho}\displaystyle\sum_{j=1}^{2}\dfrac{\partial}{\partial x_j}\Big[\nu\Big(\dfrac{\partial u_i}{\partial x_j}+\dfrac{\partial u_j}{\partial x_i}\Big)\Big] + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i} = f_i,\ i=1,2, \\ \dfrac{\partial T}{\partial t} + (u\cdot\nabla)T - \dfrac{1}{\rho T S_T}(\nabla\cdot\mu\nabla)T - \dfrac{\nu}{2\rho T S_T}\displaystyle\sum_{i,j=1}^{2}\Big(\dfrac{\partial u_j}{\partial x_i}+\dfrac{\partial u_i}{\partial x_j}\Big)^2 \\ \qquad -\dfrac{\kappa}{\rho T S_T}(\nabla\cdot u)^2 - \dfrac{\rho S_\rho}{S_T}(\nabla\cdot u) = 0, \\ \dfrac{\partial\rho}{\partial t} + (u\cdot\nabla)\rho + \rho(\nabla\cdot u) = 0, \end{cases} \tag{1.1}$$

where u is the velocity, $u=(u_1,u_2)^*$, T is the absolute temperature, $\nu(T,\rho)$ is the viscous coefficient, $\kappa(T,\rho)=\nu'(T,\rho)-\frac{2}{3}\nu(T,\rho)$ with $\nu'(T,\rho)$ being the second viscous coefficient. $\mu(T,\rho)$ is the coefficient of heat conduction, $S(T,\rho)$ is the entropy, $S_T=\frac{\partial S}{\partial T}$, $S_\rho=\frac{\partial S}{\partial\rho}$.

Under certain conditions, Tani[2] proved that the first boundary problem of (1.1) possesses unique local classical solution. Towards the numerical solution of this problem, the classical difference method is convenient, but it has lower approximate accuracy. Finite element method is particularly suitable for problems with irregular domains.

* Received February 18, 1995.

In the past sixteen years, spectral method for P.D.E. has been developed rapidly[6−10]. In [7], spectral method was used to solve (1.1), but only for periodical problems.

For many practical problems, the boundary conditions are neither fully periodic nor fully nonperiodic. An effective strategy to deal with such problems is to combine the Fourier spectral method with finite difference method or finite element method. In [6], Guo and Cao established spectral-finite element scheme for solving such semi-periodic problems, but it lacks boundary errors analysis. This paper is devoted to a Fourier spectral-finite difference method to solve such problems, we strictly analyse the errors induced by the initial values and boundary conditions.

Let $\Omega = I \times I^*$, $I = (0,1)$ and $I^* = (0, 2\pi)$, we consider the solution of (1.1) in the domain $\Omega \times [0, t_0]$. We suppose that all functions in (1.1) have the periodicity 2π in the x_2 direction, and that u, T satisfy the first kind boundary conditions. These mean that

$$\begin{cases} \eta|_{x_2=0} = \eta|_{x_2=2\pi}, \qquad \forall (x_1,t) \in I \times [0,t_0], \quad \eta = u, T, \rho, \\ u|_{x_1 \in \partial I} = g_1(x_2,t). \quad T|_{x_1 \in \partial I} = g_2(x_2,t), \quad \forall (x_2,t) \in I^* \times [0,t_0]. \end{cases} \tag{1.2}$$

Besides, we assume that the initial values of (1.1) are the following,

$$\eta|_{t=0} = \eta_0, \qquad \eta = u, T, \rho. \tag{1.3}$$

To avoid "negative density"(i.e. $\rho < 0$), which is likely caused by the round off errors during the computations, and which generates a non-physical solution and instablize the computations, we adopt the idea of Guo Ben-yu[5−7], i.e., we seek $\varphi = \ln \rho$ by (1.1) instead of calculating ρ directly. besides, we assume the fluid satisfies the following state equation, $p = R\rho T$, where R is a positive constant. Consequently (1.1) can be rewritten into the following form,

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + (u\cdot\nabla)u_i - \mathrm{e}^{-\varphi}\dfrac{\partial}{\partial x_i}(\kappa \nabla\cdot u) - \mathrm{e}^{-\varphi}\displaystyle\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\Big[\nu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big] \\ \qquad + R\dfrac{\partial T}{\partial x_i} + RT\dfrac{\partial \varphi}{\partial x_i} = f_i, \qquad i = 1,2, \\ \dfrac{\partial T}{\partial t} + (u\cdot\nabla)T - \mathrm{e}^{-\varphi}T^{-1}S_T^{-1}(\nabla\cdot\mu\nabla)T - \dfrac{1}{2}\nu \mathrm{e}^{-\varphi}T^{-1}S_T^{-1}\displaystyle\sum_{i,j=1}^{2}\Big(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\Big)^2 \\ \qquad -\kappa \mathrm{e}^{-\varphi}T^{-1}S_T^{-1}(\nabla\cdot u)^2 - S_\varphi S_T^{-1}(\nabla\cdot u) = 0, \\ \dfrac{\partial \varphi}{\partial t} + (u\cdot\nabla)\varphi + (\nabla\cdot u) = 0, \end{cases} \tag{1.4}$$

We suppose ν, μ, κ and S are sufficiently smooth for each of their variables, and there exist positive constants B_0, B_1, B_2, ν_0, ν_1, μ_0, μ_1, κ_1, A_0, A_1, S_0, S_1, S_2, Φ_0 and Φ_1, such that for $B_0 < T < B_1$ and $|\varphi| \le B_2$,

$$\begin{cases} \nu_0 < \nu < \nu_1, \ \mu_0 < \mu < \mu_1, \ |\kappa| < \kappa_1, \ \min(2\kappa + 3\nu, \nu) > A_0, \\ S_0 < S_T < S_1, \ |S_\varphi| < S_2, \ \Phi_0 < \mathrm{e}^{-\varphi} < \Phi_1, \\ \Big|\dfrac{\partial \eta}{\partial z}\Big| \le A_1, \quad \text{where} \ \ \eta = \nu, \kappa, \mu, S_T, S_\varphi, \quad z = T, \varphi. \end{cases} \tag{1.5}$$

2. Notations and Scheme

Let h be the mesh spacing of x_1, $\Omega_h = I_h \times I^*$, $\Gamma_{1h} = \partial I_h \times I^*$. Let τ be the mesh spacing of t, $\Theta_\tau = \{t = k\tau / k = 0, 1, 2, \cdots\}$. We define

$$
\begin{aligned}
&\eta_{x_1}(x_1, x_2, t) = \frac{1}{h}(\eta(x_1 + h, x_2, t) - \eta(x_1, x_2, t)),\\
&\eta_{\bar{x}_1}(x_1, x_2, t) = \eta_{x_1}(x_1 - h, x_2, t),\\
&\eta_{\hat{x}_1}(x_1, x_2, t) = \frac{1}{2}(\eta_{x_1}(x_1, x_2, t) + \eta_{\bar{x}_1}(x_1, x_2, t)),\\
&\triangle^\nu_{x_1}\eta(x_1, x_2, t) = \frac{1}{2}(\nu\eta_{x_1}(x_1, x_2, t))_{\bar{x}_1} + \frac{1}{2}(\nu\eta_{\bar{x}_1}(x_1, x_2, t))_{x_1},\\
&\triangle^\nu \eta = \triangle^\nu_{x_1}\eta + \frac{\partial}{\partial x_2}\Big(\nu\frac{\partial \eta}{\partial x_2}\Big),\\
&\eta_t(x_1, x_2, t) = \frac{1}{\tau}(\eta(x_1, x_2, t + \tau) - \eta(x_1, x_2, t)).
\end{aligned}
$$

We define that

$$
\begin{aligned}
&(\eta(x_1), \xi(x_1))_{I^*} = \frac{1}{2\pi}\int_{I^*} \eta(x_1, x_2)\bar{\xi}(x_1, x_2)dx_2,\\
&(\eta(x_2), \xi(x_2))_{I_h} = h\sum_{x_1 \in I_h} \eta(x_1, x_2)\bar{\xi}(x_1, x_2),\\
&(\eta, \xi) = h\sum_{x_1 \in I_h} (\eta(x_1), \xi(x_1))_{I^*},\\
&\|\eta(x_1)\|^2_{I^*} = (\eta(x_1), \eta(x_1))_{I^*}, \quad \|\eta(x_2)\|^2_{I_h} = (\eta(x_2), \eta(x_2))_{I_h}, \quad \|\eta\|^2 = (\eta, \eta),\\
&|\eta|^2_1 = \frac{1}{2}\|\eta_{x_1}\|^2 + \frac{1}{2}\|\eta_{\bar{x}_1}\|^2 + \Big\|\frac{\partial \eta}{\partial x_2}\Big\|^2,\\
&|\eta|^2_2 = \frac{1}{2}\|\eta_{x_1\bar{x}_1}\|^2 + \frac{h}{4}\sum_{x_1 \in I_h, x_1 \le 1-2h} \|\eta_{x_1x_1}(x_1)\|^2_{I^*} + \frac{h}{4}\sum_{x_1 \in I_h, x_1 \ge 2h} \|\eta_{\bar{x}_1\bar{x}_1}(x_1)\|^2_{I^*}\\
&\qquad + \frac{1}{2}\Big\|\frac{\partial}{\partial x_2}\eta_{x_1}\Big\|^2 + \frac{1}{2}\Big\|\frac{\partial}{\partial x_2}\eta_{\bar{x}_1}\Big\|^2 + \Big\|\frac{\partial^2 \eta}{\partial x_2^2}\Big\|^2,\\
&\|\eta(t)\|_{q,\infty} = \max_{r_0 + r_1 + r_2 \le q}\ \max_{(x_1, x_2) \in I_h \times I^*} \Big|\Big(\frac{\partial^{r_0}\eta}{\partial x_2}\Big)_{\underbrace{x_1 \cdots x_1}_{r_1}\underbrace{\bar{x}_1 \cdots \bar{x}_1}_{r_2}}\Big|,\\
&|||\eta|||_{q,\infty} = \max_{t \in \Theta_\tau, t \le t_0} \|\eta(t)\|_{q,\infty}.
\end{aligned}
$$

Similarly, we can define $\|\eta(t)\|_{q,\infty,I_h}$, $\|\eta(t)\|_{q,\infty,I^*}$ and so on. Let N be any positive integer, $V_N = \mathrm{span}\{\mathrm{e}^{inx_2}/|n| \le N\}$ and P_N be the orthogonal projection operator, i.e.,

$$
\int_{I^*} P_N\eta \cdot \bar{\xi}dx_2 = \int_{I^*} \eta \cdot \bar{\xi}dx_2, \qquad \forall \xi \in V_N.
$$

We also define

$$
d^{(\alpha)}(\eta, \xi) = \alpha\xi_1\eta_{\hat{x}_1} + (1 - \alpha)(\eta\xi_1)_{\hat{x}_1} + \alpha\xi_2\frac{\partial \eta}{\partial x_2} + (1 - \alpha)\frac{\partial(\eta\xi_2)}{\partial x_2},
$$

$$
\begin{aligned}
&D^+(\eta)=\eta_{1,x_1}+\frac{\partial\eta_2}{\partial x_2},\qquad D^-(\eta)=\eta_{1,\bar{x}_1}+\frac{\partial\eta_2}{\partial x_2},\\
&D(\eta)=\frac{1}{2}D^+(\eta)+\frac{1}{2}D^-(\eta),\\
&D_i\eta=\begin{cases}\eta_{\hat{x}_1}, & i=1\\ \dfrac{\partial\eta}{\partial x_2}, & i=2\end{cases}
\end{aligned}
$$

Let u^N, T^N and φ^N be the approximations to u, T and φ respectively, where

$$
z^N(x_1,x_2,t)=\sum_{|n|\le N} z_n^N(x_1,t)\mathrm{e}^{inx_2},\quad z=u,\ T \text{ or } \varphi.
$$

For simplicity of expression, we denote $P_N(\eta\xi)$ by $P_N\eta\xi$. The spectral-finite difference scheme for (1.4) is the following:

$$
\begin{cases}
L_1^{(i)}(u^N,T^N,\varphi^N)=u_{i,t}^N+P_Nd^{(\frac{1}{2})}(u_i^N,u^N)-\dfrac{1}{2}P_Nu_i^ND(u^N)\\
\qquad -P_N\mathrm{e}^{-\varphi^N}H_1^{(i)}(\kappa(T^N,\varphi^N),u^N)-P_N\mathrm{e}^{-\varphi^N}H_2^{(i)}(\nu(T^N,\varphi^N),u^N)\\
\qquad -P_N\mathrm{e}^{-\varphi^N}\triangle^{\nu(T^N,\varphi^N)}u_i^N+RD_iT^N+RP_NT^ND_i\varphi^N=P_Nf_i, \qquad i=1,2\\
L_2(u^N,T^N,\varphi^N)=T_t^N+P_Nd^{(0)}(T^N,u^N)-P_NT^ND(u^N)\\
\qquad\quad -P_N\mathrm{e}^{-\varphi^N}(T^N)^{-1}S_T^{-1}(T^N,\varphi^N)\triangle^{\mu(T^N,\varphi^N)}T^N-P_N\mathrm{e}^{-\varphi^N}H_3(u^N,T^N,\varphi^N)=0,\\
L_3(u^N,T^N,\varphi^N)=\varphi_t^N+P_Nd^{(\frac{1}{2})}(\varphi^N,u^N)+P_N\Big(1-\dfrac{1}{2}\varphi^N\Big)D(u^N)=0,
\end{cases}
$$

where

$$
H_1^{(i)}(\kappa(T^N,\varphi^N),u^N)=\begin{cases}
\dfrac{1}{2}(\kappa(T^N,\varphi^N)D^+(u^N))_{\bar{x}_1}\\
\qquad +\dfrac{1}{2}(\kappa(T^N,\varphi^N)D^-(u^N))_{x_1}, \qquad i=1\\
\dfrac{1}{2}\dfrac{\partial}{\partial x_2}(\kappa(T^N,\varphi^N)D^+(u^N))\\
\qquad +\dfrac{1}{2}\dfrac{\partial}{\partial x_2}(\kappa(T^N,\varphi^N)D^-(u^N)), \qquad i=2
\end{cases}
$$

$$
H_2^{(i)}(\nu(T^N,\varphi^N),u^N)=\begin{cases}
\dfrac{1}{2}(\nu u_{1,x_1}^N)_{\bar{x}_1}+\dfrac{1}{2}(\nu u_{1,\bar{x}_1}^N)_{x_1}+\dfrac{1}{2}\dfrac{\partial}{\partial x_2}(\nu u_{2,x_1}^N)\\
\qquad +\dfrac{1}{2}\dfrac{\partial}{\partial x_2}(\nu u_{2,\bar{x}_1}^N), \qquad i=1\\
\dfrac{1}{2}\Big(\nu\dfrac{\partial u_1^N}{\partial x_2}\Big)_{\bar{x}_1}+\dfrac{1}{2}\Big(\nu\dfrac{\partial u_1^N}{\partial x_2}\Big)_{x_1}+\dfrac{\partial}{\partial x_2}\Big(\nu\dfrac{\partial u_2^N}{\partial x_2}\Big), \quad i=2
\end{cases}
$$

$$
\begin{aligned}
H_3(u^N,T^N,\varphi^N)={}&\frac{1}{2}\nu(T^N,\varphi^N)(T^N)^{-1}S_T^{-1}(T^N,\varphi^N)\sum_{i,j=1}^{2}(D_iu_j^N+D_ju_i^N)^2\\
&-\kappa(T^N,\varphi^N)T^{-1}S_T^{-1}(T^N,\varphi^N)[D(u^N)]^2+\mathrm{e}^{\varphi^N}S_\varphi(T^N,\varphi^N)S_T^{-1}(T^N,\varphi^N)D(u^N).
\end{aligned}
$$

3. Lemmas

Lemma 1[3]**.** *For all* $\eta(x_1,x_2,t)$*, we have* $2(\eta(t),\eta_t(t))_{I^*}=(\|\eta(t)\|^2_{I^*})_t-\tau\|\eta_t(t)\|^2_{I^*}$, $2(\eta(t),\eta_t(t))=(\|\eta(t)\|^2)_t-\tau\|\eta_t(t)\|^2$.

Lemma 2.

$$\begin{aligned}&2(\triangle^a_{x_1}\eta,b\xi)+(a\eta_{x_1},b\xi_{x_1})+(a\eta_{\bar{x}_1},b\xi_{\bar{x}_1})+(a\eta_{x_1},\xi(x_1+h)b_{x_1})\\&\qquad+(a\eta_{\bar{x}_1},\xi(x_1-h)b_{\bar{x}_1})=2D_1(a,b,\eta,\xi).\end{aligned}\tag{3.1}$$

$$\begin{aligned}&2(\triangle^a\eta,b\xi)+\Big(ab,\eta_{x_1}\xi_{x_1}+\eta_{\bar{x}_1}\xi_{\bar{x}_1}+2\frac{\partial\eta}{\partial x_2}\frac{\partial\xi}{\partial x_2}\Big)+\Big(a\frac{\partial\eta}{\partial x_2},\xi\frac{\partial b}{\partial x_2}\Big)\\&\qquad+(a\eta_{x_1},\xi(x_1+h)b_{x_1})+(a\eta_{\bar{x}_1},\xi(x_1-h)b_{\bar{x}_1})=2D_1(a,b,\eta,\xi),\end{aligned}\tag{3.2}$$

where $D_1(a,b,\eta,\xi)=\frac{1}{2}(((a\eta_{\bar{x}_1})(1),b\xi(1-h))_{I^*}+((a\eta_{x_1})(1-h),b\xi(1))_{I^*})-\frac{1}{2}(((a\eta_{\bar{x}_1})(h),$ $b\xi(0))_{I^*}+((a\eta_{x_1})(0),\ b\xi(h))_{I^*})$*. Particularly, if* $a\equiv b\equiv 1$*,* $\eta_t\equiv\xi$*, we have*

$$2(\eta_t,\triangle\eta)+(\|\eta\|_1^2)_t-\tau\|\eta_t\|_1^2=2D_1(1,1,\eta,\eta_t).\tag{3.3}$$

Proof. By Abel's formula $(v_{\bar{x}_1},w)+(v,w_{x_1})=(v(1-h),w(1))_{I^*}-(v(0),w(h))_{I^*}$, let $v=a\eta_{x_1}$, $w=b\xi$ and $v=b\xi$, $w=a\eta_{\bar{x}_1}$ respectively, we have

$$\begin{aligned}&((a\eta_{x_1})_{\bar{x}_1},b\xi)+(a\eta_{x_1},(b\xi)_{x_1})=((a\eta_{x_1})(1-h),b\xi(1))_{I^*}-((a\eta_{x_1})(0),b\xi(h))_{I^*},\\&((b\xi)_{\bar{x}_1},a\eta_{\bar{x}_1})+(b\xi,(a\eta_{\bar{x}_1})_{x_1})=((a\eta_{\bar{x}_1}(1),b\xi(1-h))_{I^*}-((a\eta_{\bar{x}_1})(h),b\xi(0))_{I^*},\end{aligned}$$

it is easy to verify that $(b\xi)_{x_1}=b_{x_1}\xi(x_1+h)+b\xi_{x_1}$, $(b\xi)_{\bar{x}_1}=b\xi_{\bar{x}_1}+b_{\bar{x}_1}\xi(x_1-h)$. On the other hand,

$$\Big(\frac{\partial}{\partial x_2}\Big(a\frac{\partial\eta}{\partial x_2}\Big),b\xi\Big)+\Big(a\frac{\partial\eta}{\partial x_2},b\frac{\partial\xi}{\partial x_2}\Big)+\Big(a\frac{\partial\eta}{\partial x_2},\xi\frac{\partial b}{\partial x_2}\Big)=0.$$

Therefore, by the definitions of $\triangle^a_{x_1}\eta$ and $\triangle^a\eta$, we can complete the proof of this lemma.

Lemma 3[11]**.** *If* $0\le\bar{\mu}\le\beta$ *and* $\eta\in H^\beta(I^*)$*, then* $\|P_N\eta-\eta\|_{H^{\bar{\mu}}(I^*)}\le cN^{\bar{\mu}-\beta}\|\eta\|_{H^\beta(I^*)}$, $\|P_N\eta\|_{H^{\bar{\mu}}(I^*)}\le c\|\eta\|_{H^{\bar{\mu}}(I^*)}$.

Lemma 4. *There exists a positive constant* C_0 *independent of* h *and* N*, such that for any* $\eta\in V_N$*,*

(i) $\|\eta\|^2_{0,\infty,I_h}\le c_0h^{-1}\|\eta\|^2_{I_h}$, $\quad\|\eta\|^2_{0,\infty,I^*}\le c_0N\|\eta\|^2_{I^*}$.

(ii) $\|\eta\|^2_{0,\infty}\le c_0h^{-1}N\|\eta\|^2$,

Proof. (i) is easy to verify, we only prove (ii). Suppose for $x\in\Omega_h$, $\forall\eta\in V_N$, $\eta(x)=\sum\limits_{|j|\le N}\eta_j(x_1)\mathrm{e}^{ijx_2}$, thus

$$\begin{aligned}\|\eta\|_{0,\infty}&\le\sum_{|j|\le N}\|\eta_j(x_1)\|_{0,\infty,I_h}\le c_0h^{-\frac{1}{2}}\sum_{|j|\le N}\|\eta_j\|_{I_h}\\&\le c_0h^{-\frac{1}{2}}\Big(\sum_{|j|\le N}\|\eta_j\|^2_{I_h}\Big)^{\frac{1}{2}}\Big(\sum_{\||j|\le N}1\Big)^{\frac{1}{2}}\le c_0h^{-\frac{1}{2}}N^{\frac{1}{2}}\|\eta\|.\end{aligned}$$

Lemma 5[8]**.** *If* $\eta \in V_N$ *for* $x_1 \in I_h$, *then* $\left\|\dfrac{\partial \eta}{\partial x_2}\right\|^2 \le N^2\|\eta\|^2$.

Lemma 6[3]**.** *For all* $\eta(x_1,x_2)$, *we have*

$$\|\eta_{\bar{x}_1}\|^2 \le \frac{4}{h^2}\|\eta\|^2 + h\|\eta_{x_1}(0)\|^2_{I^*}, \quad \|\eta_{x_1}\|^2 \le \frac{4}{h^2}\|\eta\|^2 + h\|\eta_{\bar{x}_1}(1)\|^2_{I^*}$$

and

$$\|\eta_{\bar{x}_1}\|^2 \le \frac{4}{h^2}\|\eta\|^2 + \frac{2}{h}\|\eta(0)\|^2_{I^*}, \quad \|\eta_{x_1}\|^2 \le \frac{4}{h^2}\|\eta\|^2 + \frac{2}{h}\|\eta(1)\|^2_{I^*}.$$

Lemma 7[3]**.** *If* $h < \varepsilon$, *then for all* $x_1 \in I_h$, $\|\eta(x_1)\|^2_{I^*} \le \varepsilon(\|\eta_{x_1}\|^2 + \|\eta_{\bar{x}_1}\|^2) + c_0(\varepsilon)\|\eta\|^2$, *where* $c_0(\varepsilon)$ *is a positive constant depending only on* ε *and the domain* Ω_h.

Lemma 8[8]**.** *If* $\eta(x_1,x_2,t)$, $\xi(x_1,x_2,t) \in V_N$ *for all* $x_1 \in I_h$, *then*

$$\begin{aligned}
&\|\eta(x_1)\xi(x_1)\|^2_{I^*} \le (2N+1)\|\eta(x_1)\|^2_{I^*}\|\xi(x_1)\|^2_{I^*},\\
&\|\eta(x_2)\xi(x_2)\|^2_{I_h} \le \frac{1}{h}\|\eta(x_2)\|^2_{I_h}\|\xi(x_2)\|^2_{I_h},\\
&\|\eta\xi\|^2 \le \frac{2N+1}{h}\|\eta\|^2\|\xi\|^2.
\end{aligned}$$

Lemma 9[3]**.** $\dfrac{1}{2}(\nu, \eta^2_{x_1} + \eta^2_{\bar{x}_1}) \le \dfrac{8\max\nu}{h^2}\cdot\|\eta\|^2 + \dfrac{\max\nu}{h}\cdot\|\eta\|^2_{\Gamma_{1h}}$.

Lemma 10[3]**.** *If the following conditions are fulfilled:*

(1) η *is a non-negative function defined on* Θ_τ, ρ_0, Q_0, $a_l(h,N)$ *and* Y_l, $0 \le l \le m$, *are non-negative constants;*

(2) $\rho(t) = \rho(\eta(0),\eta(\tau),\cdots,\eta(t-\tau))$ *satisfies that* $\rho(t) \le \rho_0$ *for all* $\eta(t') \le Y_0/a_0(h,N)$, $t' = 0,\tau,\cdots,t-\tau$;

(3) $H_\eta(t) = \eta(t)[Y(\eta(t)) + a_0(h,N)Q(\eta(t))\eta(t)] + \sum_{l=1}^{m}\xi_l(\eta(t))$, *where* $Y(\eta(t)) \le Y_0$ *and* $Q(\eta(t)) \le Q_0$ *for all* $\eta(t) \le Y_0/a_0(h,N)$; *and* $\xi_l(\eta(t)) \le 0$ *for all* $\eta(t) \le Y_l/a_l(h,N)$, $1 \le l \le m$;

(4) $G_\eta(t) = G(\eta(t),\eta(t-\tau)) \ge \eta(t)$;

(5) $\eta(0) \le \rho(0) \le \rho_0$, *and*

$$G_\eta(t) \le \rho(t) + \tau\sum_{t'=0}^{t-\tau} H_\eta(t'), \quad \forall t \in \Theta_\tau;$$

(6) $\rho_0 e^{(1+Q_0)Y_0 t_0} \le \min\limits_{0\le l\le m}(Y_l/a_l(h,N))$;

Then we have for all $t \in \Theta_\tau$ *and* $t \le t_0$ *that* $\eta(t) \le \rho_0 e^{(1+Q_0)Y_0 t}$.

4. Error Estimation

Suppose that $\tau = O(h^2)$, $\tau = O\Big(\dfrac{1}{N^2}\Big)$. Let $\tilde{f}_i$ $(i=1,2)$, $\tilde{f}_3$, $\tilde{f}_4$ be the errors of f_i $(i=1,2)$ and the right ends of the last two equations of (2.1) respectively, which induce the errors of u_i^N, T^N and φ^N denoted by $\tilde{u}_i^N$, $\tilde{T}^N$ and $\tilde{\varphi}^N$. Then the errors satisfy the following equations:

$$\begin{cases} \tilde{L}_1^{(i)}(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) = \tilde{u}^N_{i,t} - \sum\limits_{j=1}^{7} \tilde{F}_j^{(i)} + \sum\limits_{j=1}^{3} \tilde{E}_j^{(i)} = P_N \tilde{f}_i, \quad i=1,2, \\ \tilde{L}_2(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) = \tilde{T}^N_t - \tilde{F}_8 - \tilde{F}_9 + \sum\limits_{j=4}^{8} \tilde{E}_j = P_N\tilde{f}_3, \\ \tilde{L}_3(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) = \tilde{\varphi}^N_t - \tilde{F}_{10} + \tilde{E}_9 = P_N\tilde{f}_4, \end{cases} \tag{4.1}$$

where

$$\begin{aligned}
&\tilde{F}_1^{(i)} = P_N \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N} H_1^{(i)}(\kappa(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N),\tilde{u}^N),\\
&\tilde{F}_2^{(i)} = P_N \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N} H_1^{(i)}(\kappa(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)-\kappa(T^N,\varphi^N),u^N),\\
&\tilde{F}_3^{(i)} = P_N \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N} \triangle^{\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)}\tilde{u}_i^N,\\
&\tilde{F}_4^{(i)} = P_N \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N} (\triangle^{\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)}u_i^N - \triangle^{\nu(T^N,\varphi^N)}u_i^N),\\
&\tilde{F}_5^{(i)} = P_N \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N} H_2^{(i)}(\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N),\tilde{u}^N),\\
&\tilde{F}_6^{(i)} = P_N \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N} H_2^{(i)}(\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)-\nu(T^N,\varphi^N),u^N),\\
&\tilde{F}_7^{(i)} = -RP_N D_i\tilde{\varphi}^N T^N,\\
&\tilde{F}_8 = P_N \mathrm{e}^{\varphi^N-\tilde{\varphi}^N}(T^N+\tilde{T}^N)^{-1}S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)\triangle^{\mu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)}\tilde{T}^N,\\
&\tilde{F}_9 = P_N \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(T^N+\tilde{T}^N)^{-1}S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)\\
&\qquad \cdot\{\triangle^{\mu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)}T^N - \triangle^{\mu(T^N,\varphi^N)}T^N\}\\
&\tilde{F}_{10} = -P_N d^{(\frac{1}{2})}(\tilde{\varphi}^N,u^N),\\
&\tilde{E}_1^{(i)} = P_N d^{(\frac{1}{2})}(\tilde{u}_i^N,u^N+\tilde{u}^N) + P_N d^{(\frac{1}{2})}(u_i^N,\tilde{u}^N)\\
&\qquad -\frac{1}{2}P_N(u_i^N+\tilde{u}_i^N)D(\tilde{u}^N) - \frac{1}{2}P_N\tilde{u}_i^N D(u^N),\\
&\tilde{E}_2^{(i)} = P_N(\mathrm{e}^{-\varphi^N}-\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N})[H_1^{(i)}(\kappa(T^N,\varphi^N),u^N) + \triangle^{\nu(T^N,\varphi^N)}u_i^N\\
&\qquad + H_2^{(i)}(\nu(T^N,\varphi^N),u^N)],\\
&\tilde{E}_3^{(i)} = RP_N D\tilde{T}^N + RP_N\tilde{T}^N D_i\varphi^N + RP_N\tilde{T}^N D_i\tilde{\varphi}^N,\\
&\tilde{E}_4 = P_N d^{(0)}(\tilde{T}^N,u^N+\tilde{u}^N) + P_N d^{(0)}(T^N,\tilde{u}^N) - P_N(T^N+\tilde{T}^N)D(\tilde{u}^N) - P_N\tilde{T}^N D(u^N),\\
&\tilde{E}_5 = P_N(\mathrm{e}^{-\varphi^N}-\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N})(T^N+\tilde{T}^N)^{-1}S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)\triangle^{\mu(T^N,\varphi^N)}T^N,\\
&\tilde{E}_6 = -P_N\mathrm{e}^{-\varphi^N}[(T^N+\tilde{T}^N)^{-1}-(T^N)^{-1}]S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)\triangle^{\mu(T^N,\varphi^N)}T^N,\\
&\tilde{E}_7 = -P_N\mathrm{e}^{-\varphi^N}(T^N)^{-1}[S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)-S_T^{-1}(T^N,\varphi^N)]\triangle^{\mu(T^N,\varphi^N)}T^N,\\
&\tilde{E}_8 = P_N\mathrm{e}^{-\varphi^N}H_3(u^N,T^N,\varphi^N) - P_N\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}H_3(u^N+\tilde{u}^N,T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N),\\
&\tilde{E}_9 = P_N d^{(\frac{1}{2})}(\varphi^N,\tilde{u}^N) + P_N d^{(\frac{1}{2})}(\tilde{\varphi}^N,\tilde{u}^N) - P_N\Big[\frac{1}{2}(\varphi^N+\tilde{\varphi}^N)-1\Big]D(\tilde{u}) + \frac{1}{2}P_N\tilde{\varphi}^N D(u^N).
\end{aligned}$$

In order to estimate the errors, we consider the following identity

$$2\sum_{i=1}^{2}(\tilde{u}_i^N+\tau\tilde{u}_{i,t}^N,\tilde{L}_1^{(i)}(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N)-P_N\tilde{f}_i)+2(\tilde{T}^N+\tau\tilde{T}_t^N,\tilde{L}_2(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N)-P_N\tilde{f}_3)$$
$$+2(\tilde{\varphi}^N+\tau\tilde{\varphi}_t^N,\tilde{L}_3(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N)-P_N\tilde{f}_4)=0. \tag{4.2}$$

By Lemma 1, Lemma 3 and ε-inequality $ab\le\varepsilon a^2+\dfrac{b^2}{4\varepsilon}$, we get that

$$(\|\tilde{u}^N\|^2+\|\tilde{T}^N\|^2+\|\tilde{\varphi}^N\|^2)_t+\tau(1-\varepsilon)(\|\tilde{u}_t^N\|^2+\|\tilde{T}_t^N\|^2+\|\tilde{\varphi}_t^N\|^2)$$
$$+2\sum_{i=1}^{2}\Big(\tilde{u}_i^N+\tau\tilde{u}_{i,t}^N,-\sum_{j=1}^{7}\tilde{F}_j^{(i)}+\sum_{j=1}^{3}\tilde{E}_j^{(i)}\Big)+2\Big(\tilde{T}^N+\tau\tilde{T}_t^N,-\tilde{F}_8-\tilde{F}_9+\sum_{j=4}^{8}\tilde{E}_j\Big)$$
$$+2(\tilde{\varphi}^N+\tau\tilde{\varphi}_t^N,-\tilde{F}_{10}+\tilde{E}_9)\le\|\tilde{u}^N\|^2+\|\tilde{T}^N\|^2+\|\tilde{\varphi}^N\|^2+c\Big(1+\frac{\tau}{\varepsilon}\Big)\sum_{m=1}^{4}\|\tilde{f}_m\|^2, \tag{4.3}$$

where ε is a suitably small constant.

Suppose that u, T and φ are sufficiently smooth, $B_0<T<B_1$ and $|\varphi|<B_2$, then we have $B_0<T^N<B_1$ and $|\varphi^N|<B_2$, if h^{-1} and N are large enough. Thus we conclude from Lemma 4 that there exists a suitably small constant $\tilde{B}>0$, such that $B_0<T^N+\tilde{T}^N<B_1$ and $|\varphi^N+\tilde{\varphi}^N|<B_2$, for all $\tilde{T}^N$ and $\tilde{\varphi}^N$ satisfying that

$$\|\tilde{T}^N\|\le\tilde{B}h^{\frac12}N^{-\frac12},\quad\|\tilde{\varphi}^N\|\le\tilde{B}h^{\frac12}N^{-\frac12},$$
$$\|\tilde{T}^N\|_{\Gamma_{1h}}\le\tilde{B}N^{-\frac12},\quad\|\tilde{\varphi}^N\|_{\Gamma_{1h}}\le\tilde{B}N^{-\frac12}. \tag{4.4}$$

Let M be a positive constant independent of h, N and τ, but may depend on ε, κ_1, ν_1, Φ_1, s_1 and some Sobolev space norms of u^N, T^N, φ^N, M may be different in different formula. Let $\lambda=c\tau(h^{-2}+N^2)$. and $\tilde{A}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)=\min(2\kappa(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)+3\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)),\ \nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N))$, Under conditions (1.5), (4.4), By Abel's formula and integrating by parts as well as Lemmas 2-9, we can prove that

$$(\|\tilde{u}^N\|^2+\|\tilde{T}^N\|^2+\|\tilde{\varphi}^N\|^2)_t+\tau\Big(\frac58-4\varepsilon\Big)(\|\tilde{u}_t^N\|^2+\|\tilde{T}_t^N\|^2+\|\tilde{\varphi}_t^N\|^2)$$
$$+\Big(\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}\tilde{A}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)-8\varepsilon-Mh^{-1}N(\|\tilde{u}^N\|^2+\|\tilde{\varphi}^N\|^2+\|\tilde{T}^N\|^2),$$
$$\sum_{i=1}^{2}\Big((\tilde{u}_{i,x_1}^N)^2+(\tilde{u}_{i,\bar{x}_1}^N)^2+2\Big(\frac{\partial\tilde{u}_i^N}{\partial x_2}\Big)^2\Big)\Big)$$
$$+\Big(\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}\mu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(T^N+\tilde{T}^N)^{-1}$$
$$\cdot S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)-4\varepsilon-Mh^{-1}N(\|\tilde{u}^N\|^2+\|\tilde{\varphi}^N\|^2+\|\tilde{T}^N\|^2),(\tilde{T}_{x_1}^N)^2$$
$$+(\tilde{T}_{\bar{x}_1}^N)^2+2\Big(\frac{\partial\tilde{T}^N}{\partial x_2}\Big)^2\Big)-64\lambda\Phi_1\mu_1^2B_0^{-2}S_0^{-2}\|\tilde{T}^N\|_1^2$$

$$- 128\lambda\Phi_1(\kappa_1^2+\nu_1^2)\|\tilde{u}^N\|_1^2 + \sum_{l=1}^{4} B_l^*(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) \le \tilde{R}(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N), \tag{4.5}$$

where

$$\begin{aligned}
B_1^*(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& (1-\varepsilon\mathrm{sign}\kappa)h^{-1}(\kappa(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(0)\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(h),\\
&(\tilde{u}_1^N(h))^2)_{I^*} + (1-\varepsilon\mathrm{sign}\kappa)h^{-1}(\kappa(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(1)\\
&\cdot \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(1-h),(\tilde{u}_1^N(1-h))^2)_{I^*} - \varepsilon\|\tilde{u}^N\|_1^2 - M\varepsilon^{-1}\|\tilde{u}^N\|^2\\
&- M\Big(h^{-1}\|\tilde{u}^N\|_{\Gamma_{1h}}^2 + h\Big\|\frac{\partial\tilde{u}^N}{\partial x_2}\Big\|_{\Gamma_{1h}}^2),\\
B_2^*(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& (1-\varepsilon)h^{-1}(\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(0)\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(h),(\tilde{u}_1^N(h))^2\\
&+(\tilde{u}_2^N(h))^2)_{I^*} + (1-\varepsilon)h^{-1}(\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(1)\\
&\cdot \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(1-h),(\tilde{u}_1^N(1-h))^2+(\tilde{u}_2^N(1-h))^2)_{I^*} - \varepsilon\|\tilde{u}^N\|_1^2\\
&- M\varepsilon^{-1}\|\tilde{u}^N\|^2 - M\Big(h^{-1}\|\tilde{u}^N\|_{\Gamma_{1h}}^2 + h\Big\|\frac{\partial\tilde{u}^N}{\partial x_2}\Big\|_{\Gamma_{1h}}^2\Big),\\
B_3^*(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& (1-\varepsilon)h^{-1}(\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(0)\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(h),(\tilde{u}_1^N(h))^2)_{I^*}\\
&+(1-\varepsilon)h^{-1}(\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(1)\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(1-h),\\
&(\tilde{u}_1^N(1-h))^2)_{I^*} - \varepsilon\|\tilde{u}^N\|_1^2 - M\varepsilon^{-1}\|\tilde{u}^N\|^2\\
&- \Big(h^{-1}\|\tilde{u}^N\|_{\Gamma_{1h}}^2 + h\Big\|\frac{\partial\tilde{u}^N}{\partial x_2}\Big\|_{\Gamma_{1h}}^2\Big),\\
B_4^*(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& (1-\varepsilon)h^{-1}(\mu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(0)\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(h)\\
&\cdot (T^N+\tilde{T}^N)^{-1}S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(h),(\tilde{T}^N(h))^2)_{I^*}\\
&+(1-\varepsilon)h^{-1}(\mu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(1)\cdot\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(1-h)\\
&\cdot (T^N+\tilde{T}^N)^{-1}S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(h),(\tilde{T}^N(1-h))^2)_{I^*}\\
&- \varepsilon\|\tilde{T}^N\|_1^2 - M\varepsilon^{-1}\|\tilde{T}^N\|^2 - M\Big(h^{-1}\|\tilde{T}^N\|_{\Gamma_{1h}}^2 + h\Big\|\frac{\partial\tilde{T}^N}{\partial x_2}\Big\|_{\Gamma_{1h}}^2\Big),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{R}(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& c\Big(1+\frac{\tau}{\varepsilon}\Big)\sum_{l=1}^{4}\|\tilde{f}_l\|^2 + M(h^{-2}N^2\|\tilde{u}^N\|^2 + h^{-2}N^2\|\tilde{T}^N\|^2 + h^{-2}N^2\|\tilde{\varphi}^N\|^2\\
&+ h^{-2}N\|\tilde{\varphi}^N\|_{\Gamma_{1h}}^2 + 1)(\|\tilde{u}^N\|^2 + \|\tilde{T}^N\|^2 + \|\tilde{\varphi}^N\|^2 + h\|\tilde{u}^N\|_{\Gamma_{1h}}^2)\\
&+ Mh\Big\|\frac{\partial\tilde{u}^N}{\partial x_2}\Big\|_{\Gamma_{1h}}^2 + Mh^{-1}(\|\tilde{\varphi}^N\tilde{u}^N\|_{\Gamma_{1h}}^2 + \|\tilde{T}^N\tilde{u}^N\|_{\Gamma_{1h}}^2 + \|\tilde{\varphi}^N\tilde{T}^N\|_{\Gamma_{1h}}^2\\
&+ \|(\tilde{T}^N)^2\|_{\Gamma_{1h}}^2) + Mh(\|\tilde{u}^N\|_{\Gamma_{1h}}^2 + \|\tilde{T}^N\|_{\Gamma_{1h}}^2 + \|\tilde{\varphi}^N\|_{\Gamma_{1h}}^2).
\end{aligned}$$

Suppose that the following condition holds

$$\lambda < \frac{1}{128}\min\Big(\frac{A_0\Phi_0}{2\Phi_1^2(\kappa_1^2+\nu_1^2)}, \frac{\mu_0\Phi_0B_0^2S_0^2}{\Phi_1^2\mu_1^2B_1S_1}\Big), \tag{4.6}$$

Define

$$
\begin{aligned}
Z_a(\eta^N,t) &= \|\eta^N(t)\|^2 + \frac{\tau}{2}\sum_{t'=0}^{t-\tau}\Big(a,(\eta^N_{x_1})^2(t') + (\eta^N_{\bar{x}_1})^2(t') + 2\Big(\frac{\partial \eta^N(t')}{\partial x_2}\Big)^2\Big) \\
&\quad + \frac{\tau^2}{4}\sum_{t'=0}^{t-\tau}\|\eta^N_t(t')\|^2, \\
\tilde{G}(t) &= Z_{\frac{\Phi_0 A_0}{2}}(\tilde{u}^N,t) + Z_{\frac{\Phi_0\mu_0 B_1^{-1}S_1^{-1}}{2}}(\tilde{T}^N,t) + Z_0(\tilde{\varphi}^N,t).
\end{aligned}
$$

By summing up (4.5) for $t' \le t-\tau$, and $t' \in \Theta_\tau$, then

$$
\tilde{G}(t) \le \rho(t) + \tau\sum_{t'=0}^{t-\tau}\Big\{\tilde{R}^*(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) + \sum_{m=1}^{3}\xi_m(\tilde{u}^N(t'),\tilde{T}^N(t'),\tilde{\varphi}^N(t'))\Big\},
$$

where

$$
\begin{aligned}
\rho(t) =& \|\tilde{u}^N(0)\|^2 + \|\tilde{T}^N(0)\|^2 + \|\tilde{\varphi}^N(0)\|^2 + \tau\sum_{t'=0}^{t-\tau}\Big\{c\Big(1+\frac{\tau}{\varepsilon}\Big)\sum_{l=1}^{4}\|\tilde{f}_l\|^2 \\
&+ Mh^{-1}(\|\tilde{\varphi}^N\tilde{u}^N\|^2_{\Gamma_{1h}} + \|\tilde{T}^N\tilde{u}^N\|^2_{\Gamma_{1h}} + \|\tilde{\varphi}^N\tilde{T}^N\|^2_{\Gamma_{1h}} + \|(\tilde{T}^N)^2\|^2_{\Gamma_{1h}}) \\
&+ Mh(\|\tilde{u}^N\|^2_{\Gamma_{1h}} + \|\tilde{T}^N\|^2_{\Gamma_{1h}} + \|\tilde{\varphi}^N\|^2_{\Gamma_{1h}}) + M\Big(h^{-1}\|\tilde{u}^N\|^2_{\Gamma_{1h}} + h\Big\|\frac{\partial \tilde{u}^N}{\partial x_2}\Big\|^2_{\Gamma_{1h}}\Big) \\
&+ M\Big(h^{-1}\|\tilde{T}^N\|^2_{\Gamma_{1h}} + h\Big\|\frac{\partial \tilde{T}^N}{\partial x_2}\Big\|^2_{\Gamma_{1h}}\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}^*(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& M(h^{-2}N^2\|\tilde{u}^N\|^2 + h^{-2}N^2\|\tilde{T}^N\|^2 + h^{-2}N^2\|\tilde{\varphi}^N\|^2 \\
&+ h^{-2}N\|\tilde{\varphi}^N\|^2_{\Gamma_{1h}} + 1)(\|\tilde{u}^N\|^2 + \|\tilde{T}^N\|^2 + \|\tilde{\varphi}^N\|^2 + h\|\tilde{u}^N\|^2_{\Gamma_{1h}}), \\
\xi_1(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& -\Big(\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}\tilde{A}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N) - \frac{1}{2}\Phi_0 A_0 \\
&- 128\lambda\Phi_1^2(\kappa_1^2+\nu_1^2) - 8\varepsilon - Mh^{-1}N(\|\tilde{u}^N\|^2 + \|\tilde{\varphi}^N\|^2 + \|\tilde{T}^N\|^2), \\
&\sum_{i=1}^{2}\Big(\Big((\tilde{u}^N_{i,x_1})^2 + (\tilde{u}^N_{i,\bar{x}_2})^2 + 2\Big(\frac{\partial \tilde{u}^N_i}{\partial x_2}\Big)^2\Big)\Big), \\
\xi_2(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& -\Big(\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}\mu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(T^N+\tilde{T}^N)^{-1} \\
&\cdot S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N) - \frac{1}{2}\Phi_0\nu_0 B_1^{-1}S_1^{-1} \\
&- 64\lambda\Phi_1^2\nu_1^2 B_0^{-2}S_0^{-2} - 4\varepsilon - Mh^{-1}N(\|\tilde{u}^N\|^2 + \|\tilde{\varphi}^N\|^2 \\
&+ \|\tilde{T}^N\|^2), (\tilde{T}^N_{x_1})^2 + (\tilde{T}^N_{\bar{x}_1})^2 + 2\Big(\frac{\partial \tilde{T}^N}{\partial x_2}\Big)^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
\xi_3(\tilde{u}^N,\tilde{T}^N,\tilde{\varphi}^N) =& -[(1-\varepsilon)h^{-1}((\kappa(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N) \\
&+ 2\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(0)\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(h),(\tilde{u}^N_1(h))^2)_{I^*} \\
&+ (1-\varepsilon)h^{-1}((\kappa(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N) + 2\nu(T^N+\tilde{t}^N,\varphi^N+\tilde{\varphi}^N)(1)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\, \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(1-h),(\tilde{u}_1^N(1-h))^2)_{I^*}+(1-\varepsilon)h^{-1}(\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(0)\\
&\cdot\, \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(h),(\tilde{u}_2^N(h))^2)_{I^*}+(1-\varepsilon)h^{-1}(\nu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(1)\\
&\cdot\, \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(1-h),(\tilde{u}_2^N(1-h))^2)_{I^*}+(1-\varepsilon)h^{-1}(\mu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(0)\\
&\cdot\, \mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(h)(T^N+\tilde{T}^N)^{-1}S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(h),(\tilde{T}^N(h))^2)_{I^*}\\
&+(1-\varepsilon)h^{-1}(\mu(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(1)\mathrm{e}^{-\varphi^N-\tilde{\varphi}^N}(1-h)\\
&\cdot\,(T^N+\tilde{T}^N)^{-1}S_T^{-1}(T^N+\tilde{T}^N,\varphi^N+\tilde{\varphi}^N)(h),(\tilde{T}^N(1-h))^2)_{I^*}].
\end{aligned}
$$

By applying Lemma 10, we get the following theorem,

Theorem 1. *If the following conditions are fulfilled:*

(1) $\tau = O(h^2)$, $\tau = O\Big(\dfrac{1}{N^2}\Big)$, *conditions* (1.5), (4.6) *hold and* $B_0 < T < B_1$, $|\varphi| < B_2$;

(2) $\|\tilde{u}^N\|_{\Gamma_{1h}} \le Mh^{3/2}N^{-1}$, $\|\tilde{T}^N\|_{\Gamma_{1h}} \le Mh^{3/2}N^{-1}$, $\|\tilde{\varphi}^N\|_{\Gamma_{1h}} \le MhN^{-1/2}$, $\Big\|\dfrac{\partial \tilde{u}^N}{\partial x_2}\Big\|_{\Gamma_{1h}} \le Mh^{1/2}N^{-1}$, $\Big\|\dfrac{\partial \tilde{T}^N}{\partial x_2}\Big\|_{\Gamma_{1h}} \le Mh^{1/2}N^{-1}$,

(3) *for all* $t' \le t_0$, $t' \in \Theta_\tau$, $\rho(t') \le Mh^2N^{-2}$,

then for all $t \in \Theta_\tau$, $\tilde{G}(t) \le M\rho(t)e^{Mt}$.

To discuss the convergence, for simplicity of expressions, for non-negative integers m and n, let

$$H^m(I,H^n(I^*)) = \Big\{\eta\Big|\frac{\partial^{p+q}\eta}{\partial x_1^p \partial x_2^q} \in L^2(\Omega),\ \ 0 \le p \le m,\ 0 \le q \le n\Big\}.$$

For any real numbers α and β, $H^\alpha(H^\beta(I),I^*)$ is defined by the interpolation of spaces with the norm $\|\cdot\|_{H^\alpha(I,H^\beta(I^*))}$. Similarly we can define the space $C^m(I,H^\alpha(I^*))$. Furthermore

$$C^m(0,t_1;H^\alpha(\Omega)) = \Big\{\eta\Big|\frac{\partial^p\eta}{\partial t^p} \in H^\alpha(\Omega) \text{ is continuous in } \ t\in(0,t_1],\ 0\le p\le m\Big\}$$

equipped with the norm $|||\cdot|||_{H^\alpha(I,H^\beta(I^*))}$. Similarly, we can define the space $C^m(0,t_1;H^\alpha(I,H^\beta(I^*))$ with the norm $|||\cdot|||_{H^\alpha(I,H^\beta(I^*))}$.

By Lemma 3 and embedding theorem as well as same arguments as in the proof of theorem 1, we can prove the following convergence theorem:

Theorem 2. *Let* $r>0$, $\alpha>0$ *and the condition* (1) *of theorem* 1 *holds, assume that the solution* (u,T,φ) *of* (1.4) *satisfies that*

$$
\begin{aligned}
&u,T,\varphi \in C(0,t_1;H^{\frac{9}{2}+r}(I,L^2(I^*))\cap H^{\frac{7}{2}+r}(I,H^1(I^*))\cap H^{\frac{3}{2}+r}(I,H^{\alpha+1}(I^*))\\
&\qquad \cap H^{\frac{1}{2}+r}(I,H^\alpha(I^*)))\\
&\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 T}{\partial t^2},\ \frac{\partial^2\varphi}{\partial t^2} \in C(0,t_1;H^{\frac{1}{2}+r}(I,L^2(I^*))),
\end{aligned}
$$

then we have for all $t \le t_0$, $t \in \Theta_\tau$ *that*

$$\|u^N(t)-u(t)\|^2 + \|T^N(t)-T(t)\|^2 + \|\varphi^N(t)-\varphi(t)\|^2 \le M^*(\tau^2+h^4+N^{-2\alpha}).$$

where M^* is a positive constant dependent on the norms appearing in the estimates of terms $\|\tilde{H}_m^{(i)}\|$, $\|\tilde{H}_m\|$.

Acknowledgement The author would like to thank his Ph.D supervisor Prof. Guo Benyu for his guidance while the author was studying at Shanghai university of science and technology.

References

[1] P.J. Roache, Computational Fluid Dynamics, 2nd edition, Hermosa Publishers, Albuquerque, 1976.

[2] Tani. Atusi, The existence and uniqueness of the solution of equations describing compressible viscous fluid flow in a domain, *Proc. Japan Acad.,* **52** (1976), 334–337.

[3] Ben-yu Guo, Difference Methods for Partial Differential Equations, Science Press, Beijing, 1988.

[4] Pen-yu Kuo, Résolution numérique de fluide compressible, *C.R. Acad. Sci. Paris,* **291A** (1980), 167–171.

[5] Ben-yu Guo, Strict error estimation of numerical solution of compressible flow in two-dimensional space, *Sciential Sinica,* **26A** (1983), 482–498.

[6] Ben-yu Guo, Wei-ming Cao, Spectral-finite element method for compressible fluid flows, *M^2AN,* **26** (1992), 469–491.

[7] Ben-yu Guo, He-ping Ma, Strict error estimation for a spectral method of compressible fluid flow, *CalColo,* **24** (1987), 263–282.

[8] Ben-yu Guo, Yue-shan Xiong, A spectral-difference method for two-dimensional viscous flow, *J. of Comput. Phys.,* **84** (1989), 259–278.

[9] Ben-yu Guo, Yue-shan Xiong, Spectral-difference method for solving three-dimensional vorticity equation (II), *Math. Numer. Sinica* **13**(1991), 331–345.

[10] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zhang, Spectral Method in Fluid Dynamics, Springer-Verlag, New York, 1988.

[11] C. Canuto, A. Quarteroni, Approximation result for orthogonal polynomials in Sobolev spaces, *Math. Comp.,* **36**(1982), 67–86.

[12] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.