Journal of Computational Mathematics, Vol.16, No.1, 1998, 51–62.

ASYMPTOTIC ERROR EXPANSION AND DEFECT CORRECTION FOR SOBOLEV AND VISCOELASTICITY TYPE EQUATIONS

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Abstract

In this paper we study the higher accuracy methods − the extrapolation and defect correction for the semidiscrete Galerkin approximations to the solutions of Sobolev and viscoelasticity type equations. The global extrapolation and the correction approximations of third order, rather than the pointwise extrapolation results are presented.

Key words: Asymptotic error, semidiscrete Galerkin approximation, global extrapolation, higher accuracy.

1. Introduction

Let Ω be a rectangular domain. We are concerned with the Richardson extrapolation and defect correction of the finite element approximations to the solutions of the following simple Sobolev type equation

$$\begin{cases} -\triangle u_t - \triangle u = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,y,0) = v(x,y) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

and viscoelasticity type equation

$$\begin{cases} u_{tt} - \triangle u_t - \triangle u = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,y,0) = v(x,y),\ u_t(x,y,0) = w(x,y) & \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

The extrapolation technique used for the finite element approximations to the solutions of the elliptic differential equations has well been investigated in [1-6, 13-17, 21, 23]. And this method has also been considered for boundary element approximations to boundary integral equations (e.g. see [25-26]). Some further investigations of the extrapolation for Galerkin method for parabolic equations have been carried out in [7-9]. Not long ago, multi-parameter parallel algorithms have been introduced into the extrapolation in order to accelerate the computational speeds [27].

* Received June 19, 1995.

Defect correction schemes of the finite element approximations for the elliptic problems have been investigated by some authors, too. For example, see [2,19,20,24].

The problems (1.1) and (1.2) can arise from many physical processes. Many authors have studied both the finite difference and the finite element methods for these problems. Especially, the error estimates of FEM for the problems (1.1) and (1.2) have been deliberated in [11] via the Ritz-Volterra projection. Here, we will use a new analysis in [12], i.e. an analysis for the "short side" in the FE-right triangle together with the sharp integral estimates of the "hypotenuse" to present an immediate analysis for extrapolation and correction for (1.1) and (1.2). Moreover, we obtain global extrapolation results by means of an interpolation postprocessing technique proposed in [18] (or [22]), instead of the pointwise extrapolation results.

2. Sobolev Type Equations

Above all, we discuss the problem (1.1). Throughout the paper, we assume that T^h is a rectangular partition over Ω with mesh size h. The weak form of (1.1) consists in finding $u \in H_0^1(\Omega)$ (the Sobolev space) such that

$$\begin{cases} (\nabla u_t, \nabla\varphi) + (\nabla u, \nabla\varphi) = (f, \varphi) \quad \forall\varphi \in H_0^1, \\ u(0) = v. \end{cases} \tag{2.1}$$

Let $S_0^h \subset H_0^1$ consist of piecewise bilinear functions. Thus, a continuous Galerkin approximation $u^h : [0, T] \to S_0^h$ is defined such that

$$\begin{cases} (\bigtriangledown u_t^h, \bigtriangledown\varphi) + (\bigtriangledown u^h, \bigtriangledown\varphi) = (f, \varphi) \quad \forall\varphi \in S_0^h, \\ u^h(0) = i_h v, \end{cases} \tag{2.2}$$

where $i_h v \in S_0^h$ is the bilinear interpolation function of v. And thus, from (2.1) and (2.2) we get the error equation

$$(\nabla(u_t^h - u_t), \nabla\varphi) + (\nabla(u^h - u), \nabla\varphi) = 0 \quad \forall\varphi \in S_0^h. \tag{2.3}$$

Set $E(x) = \dfrac{1}{2}[(x - x_\tau)^2 - h_\tau^2]$, $F(y) = \dfrac{1}{2}[(y - y_\tau)^2 - k_\tau^2]$, associated with any element $\tau = [x_\tau - h_\tau, x_\tau + h_\tau] \times [y_\tau - k_\tau,\, y_\tau + k_\tau]$ of T^h. Then, there holds by [22] for $\varphi \in S_0^h$,

$$\int_\Omega \nabla(u - i_h u)\nabla\varphi = \int_\Omega \Big\{\Big[F\varphi_x - \frac{1}{3}(F^2)_y\varphi_{xy}\Big]u_{xyy} + \Big[E\varphi_y - \frac{1}{3}(E^2)_x\varphi_{xy}\Big]u_{yxx}\Big\}. \tag{2.4}$$

What is more, for $\varphi \in S_0^h$

$$\int_\Omega \nabla(u - i_h u)\nabla\varphi = -\int_\Omega \Big[F\varphi - \frac{1}{3}(F^2)_y\varphi_y + E\varphi - \frac{1}{3}(E^2)_x\varphi_x\Big]u_{xxyy}. \tag{2.5}$$

Lemma 2.1. *For $\varphi \in S_0^h$,*

$$(\nabla(u - i_h u), \nabla\varphi) = \frac{h^2}{3}\sum_\tau \frac{h_\tau^2 + k_\tau^2}{h^2}\int_\tau \varphi u_{xxyy} + O(h^3)\|u\|_5\|\varphi\|_0.$$

Proof. From (2.5) we have, according to $F = \frac{1}{6}(F^2)_{yy} - \frac{1}{3}k_\tau$ and integration by parts, that

$$\begin{aligned}
-\int_\tau \Big[F\varphi - \frac{1}{3}(F^2)_y\varphi_y\Big]u_{xxyy} &= \frac{k_\tau^2}{3}\int_\tau \varphi u_{xxyy} - \frac{1}{6}\int_\tau (F^2)_{yy}\varphi u_{xxyy} - \frac{1}{3}\int_\tau F^2\varphi_y u_{xxyyy} \\
&= \frac{k_\tau^2}{3}\int_\tau \varphi u_{xxyy} + \frac{1}{6}\int_\tau (F^2)_y(\varphi_y u_{xxyy} + \varphi u_{xxyyy}) - \frac{1}{3}\int_\tau F^2\varphi_y u_{xxyyy} \\
&= \frac{k_\tau^2}{3}\int_\tau \varphi u_{xxyy} - \frac{1}{6}\int_\tau F^2\varphi_y u_{xxyyy} + \frac{1}{6}\int_\tau (F^2)_y\varphi u_{xxyyy} - \frac{1}{3}\int_\tau F^2\varphi_y u_{xxyyy},
\end{aligned}$$

and hence, for $\varphi \in S_0^h$

$$-\int_\Omega \Big[F\varphi - \frac{1}{3}(F^2)_y\varphi_y\Big]u_{xxyy} = \frac{1}{3}\sum_\tau k_\tau^2\int_\tau \varphi u_{xxyy} + O(h^3)\|u\|_5\|\varphi\|_0.$$

Similarly, for $\varphi \in S_0^h$

$$-\int_\Omega \Big[E\varphi - \frac{1}{3}(E^2)_x\varphi_x\Big]u_{xxyy} = \frac{1}{3}\sum_\tau h_\tau^2\int_\tau \varphi u_{xxyy} + O(h^3)\|u\|_5\|\varphi\|_0. \quad Q.E.D.$$

Remark. From (2.4) we derive by the same argument as that of Lemma 2.1

$$(\nabla(u - i_h u), \nabla\varphi) = -\frac{h^2}{3}\sum_\tau\Big(\frac{k_\tau^2}{h^2}\int_\tau \varphi_x u_{xyy} + \frac{h_\tau^2}{h^2}\int_\tau \varphi_y u_{yxx}\Big) + O(h^3)\|u\|_4|\varphi|_1. \quad (2.6)$$

Theorem 2.1. *There holds the asymptotic error expansion, in the sense of H^1- norm, $u^h = i_h u + h^2\omega^h + O(h^3)$, where $\omega \in S_0^h$.*

Proof. Let $\theta = u^h - i_h u$. Then, we have by (2.3) and (2.6), for $\varphi \in S_0^h$

$$\begin{aligned}
(\nabla\theta_t, \nabla\varphi) + (\nabla\theta, \nabla\varphi) =& (\nabla(u_t - i_h u_t), \nabla\varphi) + (\nabla(u - i_h u), \nabla\varphi) \\
=& -\frac{h^2}{3}\sum_\tau\Big[\frac{k_\tau^2}{h^2}\int_\tau (u_{txyy} + u_{xyy})\varphi_x + \frac{h_\tau^2}{h^2}\int_\tau (u_{tyxx} + u_{yxx})\varphi_y\Big] \\
&+ O(h^3)(\|u_t\|_4 + \|u\|_4)|\varphi|_1. \qquad (2.7)
\end{aligned}$$

Let $\omega \in H_0^1(\Omega)\bigcap H^2(\Omega)$ and $\omega^h \in S_0^h$ be the exact solution and the finite element solution of the auxiliary problem, respectively

$$\begin{cases} (\nabla\omega_t, \nabla\varphi) + (\nabla\omega, \nabla\varphi) = L_h(\varphi) \qquad \forall\varphi \in H_0^1(\Omega), \\ \omega(0) = 0, \end{cases} \quad (2.8)$$

where

$$L_h(\varphi) = -\frac{1}{3}\sum_\tau\Big[\frac{k_\tau^2}{h^2}\int_\tau (u_{txyy} + u_{xyy})\varphi_x + \frac{h_\tau^2}{h^2}\int_\tau (u_{tyxx} + u_{yxx})\varphi_y\Big].$$

Therefore, for $\varphi \in S_0^h$

$$(\nabla(\theta_t - h^2\omega_t^h), \nabla\varphi) + (\nabla(\theta - h^2\omega^h), \nabla\varphi) = O(h^3)(\|u_t\|_4 + \|u\|_4)|\varphi|_1. \quad (2.9)$$

Taking $\Theta = \theta - h^2\omega^h$ and $\varphi = \Theta_t$, we get

$$|\Theta_t|_1^2 + \frac{1}{2}\frac{d}{dt}|\Theta|_1^2 \le ch^6(\|u_t\|_4 + \|u\|_4)^2 + |\Theta_t|_1^2$$

or

$$\frac{d}{dt}|\Theta|_1^2 \le ch^6(\|u_t\|_4 + \|u\|_4)^2.$$

By integration with respect to t and $\Theta(0) = 0$, we finally obtain $\|\Theta\|_1 \le ch^3\Big[\int_0^t(\|u_t\|_4 + \|u\|_4)^2 ds\Big]^{1/2}$. Q.E.D.

Lemma 2.2. *If ω and ω^h are the exact solution and the finite element solution of* (2.8)*, respectively, there holds*

$$\|\omega^h - i_h\omega\|_1 \le ch\Big[\int_0^t(\|\omega_t\|_2 + \|\omega\|_2)^2 ds\Big]^{1/2}.$$

Proof. Let $\theta = \omega^h - i_h\omega$. Then

$$(\nabla\theta_t, \nabla\varphi) + (\nabla\theta, \nabla\varphi) = (\nabla(\omega_t - i_h\omega_t), \nabla\varphi) + (\nabla(\omega - i_h\omega), \nabla\varphi),$$

and hence, with $\varphi = \theta_t$ and (2.6)

$$|\theta_t|_1^2 + \frac{1}{2}\frac{d}{dt}|\theta|_1^2 \le ch^2(\|\omega_t\|_2 + \|\omega\|_2)^2 + |\theta_t|_1^2.$$

Then, we have, by integration about t and $\theta(0) = 0$, that $\|\theta\|_1 \le ch\Big[\int_0^t(\|\omega_t\|_2 + \|\omega\|_2)^2 ds\Big]^{1/2}$. Q.E.D.

Now we use an interpolation postprocessing technique to get the global extrapolation approximations of high accuracy. For this end, we assume that T^h has been gained from T^{2h} (or T^{3h}) with mesh size $2h$ (or $3h$) by subdividing each element of T^{2h} (or T^{3h}) into four (or nine) congruent elements. Thus, we can define a Lagrange biquadratic interpolation operator I_{2h}^2 associated with T^{2h} and a Lagrange bicubic interpolation operator I_{3h}^3 associated with T^{3h}. It is easy to check that $I_{2h}^2 i_h = I_{2h}^2$, $I_{3h}^3 i_h = I_{3h}^3$, $\|I_{2h}^2\varphi\|_{r,p} \le c\|\varphi\|_{r,p}$, $\|I_{3h}^3\varphi\|_{r,p} \le c\|\varphi\|_{r,p}$ $\forall\varphi \in S_0^h$, $\|I_{2h}^2\varphi - \varphi\|_{r,p} \le ch^{3-r}\|\varphi\|_{3,p}$, $\|I_{3h}^3\varphi - \varphi\|_{r,p} \le ch^{4-r}\|\varphi\|_{4,p}$, where $r = 0$ or 1 and $p = 2$ or ∞. And thus, we have

Theorem 2.2. *There holds* $\Big\|\frac{1}{3}(4I_{3h/2}^3 u^{h/2} - I_{3h}^3 u^h) - u\Big\|_1 \le c(u)h^3$*, where $u^{h/2}$ is the FE solution of* (1.1) *corresponding to the partition $T^{h/2}$ which is derived by subdividing each element of T^h into four congruent elements and $I_{3h/2}^3$ is a Lagrange bicubic interpolation operator whose definition is similar to that of I_{3h}^3.*

Proof. Denote that $r^h = u^h - i_h u - h^2 i_h\omega$, where ω is the variational solution of (2.8). Then, it follows from Theorem 2.1 and Lemma 2.2 that $\|r^h\|_1 \le c(u)h^3$. And

$$\begin{aligned} I_{3h}^3 u^h - u &= I_{3h}^3(u^h - i_h u) + (I_{3h}^3 u - u) = I_{3h}^3(h^2 i_h\omega + r^h) + (I_{3h}^3 u - u) \\ &= h^2 I_{3h}^3\omega + I_{3h}^3 r^h + (I_{3h}^3 u - u) = h^2\omega + h^2(I_{3h}^3\omega - \omega) + I_{3h}^3 r^h + (I_{3h}^3 u - u) \end{aligned}$$

$$= h^2\omega + q^h, \tag{2.10}$$

where

$$q^h = h^2(I_{3h}^3\omega - \omega) + I_{3h}^3 r^h + (I_{3h}^3 u - u)$$

with $\|q^h\|_1 \leq c(u)h^3$. Therefore, $I_{3h}^3 u^h = u + h^2\omega + O(h^3)$, which yields, in the sense of $H^1 - norm$ that $\dfrac{4I_{3h/2}^3 u^{h/2} - I_{3h}^3 u^h}{3} = u + O(h^3)$. Q.E.D.

Next we turn to the extrapolation in version of maximum norm . For this purpose we introduce the discrete Green function $G_z^h \in S_0^h$ at any point $z = (x, y) \in \bar{\Omega}$ such that, for $\varphi \in S_0^h$, $(\nabla G_z^h, \nabla\varphi) = \varphi(z)$ and $(\nabla D_z G_z^h, \nabla\varphi) = D_z\varphi$. We have the following ([28])

Lemma 2.3. $\|G_z^h\|_0 \leq c$, $\|D_z G_z^h\|_0 \leq c\Big(\log\dfrac{1}{h}\Big)^{1/2}$.

Theorem 2.3. *The following asymptotic error expansions hold for all points in* Ω, $u^h = i_h u + h^2\omega^h + O(h^3)$, $u_t^h = i_h u_t + h^2\omega_t^h + O(h^3)$, *where* $\omega^h \in S_0^h$.

Proof. Setting $\theta = u^h - i_h u$, we get according to Lemma 2.1, for $\varphi \in S_0^h$

$$\begin{aligned}(\nabla\theta_t, \nabla\varphi) + (\nabla\theta, \nabla\varphi) &= (\nabla(u_t - i_h u_t), \nabla\varphi) + (\nabla(u - i_h u), \nabla\varphi)\\ &= \frac{h^2}{3}\sum_\tau \frac{h_\tau^2 + k_\tau^2}{h^2}\int_\tau \varphi(u_{txxyy} + u_{xxyy})\\ &\quad + O(h^3)(\|u_t\|_5 + \|u\|_5)\|\varphi\|_0. \end{aligned}\tag{2.11}$$

Let $\omega \in H_0^1(\Omega)\bigcap H^2(\Omega)$ and $\omega^h \in S_0^h$ be the variational solution and the finite element solution of the auxiliary problem, respectively

$$\begin{cases} (\nabla\omega_t, \nabla\varphi) + (\nabla\omega, \nabla\varphi) = M_h(\varphi) \qquad \forall\varphi \in H_0^1(\Omega),\\ \omega(0) = 0, \end{cases}\tag{2.12}$$

where $M_h(\varphi) = \dfrac{1}{3}\displaystyle\sum_\tau \frac{h_\tau^2 + k_\tau^2}{h^2}\int_\tau (u_{txxyy} + u_{xxyy})\varphi$. And thus, for $\varphi \in S_0^h$

$$(\nabla(\theta_t - h^2\omega_t^h), \nabla\varphi) + (\nabla(\theta - h^2\omega^h), \nabla\varphi) = O(h^3)(\|u_t\|_5 + \|u\|_5)\|\varphi\|_0. \tag{2.13}$$

Denoting $\theta - h^2\omega^h$ by Θ and taking $\varphi = G_z^h$ in (2.13), we obtain by Lemma 2.3

$$\Theta_t(z, t) + \Theta(z, t) \leq ch^3(\|u_t\|_5 + \|u\|_5). \tag{2.14}$$

And hence, with integration about t, there holds due to $\Theta(z, 0) = 0$

$$\Theta_t(z, t) + \int_0^t \Theta(z, s)ds \leq ch^3\int_0^t (\|u_t\|_5 + \|u\|_5)ds$$

or

$$|\Theta_t(z, t)| \leq \int_0^t |\Theta(z, s)|ds + ch^3\int_0^t (\|u_t\|_5 + \|u\|_5)ds.$$

Then, it follows from Gronwall's Lemma that

$$|\Theta_t(z, t)| \leq ch^3\int_0^t (\|u_t\|_5 + \|u\|_5)ds. \tag{2.15}$$

From (2.14) and (2.15) we derive

$$
\begin{aligned}
|\Theta_t(z,t)| &\le |\Theta(z,t)| + ch^3(\|u_t\|_5 + \|u\|_5)\\
&\le ch^3\Big[\|u_t\|_5 + \|u\|_5 + \int_0^t (\|u_t\|_5 + \|u\|_5)ds\Big]. \quad Q.E.D.
\end{aligned}
$$

Theorem 2.4. *There hold, in* $W^{1,\infty}-norm$, $u^h = i_h u + h^2\omega^h + O\Big(h^3\Big(\log\frac{1}{h}\Big)^{1/2}\Big)$ *and* $u_t^h = i_h u_t + h^2\omega_t^h + O\Big(h^3\Big(\log\frac{1}{h}\Big)^{1/2}\Big)$, *where* ω^h *is the FE solution of* (2.12).

Proof. Taking $\varphi = D_z G_z^h$ and $\Theta = \theta - h^2\omega^h$ in (2.13), we have by means of Lemma 2.3

$$
D_z\Theta_t + D_z\Theta \le ch^3\Big(\log\frac{1}{h}\Big)^{1/2}(\|u_t\|_5 + \|u\|_5) \tag{2.16}
$$

or

$$
D_z\Theta_t + \int_0^t D_z\Theta_t(z,s)ds \le ch^3\Big(\log\frac{1}{h}\Big)^{1/2}(\|u_t\|_5 + \|u\|_5).
$$

Hence, it follows from Gronwall's Lemma that

$$
|D_z\Theta_t| \le ch^3\Big(\log\frac{1}{h}\Big)^{1/2}(\|u_t\|_5 + \|u\|_5). \tag{2.17}
$$

and thus, by (2.16) and (2.17)

$$
|D_z\Theta| \le ch^3\Big(\log\frac{1}{h}\Big)^{1/2}(\|u_t\|_5 + \|u\|_5). \quad Q.E.D.
$$

Let $E_1^h u = \frac{1}{3}(4I_h^2 u^{h/2} - I_{2h}^2 u^h)$, $E_2^h u = \frac{1}{3}(4I_{3h/2}^3 u^{h/2} - I_{3h}^3 u^h)$. Then, in terms of Theorem 2.3 and 2.4, we get the following by the same argument as that in Theorem 2.2

Theorem 2.5. *There hold* $\|E_1^h u - u\|_{0,\infty} \le c(u)h^3$ *and* $\|(E_1^h u)_t - u_t\|_{0,\infty} \le c(u)h^3$.

Theorem 2.6. *There hold* $\|E_2^h u - u\|_{1,\infty} \le c(u)h^3\Big(\log\frac{1}{h}\Big)^{1/2}$ *and* $\|(E_2^h u)_t - u_t\|_{1,\infty} \le c(u)h^3\Big(\log\frac{1}{h}\Big)^{1/2}$.

Now we discuss a defect correction scheme proposed in [20] (or [22]) for the problem (1.1). Let P_{1h} be a projection operator $P_{1h} : H_0^1(\Omega) \to S_0^h$ defined by, for $\varphi \in S_0^h$

$$
\begin{cases}
(\nabla((P_{1h}u)_t - u_t), \nabla\varphi) + (\nabla(P_{1h}u - u), \nabla\varphi) = 0,\\
P_{1h}u(0) = i_h u(0).
\end{cases}
$$

Then $P_{1h}u$ is the solution of (2.2) if u is the solution of (1.1).

Set $u_{1*}^h = I_{2h}^2 u^h + u^h - P_{1h}I_{2h}^2 u^h$ and $u_{2*}^h = I_{3h}^3 u^h + u^h - P_{1h}I_{3h}^3 u^h$. Then, we have

Theorem 2.7. *There holds* $\|u_{2*}^h - u\|_1 \le c(u)h^3$.

Proof. Multiplying (2.6) by $(I - P_{1h})$, which I is the identical operator, we get

$$
(I - P_{1h})(I_{3h}^3 u^h - u) = (I - P_{1h})(h^2\omega + q^h) = h^2(\omega - P_{1h}\omega) + (q^h - P_{1h}q^h),
$$

hence

$$
\|(I - P_{1h})(I_{3h}^3 u^h - u)\|_1 \le h^2(\|\omega - i_h\omega\|_1 + \|i_h\omega - P_{1h}\omega\|_1) + c\|q^h\|_1 \le c(u)h^3.
$$

Here, the left hand side is nothing but $(I - P_{1h})(I_{3h}^3 u^h - u) = u_{2*}^h - u$. Q.E.D.

Similarly, we have by means of Theorem 2.3 and 2.4

Theorem 2.8. *There hold* $\|u_{1*}^h - u\|_{0,\infty} \le c(u)h^3$ *and* $\|(u_{1*}^h)_t - u_t\|_{0,\infty} \le c(u)h^3$.

Theorem 2.9. *There hold* $\|u_{2*}^h - u\|_{1,\infty} \le c(u)h^3\Big(\log\dfrac{1}{h}\Big)^{1/2}$ *and* $|(u_{2*}^h)_t - u_t\|_{1,\infty} \le c(u)h^3\Big(\log\dfrac{1}{h}\Big)^{1/2}$.

3. Viscoelasticity Type Equations

In this section, we study the problem (1.2). The weak form of (1.2) reads as follows: Find $u \in H_0^1(\Omega)$ such that

$$\begin{cases} (u_{tt},\varphi) + (\nabla u_t, \nabla\varphi) + (\nabla u, \nabla\varphi) = (f,\varphi) \quad \forall\varphi \in H_0^1, \\ u(0) = v, \quad u_t(0) = w. \end{cases} \tag{3.1}$$

Thus, a semidiscrete Galerkin approximation $u^h : [0,T] \to S_0^h$ is defined such that

$$\begin{cases} (u_{tt}^h,\varphi) + (\nabla u_t^h, \nabla\varphi) + (\nabla u^h, \nabla\varphi) = (f,\varphi) \quad \forall\varphi \in S_0^h, \\ u^h(0) = i_h v, \quad u_t^h(0) = i_h w, \end{cases} \tag{3.2}$$

where $i_h v,\ i_h w \in S_0^h$ stand for the bilinear interpolation functions of v and w, respectively. Then, we obtain the error equation from (3.1) and (3.2)

$$(u_{tt} - u_{tt}^h, \varphi) + (\nabla(u_t - u_t^h), \nabla\varphi) + (\nabla(u - u^h), \nabla\varphi) = 0 \quad \forall\varphi \in S_0^h. \tag{3.3}$$

Lemma 3.1. *For* $\varphi \in S_0^h$,

$$(u_{tt} - i_h u_{tt}, \varphi) = -\frac{h^2}{3}\sum_\tau \Big(\frac{h_\tau^2}{h^2}\int_\tau u_{ttxx}\varphi + \frac{k_\tau^2}{h^2}\int_\tau u_{ttyy}\varphi\Big) + O(h^3)\|u_{tt}\|_3\|\varphi\|_0.$$

Proof. Due to

$$\varphi(x,y) = \varphi(x_\tau,y_\tau) + \varphi_x(x,y_\tau)(x-x_\tau) + \varphi_y(x_\tau,y)(y-y_\tau) + \varphi_{xy}(x-x_\tau)(y-y_\tau),$$

therefore

$$\begin{aligned}\int_\tau (u_{tt} - i_h u_{tt})\varphi = &\int_\tau (u_{tt} - i_h u_{tt})\varphi(x_\tau,y_\tau) + \int_\tau (u_{tt} - i_h u_{tt})E_x\varphi_x(x,y_\tau) \\ &+ \int_\tau (u_{tt} - i_h u_{tt})F_y\varphi_y(x_\tau,y) + \int_\tau (u_{tt} - i_h u_{tt})E_xF_y\varphi_{xy} \\ \equiv &A_1 + A_2 + A_3 + A_4.\end{aligned} \tag{3.4}$$

By [10], there is

$$\int_\Omega (u - i_h u) = \int_\Omega E u_{xx} + \int_\Omega F u_{yy} - \int_\Omega E_xF_y u_{xy},$$

hence

$$A_1 = \int_\tau (u_{tt} - i_h u_{tt})\varphi(x_\tau,y_\tau)$$

$$
\begin{aligned}
&=\varphi(x_\tau,y_\tau)\int_\tau Eu_{ttxx}+\varphi(x_\tau,y_\tau)\int_\tau Fu_{ttyy}-\varphi(x_\tau,y_\tau)\int_\tau E_xF_yu_{ttxy}\\
&=-\frac{h_\tau^2}{3}\int_\tau u_{ttxx}\varphi(x_\tau,y_\tau)+\frac{1}{6}\int_\tau(E^2)_{xx}u_{ttxx}\varphi(x_\tau,y_\tau)-\frac{k_\tau^2}{3}\int_\tau u_{ttyy}\varphi(x_\tau,y_\tau)\\
&\quad+\frac{1}{6}\int_\tau(F^2)_{yy}u_{ttyy}\varphi(x_\tau,y_\tau)+\int_\tau EF_yu_{ttxxy}\varphi(x_\tau,y_\tau)\\
&\equiv B_1+B_2+B_3+B_4+B_5,
\end{aligned}
$$

where

$$
\begin{aligned}
B_1&=-\frac{h_\tau^2}{3}\int_\tau u_{ttxx}\varphi(x_\tau,y_\tau)=-\frac{h_\tau^2}{3}\int_\tau u_{ttxx}[\varphi(x,y)-E_x\varphi_x-F_y\varphi_y+E_xF_y\varphi_{xy}]\\
&=-\frac{h_\tau^2}{3}\int_\tau u_{ttxx}\varphi-\frac{h_\tau^2}{3}\int_\tau u_{ttxxx}E\varphi_x-\frac{h_\tau^2}{3}\int_\tau u_{ttxxy}F\varphi_y+\frac{h_\tau^2}{3}\int_\tau u_{ttxxx}EF_y\varphi_{xy},\\
B_3&=-\frac{k_\tau^2}{3}\int_\tau u_{ttyy}\varphi(x_\tau,y_\tau)=-\frac{k_\tau^2}{3}\int_\tau u_{ttyy}\varphi-\frac{k_\tau^2}{3}\int_\tau u_{ttyyy}F\varphi_y-\frac{k_\tau^2}{3}\int_\tau u_{ttyyx}E\varphi_x\\
&\quad+\frac{k_\tau^2}{3}\int_\tau u_{ttyyy}E_xF\varphi_{xy},\\
B_2&=\frac{1}{6}\int_\tau(E^2)_{xx}u_{ttxx}\varphi(x_\tau,y_\tau)=-\frac{1}{6}\int_\tau(E^2)_xu_{ttxxx}[\varphi(x,y)-E_x\varphi_x-F_y\varphi_y+E_xF_y\varphi_{xy}],\\
B_4&=\frac{1}{6}\int_\tau(F^2)_{yy}u_{ttyy}\varphi(x_\tau,y_\tau)=-\frac{1}{6}\int_\tau(F^2)_yu_{ttyyy}[\varphi(x,y)-E_x\varphi_x-F_y\varphi_y+E_xF_y\varphi_{xy}],\\
B_5&=\int_\tau EF_yu_{ttxxy}\varphi(x_\tau,y_\tau)=\int_\tau EF_yu_{ttxxy}[\varphi(x,y)-E_x\varphi_x-F_y\varphi_y+E_xF_y\varphi_{xy}],
\end{aligned}
$$

and hence, there is according to the inverse estimates

$$
A_1=-\frac{h_\tau^2}{3}\int_\tau u_{ttxx}\varphi-\frac{k_\tau^2}{3}\int_\tau u_{ttyy}\varphi+O(h^3)\|u_{tt}\|_{3,\tau}\|\varphi\|_{0,\tau}. \tag{3.5}
$$

By the same way

$$
\begin{aligned}
A_2&=\int_\tau(u_{tt}-i_hu_{tt})E_x\varphi_x(x,y_\tau)=-\int_\tau E(u_{tt}-i_hu_{tt})_x\varphi_x(x,y_\tau)\\
&=\frac{h_\tau^2}{3}\int_\tau(u_{tt}-i_hu_{tt})_x\varphi_x(x,y_\tau)-\frac{1}{6}\int_\tau(E^2)_{xx}(u_{tt}-i_hu_{tt})_x[\varphi_x(x,y)-F_y\varphi_{xy}]\\
&=\frac{h_\tau^2}{3}\int_\tau Fu_{ttxyy}\varphi_x(x,y_\tau)-\frac{1}{6}\int_\tau E^2u_{ttxxx}(\varphi_x-F_y\varphi_{xy})=O(h^3)\|u_{tt}\|_{3,\tau}\|\varphi\|_{0,\tau}.
\end{aligned} \tag{3.6}
$$

Analogously

$$
A_3=\int_\tau(u_{tt}-i_hu_{tt})F_y\varphi_y(x_\tau,y)=O(h^3)\|u_{tt}\|_{3,\tau}\|\varphi\|_{0,\tau}. \tag{3.7}
$$

Notice that $\int_\tau(u_{tt}-i_hu_{tt})_{xy}=0$. Then, we have

$$
A_4=\int_\tau(u_{tt}-i_hu_{tt})E_xF_y\varphi_{xy}=\int_\tau(u_{tt}-i_hu_{tt})_{xy}EF\varphi_{xy}
$$

$$= -\frac{1}{36}\int_\tau (E^2)_x(F^2)_{yy}u_{ttxxy}\varphi_{xy} + \frac{k_\tau^2}{18}\int_\tau (E^2)_x u_{ttxxy}\varphi_{xy} + \frac{h_\tau^2}{18}\int_\tau (F^2)_y u_{ttxyy}\varphi_{xy}$$
$$= O(h^3)\|u_{tt}\|_{3,\tau}\|\varphi\|_{0,\tau}. \tag{3.8}$$

(3.4)−(3.8) complete the argument of Lemma 3.2. Q.E.D.

Theorem 3.1. *There holds the asymptotic error expansion, in H^1-norm, $u^h = i_h u + h^2\alpha^h + O(h^3)$, where $\alpha^h \in S_0^h$.*

Proof. Let $\theta = u^h - i_h u$. Then, for $\varphi \in S_0^h$, we have according to (3.3), Lemma 2.1 and 3.2

$$\begin{aligned}(\theta_{tt},\varphi) &+ (\nabla\theta_t,\nabla\varphi) + (\nabla\theta,\nabla\varphi) = (u_{tt} - i_h u_{tt},\varphi)\\ &+ (\nabla(u_t - i_h u_t),\nabla\varphi) + (\nabla(u - i_h u),\nabla\varphi)\\ &= -\frac{h^2}{3}\sum_\tau \Big(\frac{h_\tau^2}{h^2}\int_\tau u_{ttxx}\varphi + \frac{k_\tau^2}{h^2}\int_\tau u_{ttyy}\varphi\Big) + O(h^3)\|u_{tt}\|_3\|\varphi\|_0\\ &+ \frac{h^2}{3}\sum_\tau \frac{h_\tau^2 + k_\tau^2}{h^2}\int_\tau (u_{txxyy} + u_{xxyy})\varphi + O(h^3)(\|u_t\|_5 + \|u\|_5)\|\varphi\|_0.\end{aligned} \tag{3.9}$$

Let $\alpha \in H_0^1(\Omega)\bigcup H^2(\Omega)$ and $\alpha^h \in S_0^h$ be the variational solution and the finite element solution of the auxiliary problem, respectively

$$\begin{cases}(\alpha_{tt},\varphi) + (\nabla\alpha_t,\nabla\varphi) + (\nabla\alpha,\nabla\varphi) = N_h(\varphi) \quad \forall\varphi \in H_0^1(\Omega),\\ \alpha(0) = 0, \quad \alpha_t(0) = 0,\end{cases} \tag{3.10}$$

where

$$N_h(\varphi) = -\frac{1}{3}\sum_\tau \Big(\frac{h_\tau^2}{h^2}\int_\tau u_{ttxx}\varphi + \frac{k_\tau^2}{h^2}\int_\tau u_{ttyy}\varphi\Big) + \frac{1}{3}\sum_\tau \frac{h_\tau^2 + k_\tau^2}{h^2}\int_\tau (u_{txxyy} + u_{xxyy})\varphi.$$

And hence, with $\Theta = \theta - h^2\alpha^h$, for $\varphi \in S_0^h$

$$(\Theta_{tt},\varphi) + (\nabla\Theta_t,\nabla\varphi) + (\nabla\Theta,\nabla\varphi) = O(h^3)(\|u_{tt}\|_3 + \|u_t\|_5 + \|u\|_5)\|\varphi\|_0. \tag{3.11}$$

Taking $\varphi = \Theta_t$ in (3.11), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta_t\|_0^2 + c\|\Theta_t\|_1^2 + \frac{1}{2}\frac{d}{dt}|\Theta|_1^2 &\le \frac{1}{2}\frac{d}{dt}\|\Theta_t\|_0^2 + |\Theta_t|_1^2 + \frac{1}{2}\frac{d}{dt}|\Theta|_1^2\\ &\le ch^6(\|u_{tt}\|_3 + \|u_t\|_5 + \|u\|_5)^2 + \frac{c}{2}\|\Theta_t\|_0^2\end{aligned}$$

or

$$\frac{d}{dt}(\|\Theta_t\|_0^2 + |\Theta|_1^2) \le ch^6(\|u_{tt}\|_3 + \|u_t\|_5 + \|u\|_5)^2.$$

By integration with respect to t and noticing that $\Theta(0) = 0$, we have $\|\Theta_t\|_0^2 + |\Theta|_1^2 \le ch^6\int_0^t(\|u_{tt}\|_3+\|u_t\|_5+\|u\|_5)^2ds$, or $\|\Theta_t\|_0+|\Theta|_1 \le ch^3\Big[\int_0^t(\|u_{tt}\|_3+\|u_t\|_5+\|u\|_5)^2ds\Big]^{1/2}$. Q.E.D.

Theorem 3.2. *There hold $u_t^h = i_h u_t + h^2\alpha_t^h + O(h^3)$ in H^1-norm and $u_{tt}^h = i_h u_{tt} + h^2\alpha_{tt}^h + O(h^3)$ in L^2-norm, where α^h is the FE solution of* (3.10).

Proof. Differentiating (3.9) about t, we obtain by Lemma 2.1, 3.2 and (3.10)

$$\begin{aligned}(\theta_{ttt},\varphi)&+(\nabla\theta_{tt},\nabla\varphi)+(\nabla\theta_t,\nabla\varphi)=(u_{ttt}-i_hu_{ttt},\varphi)\\&+(\nabla(u_{tt}-i_hu_{tt}),\nabla\varphi)+(\nabla(u_t-i_hu_t),\nabla\varphi)\\&=-\frac{h^2}{3}\sum_\tau\Big(\frac{h_\tau^2}{h^2}\int_\tau u_{tttxx}\varphi+\frac{k_\tau^2}{h^2}\int_\tau u_{tttyy}\varphi\Big)+O(h^3)\|u_{ttt}\|_3\|\varphi\|_0\\&+\frac{h^2}{3}\sum_\tau\frac{h_\tau^2+k_\tau^2}{h^2}\int_\tau(u_{ttxxyy}+u_{txxyy})\varphi+O(h^3)(\|u_{tt}\|_5+\|u_t\|_5)\|\varphi\|_0\\&=(h^2\alpha^h_{ttt},\varphi)+(h^2\nabla\alpha^h_{tt},\nabla\varphi)+(h^2\nabla\alpha^h_t,\nabla\varphi)\\&+O(h^3)(\|u_{ttt}\|_3+\|u_{tt}\|_5+\|u_t\|_5)\|\varphi\|_0,\end{aligned}$$

and hence, with $\Theta=\theta-h^2\alpha^h$, for $\varphi\in S_0^h$

$$(\Theta_{ttt},\varphi)+(\nabla\Theta_{tt},\nabla\varphi)+(\nabla\Theta_t,\nabla\varphi)=O(h^3)(\|u_{ttt}\|_3+\|u_{tt}\|_5+\|u_t\|_5)\|\varphi\|_0.$$

Taking $\varphi=\Theta_{tt}$, we have

$$\begin{aligned}\frac12\frac{d}{dt}\|\Theta_{tt}\|_0^2+c\|\Theta_{tt}\|_1^2+\frac12\frac{d}{dt}|\Theta_t|_1^2&\le\frac12\frac{d}{dt}\|\Theta_{tt}\|_0^2+|\Theta_{tt}|_1^2+\frac12\frac{d}{dt}|\Theta_t|_1^2\\&\le ch^6(\|u_{ttt}\|_3+\|u_{tt}\|_5+\|u_t\|_5)^2+\frac{c}{2}\|\Theta_{tt}\|_0^2,\end{aligned}$$

or

$$\frac{d}{dt}(\|\Theta_{tt}\|_0^2+|\Theta_t|_1^2)\le ch^6(\|u_{ttt}\|_3+\|u_{tt}\|_5+\|u_t\|_5)^2.$$

Therefore, there holds according to $\Theta_t(0)=0$

$$\|\Theta_{tt}\|_0^2+|\Theta_t|_1^2\le\|\Theta_{tt}(0)\|_0^2+ch^6\int_0^t(\|u_{ttt}\|_3+\|u_{tt}\|_5+\|u_t\|_5)^2ds. \tag{3.12}$$

Let $t=0$ and $\varphi=\Theta_{tt}(0)$ in (3.11). Then, $\|\Theta_{tt}(0)\|_0\le ch^3(\|u_{tt}(0)\|_3+\|u_t(0)\|_5+\|u(0)\|_5)$ which, together with (3.12), leads to Theorem 3.2. Q.E.D.

Theorem 3.3. *There hold, for all points in* Ω, $u^h=i_hu+h^2\alpha^h+O(h^3)$ *and* $u_t^h=i_hu_t+h^2\alpha_t^h+O(h^3)$, *where* α^h *is the FE solution of* (3.10).

Proof. Setting $\varphi=G_z^h$ in (3.11), we get according to Lemma 2.3 that $(\Theta_{tt},G_z^h)+\Theta_t(z,t)+\Theta(z,t)\le ch^3(\|u_{tt}\|_3+\|u_t\|_5+\|u\|_5)$, or $\Theta_t(z,t)+\int_0^t\Theta_t(z,s)ds\le c\|\Theta_{tt}\|_0+ch^3(\|u_{tt}\|_3+\|u_t\|_5+\|u\|_5)$. And thus, we complete the theorem by Theorem 3.2 and Gronwall' Lemma. Q.E.D.

Theorem 3.4. *There hold, in* $W^{1,\infty}-$ *norm,* $u^h=i_hu+h^2\alpha^h+O(h^3(\log\frac1h)^{1/2})$ *and* $u_t^h=i_hu_t+h^2\alpha_t^h+O(h^3(\log\frac1h)^{1/2})$, *where* α^h *is the FE solution of* (3.10).

Proof. Taking $\varphi=D_zG_z^h$ in (3.11), we obtain by means of Lemma 2.3

$$(\Theta_{tt},D_zG_z^h)+D_z\Theta_t(z,t)+D_z\Theta(z,t)\le ch^3\Big(\log\frac1h\Big)^{1/2}(\|u_{tt}\|_3+\|u_t\|_5+\|u\|_5),$$

or

$$|D_z\Theta_t(z,t)|\le\int_0^t|D_z\Theta_t(z,s)|ds+c\Big(\log\frac1h\Big)^{1/2}\|\Theta_{tt}\|_0$$

$$+ ch^3\Big(\log\frac{1}{h}\Big)^{1/2}(\|u_{tt}\|_3 + \|u_t\|_5 + \|u\|_5).$$

And hence, Theorem 3.4 follows Theorem 3.2 and Gronwall' Lemma. Q.E.D.

Analogous with Section 2, we have the following main according to Theorem 3.1 −3.4

Theorem 3.5. *There hold* $\|E_2^h u - u\|_1 \le c(u)h^3$ *and* $\|(E_2^h u)_t - u_t\|_1 \le c(u)h^3$.

Theorem 3.6. *There holds* $\|(E_1^h u)_{tt} - u_{tt}\|_0 \le c(u)h^3$.

Theorem 3.7. *There hold* $\|E_1^h u - u\|_{0,\infty} \le c(u)h^3$ *and* $\|(E_1^h u)_t - u_t\|_{0,\infty} \le c(u)h^3$.

Theorem 3.8. *There hold* $\|E_2^h u - u\|_{1,\infty} \le c(u)h^3\Big(\log\frac{1}{h}\Big)^{1/2}$ *and* $\|(E_2^h u)_t - u_t\|_{1,\infty} \le c(u)h^3\Big(\log\frac{1}{h}\Big)^{1/2}$.

Also we apply the correction techniques in Section 2 to the problem (1.2). For this purpose we introduce a projection operator

$$P_{2h} : H_0^1(\Omega) \to S_0^h$$

defined by

$$\begin{cases} ((P_{2h}u)_{tt} - u_{tt}, \varphi) + (\nabla((P_{2h}u)_t - u_t), \nabla\varphi) + (\nabla(P_{2h}u - u), \nabla\varphi) = 0 \quad \forall\varphi \in S_0^h, \\ P_{2h}u(0) = i_h u(0), \quad (P_{2h}u)_t(0) = i_h u_t(0). \end{cases}$$

Denote that $u_{1*}^h = I_{2h}^2 u^h + u^h - P_{2h}I_{2h}^2 u^h$, $u_{2*}^h = I_{3h}^3 u^h + u^h - P_{2h}I_{3h}^3 u^h$. Then, we have in terms of Theorem 3.1−3.4

Theorem 3.9. *There hold* $\|u_{2*}^h - u\|_1 \le c(u)h^3$ *and* $|(u_{2*}^h)_t - u_t\|_1 \le c(u)h^3$.

Theorem 3.10. *There holds* $\|(u_{1*}^h)_{tt} - u_{tt}\|_0 \le c(u)h^3$.

Theorem 3.11. *There hold* $\|u_{1*}^h - u\|_{0,\infty} \le c(u)h^3$ *and* $\|(u_{1*}^h)_t - u_t\|_{0,\infty} \le c(u)h^3$.

Theorem 3.12. *There hold* $\|u_{2*}^h - u\|_{1,\infty} \le c(u)h^3(\log\frac{1}{h})^{1/2}$ *and* $\|(u_{2*}^h)_t - u_t\|_{1,\infty} \le c(u)h^3(\log\frac{1}{h})^{1/2}$.

Remark. When Ω is a convex quadrilateral domain, there are the corresponding global extrapolation and correction results for (1.1) and (1.2) if the quadrilateral meshes are almost uniform and are constructed by connecting the equi-proportional points of two opposite boundaries (see [22]).

Acknowledgement The work in the FEMs of Sobolev and viscoelastic equations done by Professor Y.P. Lin aroused our interest in such problems. The authors would like to thank Professor C.M. Chen, Prof. Q.D. Zhu and Dr. Y.Q. Huang whose comments improved the version of the paper.

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