Journal of Computational Mathematics, Vol.16, No.1, 1998, 81–96.

SOME SUPERCONVERGENCE RESULTS OF WILSON-LIKE ELEMENTS*

Lin Zhang
(*Institute of Mathematics, Fudan University, Shanghai, China. Department Mathematics, Shangdong Mining Institute, Taian, China*)
Li-kang LI
(*Department of Mathematics, Fudan University, Shanghai, China*)

Abstract

In this paper, the superconvergence of a class of Wilson-like elements is considered. A superconvergent estimate on the centers of elements and some superconvergence recoveries on the four vertices and the four midpoints of edges of elements are also obtained for piecewise strongly regular quadrilateral subdivisions.
Key words: Superconvergence, Wilson-like element.

1. Introduction

The superconvergence of finite element methods is an important property both in theory and in practice. Many superconvergence results about conforming finite element methods have been obtained (see [4] [17]). But there are also many nonconforming finite element methods in computational practice, in addition to conforming ones. Do the superconvergence results still hold for those nonconforming finite element methods ? The Wilson element is one of the most important nonconforming finite element. [14] first studied the superconvergence property of Wilson element, and obtained the superconvergent estimate of the gradient error on the centers of elements, on an average sense. [1] and [12] sharpened this result, and obtained the pointwise superconvergent estimate. But, as we know, Wilson element may cause divergence for the arbitrary quadrilateral meshes (see [5] [10]), and the results obtained in [14], [1] and [12] only hold for rectangular meshes, and only for the partial differential equations which do not involve mixed derivative terms. [3] and [11] presented a class of so-called Wilson-like nonconforming elements, which converge for arbitrary quadrilateral meshes. [15] studied the superconvergence property of those Wilson-like elements, and obtained a superconvergent estimate of gradient error on the centers of elements, on the average sense. In this paper, we study the pointwise superconvergence property of those

* Received February 8, 1996.

1) The research was supported by the Doctoral Point Foundation of China Universities and by State Major Key Project for Basic Research of China.

Wilson-like elements, and prove that Wilson-like elements, if the nonconforming basis functions are even polynomials, i.e., $b_i = 0$ $(1 \leq i \leq n)$ (see Theorem 4.1), have the superconvergence at the centers of elements, at the four vertices and the midpoints of four edges of elements, provided that the quadrilateral subdivision is piecewise strongly regular. In the last section, some numerical examples are presented to illustrate the theoretical results and the necessity of the condition $b_i = 0$ $(1 \leq i \leq n)$.

2. Basic Notions

Consider the following Dirichlet problem of a second order elliptic equation,

$$\begin{cases} -\sum_{i,j=1}^{2} D_j(\alpha_{ij} D_i u) + \beta u = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^2$ is a convex polygonal domain, $\partial\Omega$ is the boundary of Ω, $f \in L^2(\Omega)$, functions α_{ij}, β are sufficiently smooth, α_{ij} satisfy the ellipticity condition and $\beta \geq 0$. The partial differential operators $D_j (j = 1, 2)$ mean $D_1 = \dfrac{\partial}{\partial x}$, $D_2 = \dfrac{\partial}{\partial y}$ respectively.

The variational problem of the equation (2.1) is: Find $u \in H_0^1(\Omega)$ so that

$$a(u, v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u, v) = \int_\Omega \Big(\sum_{i,j=1}^{2} \alpha_{ij} D_i u D_j v + \beta u v \Big) dxdy, \quad (f, v) = \int_\Omega f v dxdy.$$

Applying the definition of strongly regular subdivision (see Lesaint and Zlámal [6], Zhu and Lin [17], and Zlámal [18]), similar to the definition of piecewise strongly regular triangulation (see Lin and Lü [7], Lin and Xu [8]), we define the piecewise strongly regular quadrilateral subdivision by (also see Zhang and Li [16])

Definition. *A quadrilateral subdivision T_h on Ω is called piecewise strongly regular subdivision, if Ω is divided into finite quadrilateral subdomains Ω_i $(1 \leq i \leq N)$ without inner vertices, and the subdivision restricted on each Ω_i is strongly regular.*

$\forall K \in T_h$, let $A_i(x_i, y_i)(1 \leq i \leq 4)$ denote the four vertices of the element K, $A(x_0, y_0)$ be the center. h_K means the diameter of K, i.e., $h_K = \operatorname{diam}(K)$, $h = \max_{K \in T_h} h_K$. In this paper, C means a generic constant independent of K and h, and may have different values at different places. The notations of Sobolev spaces and their norms used in this paper are as the same as those in Ciarlet [2].

Let $\widehat{K} = [-1, 1] \times [-1, 1]$ be the reference square having the vertices $\widehat{A}_i(1 \leq i \leq 4)$. For every element K, there exists a unique one-to-one mapping $F_K : \widehat{K} \longrightarrow K$, given by $x = \sum_{i=1}^{4} x_i \widehat{N}_i(\xi, \eta)$, $y = \sum_{i=1}^{4} y_i \widehat{N}_i(\xi, \eta)$, where $\widehat{N}_i(\xi, \eta) = (1 \pm \xi)(1 \pm \eta)/4$ $(1 \leq i \leq 4)$ are the bilinear shape functions on $\widehat{K}$.

Every function $\widehat{v}$ defined on $\widehat{K}$, through mapping F_K, corresponds a unique function v defined on K: $v = \widehat{v} \circ F_K^{-1}$, or written as $\widehat{v} = v \circ F_K$. Besides, let $|J|$ denote the Jacobian determinant of the mapping F_K.

Let us introduce some notations and results from [11]. Define

$$\Phi = \Big\{\varphi \in C_0^1([-1,1]) \cap C^2([-1,1]) | \varphi''(0) = 1,\ \int_{-1}^{1} \varphi(t)dt = 0\Big\}.$$

Functions in the space Φ satisfy (see Lemma 2.1 in [11])

$$\int_{-1}^{1} \varphi'(t)dt = 0, \quad \int_{-1}^{1} t\varphi'(t)dt = 0, \quad \forall\ \varphi \in \Phi, \tag{2.4}$$

and let P be the set of all polynomials, then $\forall v \in \Phi \cap P$, v has the following form (see Theorem 4.1 in [11])

$$v(t) = \sum_{i=1}^{m} a_i(t^{2i} - 1) + t\sum_{i=1}^{n} b_i(t^{2i} - 1),$$

where, $b_i(1 \le i \le n)$ are arbitrary real numbers, $m \ge 2$ and n are arbitrary positive whole numbers, $a_i(1 \le i \le m)$ satisfy

$$a_1 = \frac{1}{2},\ \sum_{i=2}^{m} \frac{ia_i}{2i+1} = -\frac{1}{6}. \tag{2.5}$$

On reference element $\widehat{K}$, the finite element $(\widehat{K}, \widehat{\sum}, \widehat{P})$ is defined as: $\forall \widehat{v} \in \widehat{P}$, $\widehat{v}$ is uniquely determined by the set of 6 degrees of freedom, $\widehat{\sum} = \{\widehat{v}(\widehat{A}_i),\ 1 \le i \le 4,$ $\dfrac{\partial^2 \widehat{v}}{\partial \xi^2}(0,0),\ \dfrac{\partial^2 \widehat{v}}{\partial \eta^2}(0,0)\}$, that is,

$$\widehat{v}(\xi,\eta) = \sum_{i=1}^{4} \widehat{v}(\widehat{A}_i)\widehat{N}_i(\xi,\eta) + \frac{\partial^2 \widehat{v}}{\partial \xi^2}(0,0)\widehat{\psi}_1(\xi,\eta) + \frac{\partial^2 \widehat{v}}{\partial \eta^2}(0,0)\widehat{\psi}_2(\xi,\eta), \tag{2.6}$$

in which $\widehat{\psi}_1(\xi,\eta) = \widehat{\psi}(\xi)$, $\widehat{\psi}_2(\xi,\eta) = \widehat{\psi}(\eta)$, $\widehat{\psi} \in \Phi$.

Define the finite element space as:

$$P(K) = \{v_h | v_h \circ F_K \in \widehat{P}, \forall K \in T_h\},$$

$$V_h = \{v_h | v_h|_K \in P(K), \forall K \in T_h,\ v_h \text{ vanishes at boundary nodes}\}.$$

The Wilson-like approximation $u_h \in V_h$ of the variational problem (2.2) is defined by the following variational equation

$$a_h(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h, \tag{2.7}$$

where

$$a_h(u_h, v_h) = \sum_{K \in T_h} \int_K \Big(\sum_{i,j=1}^{2} \alpha_{ij} D_i u_h D_j v_h + \beta u_h v_h\Big) dxdy.$$

In view of (2.6), u_h has the following decomposition

$$u_h = w_h + z_h, \tag{2.8}$$

where w_h is the conforming part, and z_h is the nonconforming part.

Let W_h be the isoparametric bilinear finite element space, then the isoparametric bilinear approximation $u_h^* \in W_h$ is defined by the following variational equation

$$a(u_h^*, v_h) = (f, v_h), \qquad \forall v_h \in W_h. \tag{2.9}$$

3. Superconvergence on Centers of Elements

Lemma 3.1. *It holds that*

$$|a_h(z_h, v_h)| \le C \sum_{K \in T_h} (h|z_h|_{1,K}|v_h|_{1,K} + |z_h|_{0,K}|v_h|_{0,K}), \quad \forall v_h \in W_h. \tag{3.1}$$

Proof. See the formula (3.51) in [15].

Lemma 3.2. *The following estimate holds*

$$\|w_h - u_h^*\|_{1,\infty} \le Ch^2 \ln \frac{1}{h} \|u_h\|_{2,\infty,h}.$$

Proof. Let $p \in \bar{\Omega}$ be arbitrary. Define the derivative Green's function $\partial G_h^p \in W_h$ by (see [17])

$$a(v_h, \partial G_h^p) = \partial v_h(p), \qquad \forall v_h \in W_h, \tag{3.2}$$

where ∂ means the directional derivative operation $\partial = \dfrac{\partial}{\partial x}$ or $\partial = \dfrac{\partial}{\partial y}$.

Since $W_h \subset V_h$, and by (2.7), (2.9) and the decomposition (2.8), it follows that

$$a_h(u_h^* - w_h, v_h) = a_h(z_h, v_h), \qquad \forall v_h \in W_h. \tag{3.3}$$

By (3.2), (3.3) and Lemma 3.1, we have that

$$\begin{aligned} \partial(u_h^* - w_h)(p) &= a(u_h^* - w_h, \partial G_h^p) = a_h(z_h, \partial G_h^p) \\ &\le C \sum_{K \in T_h} (h|z_h|_{1,K}|\partial G_h^p|_{1,K} + |z_h|_{0,K}|\partial G_h^p|_{0,K}). \end{aligned} \tag{3.4}$$

Since $z_h = u_h - w_h$, and w_h is the piecewise isoparametric bilinear interpolation of u_h, we have that, by standard interpolation estimates and the inverse estimate,

$$|z_h|_{1,K} = |u_h - w_h|_{1,K} \le Ch|u_h|_{2,K} \le Ch^2|u_h|_{2,\infty,K}, \tag{3.5}$$

$$|z_h|_{0,K} = |u_h - w_h|_{0,K} \le Ch^2|u_h|_{2,K} \le Ch^3|u_h|_{2,\infty,K}, \tag{3.6}$$

$$|\partial G_h^p|_{1,K} \le Ch^{-1}|\partial G_h^p|_{1,1,K}, \quad |\partial G_h^p|_{0,K} \le Ch^{-1}|\partial G_h^p|_{0,1,K}. \tag{3.7}$$

By (3.5)–(3.7), we obtain that, from (3.4),

$$\partial(u_h^* - w_h)(p) \le Ch^2 |u_h|_{2,\infty,h} \|\partial G_h^p\|_{1,1}. \tag{3.8}$$

Thus, by noting that (see [17]) $\|\partial G_h^p\|_{1,1} \le C \ln \frac{1}{h}$, we obtain

$$|u_h^* - w_h|_{1,\infty,h} \le ch^2 \ln \frac{1}{h} |u_h|_{2,\infty,h}.$$

The L^∞ estimate can be similarly obtained. □

Lemma 3.3. *$\forall K \in T_h$, we have*

$$\Big|\frac{\partial^2 \widehat{u}_h}{\partial \xi^2}(0,0)\Big| \le Ch^2(\|u_h\|_{1,\infty,K} + |f|_{0,\infty}), \tag{3.9}$$

$$\Big|\frac{\partial^2 \widehat{u}_h}{\partial \eta^2}(0,0)\Big| \le Ch^2(\|u_h\|_{1,\infty,K} + |f|_{0,\infty}). \tag{3.10}$$

Proof. In (2.7), setting

$$v_h = \begin{cases} \psi_1^K = \widehat{\psi}_1 \circ F_K^{-1}, & \text{in} \quad K, \\ 0, & \text{in} \quad \Omega - K, \end{cases}$$

we get

$$\int_K \Big(\sum_{i,j=1}^2 \alpha_{ij} D_i u_h \cdot D_j \psi_1^K + \beta u_h \cdot \psi_1^K \Big) dxdy = \int_K f \cdot \psi_1^K dxdy.$$

By Taylor's expansion formula, we get

$$\alpha_{ij} = \alpha_{ij}(A_0) + O(h)|\alpha_{ij}|_{1,\infty,K},$$

thus

$$\sum_{i,j=1}^2 \alpha_{ij}(A_0) \int_K D_i u_h \cdot D_j \psi_1^K dxdy = -\sum_{i,j=1}^2 \int_K O(h)|\alpha_{ij}|_{1,\infty,K} D_i u_h D_j \psi_1^K dxdy$$
$$- \int_K \beta u_h \psi_1^K dxdy + \int_K f \psi_1^K dxdy. \tag{3.11}$$

Obviously we know that

$$\Big| \sum_{i,j=1}^2 \int_K O(h) \Big|\alpha_{ij}|_{1,\infty,K} D_i u_h D_j \psi_1^K dxdy| \le Ch^2 |u_h|_{1,\infty,K}, \tag{3.12}$$

$$\Big| \int_K \beta u_h \psi_1^K dxdy \Big| \le Ch^2 |u_h|_{0,\infty,K}, \tag{3.13}$$

$$\Big| \int_K f \psi_1^K dxdy \Big| \le Ch^2 |f|_{0,\infty,K}. \tag{3.14}$$

Hence, we focus our attention on the left hand side of (3.11).

$$
\begin{aligned}
&\sum_{i,j=1}^{2} \alpha_{ij}(A_0)\int_K D_i u_h D_j \psi_1^K dxdy \\
=&\alpha_{11}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \xi}\frac{\partial \widehat{\psi}_1}{\partial \xi}\Big(\frac{\partial \xi}{\partial x}\Big)^2 |J| d\xi d\eta + \alpha_{12}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \xi}\frac{\partial \widehat{\psi}_1}{\partial \xi}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}|J| d\xi d\eta \\
&+ \alpha_{21}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \xi}\frac{\partial \widehat{\psi}_1}{\partial \xi}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}|J| d\xi d\eta + \alpha_{22}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \xi}\frac{\partial \widehat{\psi}_1}{\partial \xi}\Big(\frac{\partial \xi}{\partial y}\Big)^2 |J| d\xi d\eta \\
&+ \alpha_{11}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \eta}\frac{\partial \widehat{\psi}_1}{\partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x}|J| d\xi d\eta + \alpha_{12}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \eta}\frac{\partial \widehat{\psi}_1}{\partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}|J| d\xi d\eta \\
&+ \alpha_{21}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \eta}\frac{\partial \widehat{\psi}_1}{\partial \xi}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x}|J| d\xi d\eta + \alpha_{22}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \eta}\frac{\partial \widehat{\psi}_1}{\partial \xi}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y}|J| d\xi d\eta,
\end{aligned}
\tag{3.15}
$$

and by the decomposition (2.8) and properties (2.4) of the nonconforming basis functions, we have

$$
\begin{aligned}
\alpha_{11}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \xi}\frac{\partial \widehat{\psi}_1}{\partial \xi}\Big(\frac{\partial \xi}{\partial x}\Big)^2 |J| d\xi d\eta =&\alpha_{11}(A_0)\int_{\widehat{K}} O(h)\frac{\partial \widehat{u}_h}{\partial \xi}\frac{\partial \widehat{\psi}_1}{\partial \xi} d\xi d\eta, \\
&+ \alpha_{11}(A_0)\Big(\frac{\partial y}{\partial \eta}\Big)^2_{A_0} |J|^{-1}_{A_0}\frac{\partial^2 \widehat{u}_h}{\partial \xi^2}(0,0)\int_{\widehat{K}}\Big(\frac{\partial \widehat{\psi}_1}{\partial \xi}\Big)^2 d\xi d\eta
\end{aligned}
\tag{3.16}
$$

and we also have

$$
\alpha_{11}(A_0)\int_{\widehat{K}} \frac{\partial \widehat{u}_h}{\partial \eta}\frac{\partial \widehat{\psi}_1}{\partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x}|J| d\xi d\eta = \alpha_{11}(A_0)\int_{\widehat{K}} O(h)\frac{\partial \widehat{u}_h}{\partial \eta}\frac{\partial \widehat{\psi}_1}{\partial \xi} d\xi d\eta, \tag{3.17}
$$

Similarly deal with the other six terms on the right hand side of (3.15), and by (3.12)–(3.17), we obtain that, from (3.11)

$$
\begin{aligned}
&\Big|\Big(\alpha_{11}(A_0)\Big(\frac{\partial y}{\partial \eta}\Big)^2_{A_0} - \alpha_{12}(A_0)\Big(\frac{\partial y}{\partial \eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial \eta}\Big)_{A_0} - \alpha_{21}(A_0)\Big(\frac{\partial y}{\partial \eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial \eta}\Big)_{A_0} \\
&\qquad + \alpha_{22}(A_0)\Big(\frac{\partial x}{\partial \eta}\Big)^2_{A_0}\Big)|J|^{-1}_{A_0}\frac{\partial^2 \widehat{u}_h}{\partial \xi^2}(0,0)\int_{\widehat{K}}\Big(\frac{\partial \widehat{\psi}_1}{\partial \xi}\Big)^2 d\xi d\eta\Big| \\
&\qquad \le Ch^2(|u_h|_{1,\infty,K} + |u_h|_{0,\infty,K} + |f|_{0,\infty,K}).
\end{aligned}
\tag{3.18}
$$

Because of the ellipticity condition, there exists a constant $\gamma > 0$ such that

$$
\begin{aligned}
&\Big(\alpha_{11}(A_0)\Big(\frac{\partial y}{\partial \eta}\Big)^2_{A_0} - \alpha_{12}(A_0)\Big(\frac{\partial y}{\partial \eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial \eta}\Big)_{A_0} - \alpha_{21}(A_0)\Big(\frac{\partial y}{\partial \eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial \eta}\Big)_{A_0} \\
&\qquad + \alpha_{22}(A_0)\Big(\frac{\partial x}{\partial \eta}\Big)^2_{A_0}\Big) \ge \gamma\Big(\Big(\frac{\partial x}{\partial \eta}\Big)^2_{A_0} + \Big(\frac{\partial y}{\partial \eta}\Big)^2_{A_0}\Big).
\end{aligned}
\tag{3.19}
$$

Combining (3.18) and (3.19), we complete the proof of (3.9). By substituting $\widehat{\psi}_1$ with $\widehat{\psi}_2$ in above proof, (3.10) can be similarly proved. □

Theorem 3.1. *Suppose $u \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$. Then, the following stability estimate holds*

$$\|u_h\|_{2,\infty,h} \le C\|u\|_{2,\infty}. \tag{3.20}$$

Proof. By the reference element and Lemma 3.3, we know

$$\begin{aligned} \|z_h\|_{2,\infty,K} &\le Ch^{-2}\|\widehat{z}_h\|_{2,\infty,\widehat{K}} \\ &\le Ch^{-2}\Big(\Big|\frac{\partial^2\widehat{u}_h}{\partial\xi^2}(0,0)\Big| + \Big|\frac{\partial^2\widehat{u}_h}{\partial\eta^2}(0,0)\Big|\Big) \\ &\le C(\|u_h\|_{1,\infty,K} + |f|_{0,\infty,K}). \end{aligned} \tag{3.21}$$

By (3.21) and the standard interpolation estimates, we have

$$\begin{aligned} \|u_h - w_h\|_{2,\infty,h} = \|z_h\|_{2,\infty,h} &\le C\|u_h\|_{1,\infty,h} + C|f|_{0,\infty} \\ &\le C\|u_h - w_h\|_{1,\infty,h} + C\|w_h\|_{1,\infty,h} + C|f|_{0,\infty} \\ &\le Ch\|u_h\|_{2,\infty,h} + C\|w_h\|_{1,\infty,h} + C|f|_{0,\infty}. \end{aligned} \tag{3.22}$$

By Lemma 3.2, the inverse estimate, and the $W^{2,\infty}$-stability of conforming finite element (see [9] [1]), we find that

$$\begin{aligned} \|w_h\|_{2,\infty,h} &\le C\|w_h - u_h^*\|_{2,\infty,h} + C\|u_h^*\|_{2,\infty,h} \\ &\le Ch\ln\frac{1}{h}\|u_h\|_{2,\infty,h} + C\|u\|_{2,\infty}. \end{aligned} \tag{3.23}$$

By (3.22) and (3.23), we obtain

$$\begin{aligned} \|u_h\|_{2,\infty,h} &\le \|u_h - w_h\|_{2,\infty,h} + \|w_h\|_{2,\infty,h} \\ &\le Ch\ln\frac{1}{h}\|u_h\|_{2,\infty,h} + C\|u\|_{2,\infty}, \end{aligned}$$

which, for sufficiently small h, yields (3.20). □

Theorem 3.2. *Suppose $u \in W^{3,\infty}(\Omega)\bigcap H_0^1(\Omega)$. If the basis functions $\widehat{\psi}_1(\xi,\eta) = \widehat{\psi}(\xi)$, $\widehat{\psi}_2(\xi,\eta) = \widehat{\psi}(\eta)$, satisfy $\widehat{\psi} \in \Phi\bigcap P$, and the cooefficients of $\widehat{\psi}$, $b_i (1 \le i \le n)$, satisfy $\sum_{i=1}^{n} b_i = 0$, then the following superconvergent estimate holds*

$$\max_{p\in C_h} |\nabla(u-u_h)(p)| \le Ch^2\ln\frac{1}{h}\|u\|_{3,\infty}$$

where C_h denotes the set of the centers of all elements.

Proof. Let $p \in C_h$. It's easy to check that $\nabla z_h(p) = 0$. By applying the superconvergence estimate for the conforming finite element method (see [17])

$$|\nabla(u - u_h^*)(p)| \le Ch^2 \ln \frac{1}{h} \|u\|_{3,\infty},$$

and noting that

$$\begin{aligned}\nabla(u - u_h)(p) &= \nabla(u - u_h^*)(p) + \nabla(u_h^* - u_h)(p)\\ &= \nabla(u - u_h^*)(p) + \nabla(u_h^* - w_h)(p),\end{aligned}$$

the proof is completed by applying Lemma 3.2 and Theorem 3.1. □

4. Superconvergence Recoveries

For the sake of explicitness, let's suppose that Ω is an arbitrary quadrilateral, being subdivided into small quadrilateral elements by bi-section scheme (see [13] [10]). Let K and K' are two neighbouring elements, $A_i(x_i, y_i)$ and $A_i'(x_i', y_i')$ $(1 \le i \le 4)$ are their vertices respectively, $A_0(x_0, y_0)$ and $A_0'(x_0', y_0')$ are their centers.

We need some Lemmas. First, from (2.6), Lemma 3.3, Theorem 3.1 and the error estimates for the isoparametric bilinear solution, we have

Lemma 4.1.

$$|z_h|_{1,\infty,h} \le Ch\|u\|_{2,\infty}, \qquad |z_h|_{0,\infty} \le Ch^2\|u\|_{2,\infty},$$

$$|u - u_h|_{1,\infty,h} \le Ch\|u\|_{2,\infty}, \qquad |u - u_h|_{0,\infty} \le Ch^2 \ln \frac{1}{h}\|u\|_{2,\infty}.$$

Lemma 4.2. *Suppose $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$ and the nonconforming basis functions $\widehat{\psi}_1 = \widehat{\psi}(\xi)$ and $\widehat{\psi}_2 = \widehat{\psi}(\eta)$ satisfy $b_i = 0$ $(1 \le i \le n)$, i.e., $\widehat{\psi}$ is an even function, then*

$$\Big|\frac{\partial^2}{\partial \xi^2}(\widehat{u_h^K} - \widehat{u_h^{K'}})(0,0)\Big| \le Ch^3\|u\|_{3,\infty}, \tag{4.1}$$

$$\Big|\frac{\partial^2}{\partial \eta^2}(\widehat{u_h^K} - \widehat{u_h^{K'}})(0,0)\Big| \le Ch^3\|u\|_{3,\infty}. \tag{4.2}$$

Proof. In (2.7), setting

$$v_h = \begin{cases} v_h^K = \widehat{\psi_1} \circ F_K^{-1}, & (x,y) \in K, \\ v_h^{K'} = -\widehat{\psi_1} \circ F_{K'}^{-1}, & (x,y) \in K', \\ 0, & (x,y) \in \Omega - K - K', \end{cases} \tag{4.3}$$

and by the decomposition $u_h = w_h + z_h$, we obtain

$$\int_{K \cup K'} \sum_{i,j=1}^{2} \alpha_{ij} D_i z_h D_j v_h dxdy = \int_{K \cup K'} \sum_{i,j=1}^{2} \alpha_{ij} D_i (u - w_h) D_j v_h dxdy$$

$$+\int_{K\cup K'}\beta(u-u_h)v_h dxdy-\int_{K\cup K'}\sum_{i,j=1}^{2}\alpha_{ij}D_iuD_jv_h dxdy$$

$$-\int_{K\cup K'}\beta uv_h dxdy+\int_{K\cup K'}fv_h dxdy=R_1+R_2+R_3+R_4+R_0. \tag{4.4}$$

Let's first analyse the left hand side of (4.4). By Taylor's expansion, we know

$$\int_{K\cup K'}\sum_{i,j=1}^{2}\alpha_{ij}D_iz_hD_jv_h dxdy=\sum_{i,j=1}^{2}\alpha_{ij}(A_0)\int_K D_iz_hD_jv_h dxdy$$

$$+\sum_{i,j=1}^{2}\alpha_{ij}(A_0')\int_{K'}D_iz_hD_jv_h dxdy+\sum_{i,j=1}^{2}\int_{K\cup K'}O(h)|\alpha_{ij}|_{1,\infty}D_iz_hD_jv_h dxdy$$

$$=L_1+L_2+R_5. \tag{4.5}$$

Obviously, by Lemma 4.1, we have

$$|R_5|\le Ch^3|z_h|_{1,\infty,h}|v_h|_{1,\infty,h}\le Ch^3\|u\|_{2,\infty}. \tag{4.6}$$

Similar to the proof of Lemma 3.3, we have

$$L_1=\Big(\alpha_{11}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)^2_{A_0}-\alpha_{12}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial\eta}\Big)_{A_0}-\alpha_{21}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial\eta}\Big)_{A_0}$$

$$+\alpha_{22}(A_0)\Big(\frac{\partial x}{\partial\eta}\Big)^2_{A_0}\Big)|J^K|^{-1}_{A_0}\frac{\partial^2\widehat{u_h^K}}{\partial\xi^2}(0,0)\int_{\widehat{K}}\Big(\frac{\partial\widehat{\psi_1}}{\partial\xi}\Big)^2d\xi d\eta+R_6,$$

$$L_2=-\Big(\alpha_{11}(A_0')\Big(\frac{\partial y'}{\partial\eta}\Big)^2_{A_0'}-\alpha_{12}(A_0')\Big(\frac{\partial y'}{\partial\eta}\Big)_{A_0'}\Big(\frac{\partial x'}{\partial\eta}\Big)_{A_0'}-\alpha_{21}(A_0')\Big(\frac{\partial y'}{\partial\eta}\Big)_{A_0'}\Big(\frac{\partial x'}{\partial\eta}\Big)_{A_0'}$$

$$+\alpha_{22}(A_0')\Big(\frac{\partial x'}{\partial\eta}\Big)^2_{A_0'}\Big)|J^{K'}|^{-1}_{A_0'}\frac{\partial^2\widehat{u_h^{K'}}}{\partial\xi^2}(0,0)\int_{\widehat{K}}\Big(\frac{\partial\widehat{\psi_1}}{\partial\xi}\Big)^2d\xi d\eta+R_7,$$

where

$$|R_6|\le Ch^2|z_h|_{1,\infty,K}\le Ch^3\|u\|_{2,\infty},\quad |R_7|\le Ch^2|z_h|_{1,\infty,K'}\le Ch^3\|u\|_{2,\infty}. \tag{4.7}$$

Hence

$$L_1+L_2=$$

$$\Big(\alpha_{11}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)^2_{A_0}-\alpha_{12}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial\eta}\Big)_{A_0}-\alpha_{21}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial\eta}\Big)_{A_0}$$

$$+\alpha_{22}(A_0)\Big(\frac{\partial x}{\partial\eta})^2_{A_0}\Big)|J^K|^{-1}_{A_0}\int_{\widehat{K}}\Big(\frac{\partial\widehat{\psi_1}}{\partial\xi}\Big)^2d\xi d\eta\Big(\frac{\partial^2\widehat{u_h^K}}{\partial\xi^2}(0,0)-\frac{\partial^2\widehat{u_h^{K'}}}{\partial\xi^2}(0,0)\Big)$$

$$
\begin{aligned}
&+\Big\{\Big(\alpha_{11}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)^2_{A_0}|J^K|^{-1}_{A_0}-\alpha_{11}(A_0')\Big(\frac{\partial y'}{\partial\eta}\Big)^2_{A_0'}|J^{K'}|^{-1}_{A_0'}\Big)\\
&-\Big(\alpha_{12}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial\eta}\Big)_{A_0}|J^K|^{-1}_{A_0}-\alpha_{12}(A_0')\Big(\frac{\partial y'}{\partial\eta}\Big)_{A_0'}\Big(\frac{\partial x'}{\partial\eta}\Big)_{A_0'}|J^{K'}|^{-1}_{A_0'}\Big)\\
&-\Big(\alpha_{21}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)_{A_0}\Big(\frac{\partial x}{\partial\eta}\Big)_{A_0}|J^K|^{-1}_{A_0}-\alpha_{21}(A_0')\Big(\frac{\partial y'}{\partial\eta}\Big)_{A_0'}\Big(\frac{\partial x'}{\partial\eta}\Big)_{A_0'}|J^{K'}|^{-1}_{A_0'}\Big)\\
&+\Big(\alpha_{22}(A_0)\Big(\frac{\partial x}{\partial\eta}\Big)^2_{A_0}|J^K|^{-1}_{A_0}-\alpha_{22}(A_0')\Big(\frac{\partial x'}{\partial\eta}\Big)^2_{A_0'}|J^{K'}|^{-1}_{A_0'}\Big)\Big\}\\
&\quad\times\frac{\partial^2\widehat{u_h^{K'}}}{\partial\xi^2}(0,0)\int_{\widehat{K}}\Big(\frac{\partial\widehat{\psi_1}}{\partial\xi}\Big)^2d\xi d\eta+R_6+R_7\\
&=L_3+R_8+R_6+R_7,
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
|L_3|&\ge\gamma\Big(\Big(\frac{\partial y}{\partial\eta}\Big)^2+\Big(\frac{\partial x}{\partial\eta}\Big)^2\Big)|J^K|^{-1}_{A_0}\int_{\widehat{K}}\Big(\frac{\partial\widehat{\psi_1}}{\partial\xi}\Big)^2d\xi d\eta\Big|\frac{\partial^2\widehat{u_h^{K}}}{\partial\xi^2}(0,0)-\frac{\partial^2\widehat{u_h^{K'}}}{\partial\xi^2}(0,0)\Big|\\
&\ge C\Big|\frac{\partial^2\widehat{u_h^{K}}}{\partial\xi^2}(0,0)-\frac{\partial^2\widehat{u_h^{K'}}}{\partial\xi^2}(0,0)\Big|.
\end{aligned}
\tag{4.9}
$$

For estimating R_8, let's consider the term in the first bracket of R_8. Noting that

$$\alpha_{11}(A_0')=\alpha_{11}(A_0)+O(h)|\alpha_{11}|_{1,\infty,K\cup K'},$$

we know

$$
\begin{aligned}
&\alpha_{11}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)^2_{A_0}|J^K|^{-1}_{A_0}-\alpha_{11}(A_0')\Big(\frac{\partial y'}{\partial\eta}\Big)^2_{A_0'}|J^{K'}|^{-1}_{A_0'}\\
=&\alpha_{11}(A_0)\Big\{\Big(\frac{\partial y}{\partial\eta}\Big)^2_{A_0}|J^K|^{-1}_{A_0}-\Big(\frac{\partial y'}{\partial\eta}\Big)^2_{A_0'}|J^{K'}|^{-1}_{A_0'}\Big\}+O(h)|\alpha_{11}|_{1,\infty,K\cup K'}\Big(\frac{\partial y'}{\partial\eta}\Big)^2_{A_0'}|J^{K'}|^{-1}_{A_0'}.
\end{aligned}
\tag{4.10}
$$

Since K and K' are two h^2-approximate parallelograms, we obtain

$$\Big(\frac{\partial y}{\partial\eta}\Big)^2_{A_0}|J^K|^{-1}_{A_0}-\Big(\frac{\partial y'}{\partial\eta}\Big)^2_{A_0'}|J^{K'}|^{-1}_{A_0'}=O(h).\tag{4.11}$$

Applying (4.11) in (4.10) yields that

$$\Big|\alpha_{11}(A_0)\Big(\frac{\partial y}{\partial\eta}\Big)^2_{A_0}|J^K|^{-1}_{A_0}-\alpha_{11}(A_0')\Big(\frac{\partial y'}{\partial\eta}\Big)^2_{A_0'}|J^{K'}|^{-1}_{A_0'}\Big|\le Ch.$$

Applying similar procedure to other three brackets of R_8, and by Lemma 3.3, we obtain

$$|R_8|\le Ch\Big|\frac{\partial^2\widehat{u_h^{K'}}}{\partial\xi^2}(0,0)\Big|\le Ch^3\|u\|_{2,\infty}.\tag{4.12}$$

Now let's estimate the right hand side of (4.4). Let u^I denote the isoparametric bilinear interpolation of the function u, then we have

$$\begin{aligned}R_1 &= \int_{K\cup K'}\sum_{i,j=1}^{2}\alpha_{ij}D_i(u-u^I)D_jv_h dxdy + \int_{K\cup K'}\sum_{i,j=1}^{2}\alpha_{ij}D_i(u^I-w_h)D_jv_h dxdy\\ &= R_{11}+R_{12}. \end{aligned}\tag{4.13}$$

Applying Taylor's expansion, we know

$$\begin{aligned}R_{11} &= \sum_{i,j=1}^{2}\alpha_{ij}(A_0)\int_K D_i(u-u^I)D_jv_h dxdy\\ &\quad+\sum_{i,j=1}^{2}\alpha_{ij}(A_0')\int_{K'} D_i(u-u^I)D_jv_h dxdy\\ &\quad+\sum_{i,j=1}^{2}\int_{K\cup K'} O(h)D_i(u-u^I)D_jv_h dxdy\\ &= R_{111}+R_{112}+R_{113}.\end{aligned}\tag{4.14}$$

Application of interpolation theory simply yields

$$|R_{113}|\le Ch^3|u-u^I|_{1,\infty,K\cup K'}|v_h|_{1,\infty,K\cup K'}\le Ch^3|u|_{2,\infty}.\tag{4.15}$$

For estimating R_{111} and R_{112}, we observe that

$$\begin{aligned}&\int_K D_1(u-u^I)D_1v_h dxdy\\ =&\int_{\widehat K}\frac{\partial(\widehat u-\widehat u^I)}{\partial\xi}\frac{\partial\widehat\psi_1}{\partial\xi}\Big(\frac{\partial y}{\partial\eta}\Big)^2|J|^{-1}d\xi d\eta-\int_{\widehat K}\frac{\partial(\widehat u-\widehat u^I)}{\partial\eta}\frac{\partial\widehat\psi_1}{\partial\xi}\Big(\frac{\partial y}{\partial\xi}\Big)\Big(\frac{\partial y}{\partial\eta}\Big)|J|^{-1}d\xi d\eta\\ =&R_{114}+R_{115},\end{aligned}\tag{4.16}$$

and

$$R_{114}=\Big(\frac{\partial y}{\partial\eta}\Big)^2_{A_0}|J|^{-1}_{A_0}\int_{\widehat K}\frac{\partial(\widehat u-\widehat u^I)}{\partial\xi}\frac{\partial\widehat\psi_1}{\partial\xi}d\xi d\eta+\int_{\widehat K}O(h)\frac{\partial(\widehat u-\widehat u^I)}{\partial\xi}\frac{\partial\widehat\psi_1}{\partial\xi}d\xi d\eta.$$

By (2.4), we find that

$$\int_K\frac{\partial(\widehat u-\widehat u^I)}{\partial\xi}\frac{\partial\widehat\psi_1}{\partial\xi}d\xi d\eta=0,\qquad\forall\widehat u\in P_2(\widehat K),$$

hence, by virtue of Bramble-Hilbert lemma, we obtain

$$|R_{114}|\le C|\widehat u|_{3,\infty,\widehat K}+Ch|\widehat u-\widehat u^I|_{1,\infty,\widehat K}$$

$$\leq Ch^3|u|_{3,\infty,K} + Ch^2|u-u^I|_{1,\infty,K}$$
$$\leq Ch^3\|u\|_{3,\infty}. \tag{4.17}$$

Similarly, we have $|R_{115}| \leq Ch^3\|u\|_{3,\infty}$, then, we obtain that, from (4.16),

$$\Big|\int_K D_1(u-u^I)D_1v_h dxdy\Big| \leq Ch^3\|u\|_{3,\infty}. \tag{4.18}$$

Similar to (4.18), we have

$$\Big|\int_K D_i(u-u^I)D_jv_h dxdy\Big| \leq Ch^3\|u\|_{3,\infty}, \quad i,j=1,2.$$

Hence

$$|R_{111}| \leq Ch^3\|u\|_{3,\infty}, \qquad |R_{112}| \leq Ch^3\|u\|_{3,\infty}. \tag{4.19}$$

Application of (4.15) and (4.19) in (4.14) yields

$$|R_{11}| \leq Ch^3\|u\|_{3,\infty}. \tag{4.20}$$

Also applying Taylor's expansion in R_{12}, we know

$$\begin{aligned}
R_{12} = &\sum_{i,j=1}^{2}\alpha_{ij}(A_0)\int_K D_i(u^I-w_h)D_jv_h dxdy \\
&+ \sum_{i,j=1}^{2}\alpha_{ij}(A_0')\int_{K'} D_i(u^I-w_h)D_jv_h dxdy \\
&+ \sum_{i,j=1}^{2}\int_{K\cup K'} O(h)D_i(u^I-w_h)D_jv_h dxdy \\
= &R_{121}+R_{122}+R_{123}.
\end{aligned} \tag{4.21}$$

The estimation of R_{123} is as follows

$$\begin{aligned}
|R_{123}| &\leq Ch^3|u^I-w_h|_{1,\infty,K\cup K'}|v_h|_{1,\infty,K\cup K'} \\
&\leq Ch^2(|u-u_h|_{1,\infty,K\cup K'} + |(u-u_h)-(u-u_h)^I|_{1,\infty,K\cup K'}) \\
&\leq Ch^3(|u|_{2,\infty} + |u-u_h|_{2,\infty}) \leq Ch^3\|u\|_{2,\infty}.
\end{aligned} \tag{4.22}$$

Because $u^I - w_h \in W_h$, applying the same technique as in (3.15)–(3.18), we obtain

$$\begin{aligned}
|R_{121}| &\leq Ch^2|u^I-w_h|_{1,\infty,K} \\
&\leq Ch^2(|u-u_h|_{1,\infty,K} + |(u-u_h)-(u-u_h)^I|_{1,\infty,K} \\
&\leq Ch^3\|u\|_{2,\infty},
\end{aligned} \tag{4.23}$$

$$|R_{122}| \le Ch^2 |u^I - w_h|_{1,\infty,K'} \le Ch^3 \|u\|_{2,\infty}. \tag{4.24}$$

Application of (4.22)-(4.24) in (4.21) yields

$$|R_{12}| \le Ch^3 \|u\|_{2,\infty}. \tag{4.25}$$

Therefore, by (4.20) and (4.25), we obtain that, from (4.13),

$$|R_1| \le Ch^3 \|u\|_{2,\infty}. \tag{4.26}$$

The estimation of R_2 is easy. In fact

$$|R_2| \le Ch^2 |u - u_h|_{0,\infty,K\cup K'} \le Ch^3 \|u\|_{2,\infty}. \tag{4.27}$$

For estimating R_3, we still need Taylor's expansion. Let $\sigma_{ij} = \alpha_{ij} D_i u$, then

$$\sigma_{ij} = \sigma_{ij}(A_0) + \frac{\partial \sigma_{ij}}{\partial x}(A_0)(x - x_0) + \frac{\partial \sigma_{ij}}{\partial y}(A_0)(y - y_0) + O(h^2)|\sigma_{ij}|_{2,\infty,K}.$$

Noting that $\widehat{\psi}_1$ is an even function and (2.4), it's easy to find that

$$\int_K D_1 v_h dxdy = 0, \quad \int_K x D_1 v_h dxdy = 0, \quad \int_K y D_1 v_h dxdy = 0,$$
$$\int_K D_2 v_h dxdy = 0, \quad \int_K x D_2 v_h dxdy = 0, \quad \int_K y D_2 v_h dxdy = 0,$$

therefore

$$\begin{aligned} |R_3| &\le \Big| \sum_{i,j=1}^{2} \int_{K\cup K'} O(h^2) |\sigma_{ij}|_{2,\infty,K\cup K'} D_j v_h dxdy \Big| \\ &\le Ch^4 |u|_{3,\infty} |v_h|_{1,\infty,K\cup K'} \le Ch^3 |u|_{3,\infty}. \end{aligned} \tag{4.28}$$

The estimation of R_4 and R_0 is also easy. In fact, by using the reference element and the properties of function $\widehat{\psi}_1$, we have

$$|R_4| \le Ch^3 \|u\|_{1,\infty}, \quad |R_0| \le Ch^3 \|f\|_{1,\infty} \le Ch^3 \|u\|_{3,\infty}. \tag{4.29}$$

Combination of (4.4)-(4.9), (4.12) and (4.26)-(4.29) completes the proof of (4.1).

The substitution of $\widehat{\psi}_2$ for $\widehat{\psi}_1$ in above proof can prove (4.2). □

Theorem 4.1. *Suppose $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$, and the basis functions $\widehat{\psi_1} = \widehat{\psi}(\xi)$, $\widehat{\psi_2} = \widehat{\psi}(\eta)$ satisfy $b_i = 0$ $(1 \le i \le n)$. Then*

$$|\overline{\nabla} z_h(p)| \le Ch^2 \|u\|_{3,\infty}, \tag{4.30}$$

where p is an inner vertex or a midpoint of an inner edge, and $\overline{\nabla}$ means taking average gradient over all the neighbouring elements of the point p.

Proof. Noting that $b_i = 0$ $(1 \le i \le n)$, we know

$$\frac{\partial \widehat{\psi_1}}{\partial \xi}(1,\eta) = -\frac{\partial \widehat{\psi_1}}{\partial \xi}(-1,\eta), \quad \frac{\partial \widehat{\psi_2}}{\partial \eta}(\xi,1) = -\frac{\partial \widehat{\psi_2}}{\partial \eta}(\xi,-1),$$

then, from Lemma 4.2, we obtain,

$$\Big|\frac{1}{2}\Big(\frac{\partial z_h^K}{\partial x} + \frac{\partial z_h^{K'}}{\partial x}\Big)(p)\Big| \le Ch^2\|u\|_{3,\infty}, \tag{4.31}$$

$$\Big|\frac{1}{2}\Big(\frac{\partial z_h^K}{\partial y} + \frac{\partial z_h^{K'}}{\partial y}\Big)(p)\Big| \le Ch^2\|u\|_{3,\infty}.$$

Combination of (4.31) and (4.32) completes the proof of (4.30). □

Lemma 4.3. (*see* [16]) *Suppose* $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$, *then we have the following superconvergent estimate*

$$|\overline{\nabla}(u - u_h^*)(p)| \le Ch^2 \ln \frac{1}{h}\|u\|_{3,\infty},$$

where p *is any inner vertex or the midpoint of any inner edge.*

Theorem 4.2. *Suppose the conditions in Theorem* 4.1 *are satisfied, then the following superconvergent estimate holds.*

$$|\overline{\nabla}(u - u_h)(p)| \le Ch^2 \ln \frac{1}{h}\|u\|_{3,\infty}, \tag{4.33}$$

where p *is any inner vertex or the midpoint of any inner edge.*

Proof. Since

$$u - u_h = (u - u_h^*) + (u_h^* - w_h) - z_h,$$

applying Lemma 4.3, Lemma 3.2, Theorem 3.1 and Theorem 4.1, we complete the proof of (4.33). □

Remark *If* Ω *is a polygonal domain, it can be divided into finite quadrilaterals. On each quadrilateral, we use the bi-section scheme to subdivide it* (*i.e., piecewise strongly regular subdivision*)*. Then, on every inner vertex and midpoint of inner edge of each quadrilateral , the superconvergence results in this section are still valid.*

5. Numerical Examples

Now let's give some numerical examples. Consider the following Dirichlet boundary problem,

$$\begin{cases} -\Delta u = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset R^2$ is a quadrilateral with the four vertices (0, 0), (0.25, 0), (0.375, 0.25), (0, 0.25).

$$f = 128(1-\xi^2) + \frac{128}{(2y+1)^2}(1-\eta^2) + \frac{1024x}{(2y+1)^2}\xi\eta$$

$$+\frac{128x}{(2y+1)^3}\xi(1-\eta^2)+\frac{512x^2}{(2y+1)^4}(1-\eta^2),$$

in which $\xi=\dfrac{8x}{2y+1}-1$, $\eta=8y-1$. The exact solution is $u=(1-\xi^2)(1-\eta^2)$. We choose the nonconforming basis functions $\widehat{\psi}_1(\xi,\eta)$ and $\widehat{\psi}_2(\xi,\eta)$ as

$$\widehat{\psi}_1(\xi,\eta)=\frac{1}{2}(\xi^2-1)-\frac{5}{12}(\xi^4-1),\quad \widehat{\psi}_2(\xi,\eta)=\frac{1}{2}(\eta^2-1)-\frac{5}{12}(\eta^4-1).$$

We use the bi-section scheme to subdivide the domain. The numerical results are listed below.

h	eg	ep	eq
0.2519456	15.61572	70.83024	52.09018
0.1328125	15.14269	70.32478	56.42291
0.0681358	15.36230	70.18006	52.83535

Here,

$$eg=\frac{\max|\nabla(u-u_h)(g)|}{h^2|\ln h|},\ ep=\frac{\max|\overline{\nabla}(u-u_h)(p)|}{h^2|\ln h|},\ eq=\frac{\max|\overline{\nabla}(u-u_h)(q)|}{h^2|\ln h|},$$

and g means the centers of the elements, p means the inner vertices and q means the midpoints of inner edges.

The following numerical examples can illustrate the necessity of the conditions $b_i=0$ $(1\le i\le n)$ or $\sum_{i=1}^n b_i=0$. Choose the nonconforming basis function $\widehat{\psi}_1(\xi,\eta)$ and $\widehat{\psi}_2(\xi,\eta)$ as

$$\widehat{\psi}_1(\xi,\eta)=\frac{1}{2}(\xi^2-1)-\frac{5}{12}(\xi^4-1)+b\xi(\xi^2-1),$$
$$\widehat{\psi}_2(\xi,\eta)=\frac{1}{2}(\eta^2-1)-\frac{5}{12}(\eta^4-1)+b\eta(\eta^2-1).$$

The numerical results are listed below.

	eg				
h	$b=0.1$	$b=0.2$	$b=0.5$	$b=0.7$	$b=1.0$
0.2519456	17.05209	19.62461	60.33922	142.6141	373.9902
0.1328125	17.40823	22.13526	85.88200	197.2402	498.3740
0.0681358	18.75752	27.95114	112.6709	246.8026	590.4153

References

[1] H.S. Chen and B. Li, "Superconvergence analysis and error expansion for the Wilson nonconforming finite element", *Numer. Math.* **69**(1994) 125-140.

[2] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[3] J.S. Jiang and X.L. Cheng, "A nonconforming element like Wilson's for second order problems", *Math. Numer. Sinica*, Vol. 14, No. 3, 274-278 (1992). (in Chinese).

[4] M. Křížek and P. Neittaanmäki, "On superconvergence techniques", *Acta Applicandae Mathematicae*, **9**, 175-198 (1987).

[5] P. Lesaint and M. Zlámal, "Convergence of the nonconforming Wilson element for arbitrary quadrilateral meshes", *Numer. Math.*, **36**, 33-52 (1980).

[6] P. Lesaint and M. Zlámal, "Superconvergence of the gradient of finite element solutions", *R.A.I.R.O. Numer. Anal.*, Vol.3, No.12, pp.139-166(1979).

[7] Q. Lin and T. Lü, "Asympotic expansions for finite element approximation of elliptic problem on polygonal domains", *Computing Methods in Applied Science and Engineering VI*, Ed. by R. Glowinski and J.-L. Lions, Elsevier Science Publishers, North-Holland, pp. 317-321, 1984.

[8] Q. Lin and J.C. Xu, "Linear finite element with high accuracy", *J. Comp. Math.*, Vol.3, No.2, pp.115-133(1985).

[9] R. Rannacher and R. Scott, "Some optimal error estimates for piecewise linear finite element approximations", *Math. Comp.*, **38** 437-445, (1982).

[10] Z.C. Shi, "A convergence condition for the quadrilateral Wilson element", *Numer. Math.*, **44**, 349-361 (1984).

[11] D.Y. Shi and S.C. Chen, "A kind of improved Wilson arbitrary quadrilateral elements", *Numer. Math. J. of Chinese Univ.*, Vol. 16, No. 2, 161-167 (1994). (in Chinese).

[12 Z.C. Shi and B. Jiang, "A new superconvergence property of Wilson nonconforming finite element", *preprint*.

[13] R.L. Taylor, P.J. Beresfors and E.L. Wilson, "A nonconforming element for stress analysis", *Int. J. Numer. Mechs. Eng.*, **10**, 1211-1219, (1976).

[14] H. Wang, "On the superconvergence estimate of Wilson element", *J. of Shandong Univ. (Natural Science)*, Vol. 21, No. 4, 89-95 (1986). (in Chinese).

[15] L. Zhang and L.K. Li, " A superconvergent estimate of Wilson-like elements", *Numer. Math. J. Chin. Univ.*. (to appear).

[16] L. Zhang and L.K. Li, "On superconvergence of isoparametric bilinear element", *Comm. Numer. Methods Engrg.*. (to appear).

[17] Q.D. Zhu and Q. Lin, *The superconvergence theory of finite element methods*, Hunan Science and Technology Press, 1989. (in Chinese).

[18] M. Zlámal, "Superconvergence and reduced integration in the finite element method", *Math. Comp.*, **32**, 663-685 (1978).