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THE ELLIPTIC TYPE NODE CONFIGURATION AND INTERPOLATION IN $R^{2*1)}$

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Abstract

In this paper, we have obtained an expression of the bivariate Vandermonde determinant for the Elliptic Type Node Configuration in R^2, and discussed the possibility of the corresponding multivariate Lagrange, Hermite and Birkhoff interpolation.

Key words: Multivariate interpolation, Polynomial interpolation, Birkhoff interpolation, Node configuration.

1. Introduction

In this paper, we use the usual multivariate notation $w^j = w_1^{j_1}\cdots w_s^{j_s}$, $|j| = j_1 + \cdots + j_s$ $(j_1, \cdots, j_s \in Z_+)$ [1,2] and let P_n be the (bivariate) polynomial space of all real (bivariate) polynomials of degree at most n.

Now we introduce the concept of the Curve Type Node Configuration (CTNC):

Definition 1. Curve Type Node Configuration A (CTNCA)[3]. *Let $L_n = (n+1)(2n+1)$. Then carry out the following steps:*

0. *Arbitrarily select a point as node x_1 in R^2;*

1. *Draw a quadratic irreducible curve X_1 such that it does not go through the node x_1 on R^2 (X_1 can be an ellipse, a hyperbola or a parabola), arbitrarily select five distinct points from X_1 as nodes $x_2, \cdots, x_6$;*

$\cdots$

n. *Draw a quadratic irreducible curve X_n such that it does not go through the nodes that have been selected on R^2 (X_n can be an ellipse, a hyperbola or a parabola), arbitrarily select $4n+1$ distinct points from X_n as nodes $x_{L_{n-1}+1}, \cdots, x_{L_n}$.*

The obtained node group $\hat{X}_n = \{x_i : i = 1, \cdots, L_n\}$ is called the Curve Type Node Configuration A (CTNCA). If every quadratic irreducible curve is an ellipse, then $\hat{X}_n$ can be called an Elliptic Type Node Configuration A (ETNCA).

Let $w = (u, v)$ be the variables in R^2 and arrange the bivariate monomial sequence $\varphi_1, \varphi_2, \varphi_3, \cdots$ as the following order:

1; u, v, u^2, uv, v^2; $u^3, u^2v, uv^2, v^3, u^4, u^3v, u^2v^2, uv^3, v^4; \cdots$.

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The multivariate Vandermonde determinant that we will study can be formulated as follows:

$$VD_n\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_n}\\ x_1, & \cdots, & x_{L_n}\end{pmatrix} = \det[\phi_1, \cdots, \phi_{L_n}]$$

where the column vector

$$\phi_i = [\varphi_1(x_i), \cdots, \varphi_{L_n}(x_i)]^T.$$

If a node distribution guarantees the existence and uniqueness of a Lagrange interpolant to any given data, we say that the set of nodes admits unique Lagrange interpolation. Hence, $\hat{X}_n$ admits unique Lagrange interpolation if and only if

$$VD_n\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_n}\\ x_1, & \cdots, & x_{L_n}\end{pmatrix} \neq 0.$$

To allow coalescence of nodes along the curves $X_1, \cdots, X_n$, we consider the following definition.

Definition 2. Curve Type Node Configuration B (CTNCB). *There exist quadratic irreducible curves $X_1, \cdots, X_n$, such that*

$$x_{L_{j-1}+1}, \cdots, x_{L_j} \in X_j \backslash (X_{j+1} \cup \cdots \cup X_n)$$

for $j = 1, \cdots, n$ as in CTNCA, where

$$x_{L_{j-1}+1}, \cdots, x_{L_j} = \underbrace{y_{j1}, \cdots, y_{j1}}_{\ell_{j1}}, \cdots, \underbrace{y_{jk_j}, \cdots, y_{jk_j}}_{\ell_{jk_j}}$$

with $\ell_{j1} + \cdots + \ell_{jk_j} = L_j - L_{j-1}$, $j = 1, \cdots, n$.

Node coalescence along X_j corresponds to Hermite interpolation with derivatives $D^k_{X_j}$ ($D^0_{X_j} := I$, the identity operator). The definition of D_{X_j} will be different according to whether X_j is an ellipse, a hyperbola or a parabola. In this paper, we will give the definition of D_{X_j} when X_j is an ellipse. We denote the column vectors by

$$D^k_{X_j}\phi_i = [D^k_{X_j}\varphi_1(x_i), \cdots, D^k_{X_j}\varphi_{L_n}(x_i)]^T.$$

Hence, the generalized Vandermonde determinant corresponding to the Hermite interpolation problem on the nodes $\hat{X}_n$ satisfying CTNCB becomes:

$$\begin{aligned}HD_n&\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_n}\\ x_1, & \cdots, & x_{L_n}\end{pmatrix}\\ &= \det\Big[\phi_1 \vdots \cdots \vdots \underbrace{\phi_{j1} \vdots D_{X_j}\phi_{j1} \vdots \cdots \vdots D^{\ell_{j1}-1}_{X_j}\phi_{j1} \vdots \cdots \vdots \phi_{jk_j} \vdots D_{X_j}\phi_{jk_j} \vdots \cdots \vdots D^{\ell_{jk_j}-1}_{X_j}\phi_{jk_j}}_{(\textit{for points on } X_j)} \vdots \cdots\Big].\end{aligned}$$

To allow coalescence of the quadratic irreducible curves $X_1, \cdots, X_n$, we consider the following definition.

Definition 3. Curve Type Node Configuration C (CTNCC). *The set $\hat{X}_n$ consists of distinct nodes $x_1, \cdots, x_{L_n}$, and there exist curves $X_1, \cdots, X_n$ where*

$$X_1, \cdots, X_n = \underbrace{Y_1, \cdots, Y_1}_{m_1}, \cdots, \underbrace{Y_d, \cdots, Y_d}_{m_d},$$

$m_1 + \cdots + m_d = n$ *and* $Y_1, \cdots, Y_d$ *distinct, such that*

$$x_{L_{j-1}+1}, \cdots, x_{L_j}$$

lie on X_j *but not on those* $X_{j+1}, \cdots, X_n$ *different from* X_j.

The corresponding interpolation problem involves interpolating values of "normal" derivatives $N^k_{X_j}$ ($N^0_{X_j} := I$, the identity operator). In this paper, we will give the definition of N_{X_j} when X_j is an ellipse. Consider the column vectors

$$N^k_{X_j}\phi_i = [N^k_{X_j}\varphi_1(x_i), \cdots, N^k_{X_j}\varphi_{L_n}(x_i)]^T.$$

The interpolation problem is:

$$\begin{aligned}&(p-f)(x_1) = 0,\\ &N^{m_j-k-1}_{Y_j}(p-f)(x_i) = 0, \quad 0 \le k \le m_j - 1,\end{aligned}$$

where $p \in P_n$ is the interpolation polynomial and

$$i = L_{m_1+\cdots+m_{j-1}+k} + 1, \cdots, L_{m_1+\cdots+m_{j-1}+k+1}; \quad j = 1, \cdots, d,$$

with the usual notation $m_1 + \cdots + m_{j-1} := 0$ for $j = 1$. Since this is a Birkhoff interpolation problem (where "normal" derivatives instead of function values are interpolated at the nodes on the curve which contains less nodes and is brought to coincide with a curve with more nodes), we use the notation BD_n to denote the determinant of the coefficient matrix. Then we have

$$BD_n\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_n}\\ x_1, & \cdots, & x_{L_n}\end{pmatrix} = \det\Big[\cdots\vdots\underbrace{N^{m_j-1}_{Y_j}\phi_{L_{m_1+\cdots+m_{j-1}}+1}\vdots\cdots\vdots N^{m_j-1}_{Y_j}\phi_{L_{m_1+\cdots+m_{j-1}+1}}}_{for\ k=0}\vdots\cdots$$
$$\vdots\underbrace{\phi_{L_{m_1+\cdots+m_{j-1}+m_j-1}+1}\vdots\cdots\vdots\phi_{L_{m_1+\cdots+m_{j-1}+m_j}}}_{for\ k=m_j-1}\vdots\cdots\Big]$$

where we have listed the columns for the points on Y_j.

2. A Group of Lemmas

Lemma 1. *Let* $\hat{X}_n = \{x_i : i = 1, \cdots, L_n\}$ *be a set of distinct nodes in* R^2 *satisfying CTNCA. Let the n-th quadratic irreducible curve* X_n *be an ellipse:*

$$\frac{(u-\lambda_n)^2}{a_n^2} + \frac{(v-\mu_n)^2}{b_n^2} = 1,$$

and set $x_i = (u_i, v_i)$. *Then*

$$VD_n\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_n}\\ x_1, & \cdots, & x_{L_n}\end{pmatrix} = c\frac{a_n^{(2n)^2}b_n^{(2n)^2}}{2^{(2n)^2}}\Bigg[\prod_{s=1}^{L_{n-1}}\frac{d(x_s,z_{ns})[d(x_s,O_n)+d(z_{ns},O_n)]}{[d(z_{ns},O_n)]^2}\Bigg].$$

$$\cdot\Big(\prod_{L_{n-1}<r\le L_n} e^{-i2n\theta_r}\Big)\Big[\prod_{L_{n-1}<p<q\le L_n}(e^{i\theta_q}-e^{i\theta_p})\Big]\cdot VD_{n-1}\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_{n-1}}\\ x_1, & \cdots, & x_{L_{n-1}}\end{pmatrix},$$

where $c = 1$ or $= -1$, O_n is the center point of the ellipse X_n, z_{ns} is the intersection point of X_n with the ray that starts from O_n and passes through x_s. The Euclidean distance between points x^* and y^* is denoted by $d(x^*, y^*)$, and θ_t is determined by the following expressions:

$$u_t = \lambda_n + a_n\cos\theta_t,\ \ v_t = \mu_n + b_n\sin\theta_t,\ \ L_{n-1} < t \le L_n.$$

Proof. Instead of giving a formal proof, we indicate the method of proof by considering the special case $n = 2$. We denote $x'_i = (u'_i, v'_i) = x_i - (\lambda_2, \mu_2)$ and indicate that the equation of the ellipse X_2 can be written as

$$u - \lambda_2 = a_2\cos\theta,\ \ v - \mu_2 = b_2\sin\theta,\ \ (0 \le \theta \le 2\pi).$$

Let $P[m, n(k)]$ be the elementary matrix that is obtained by adding k times the elements in the n-th row of the identity matrix to the m-th row of the identity matrix, and

$$VD_2\begin{bmatrix}\varphi_1, & \cdots, & \varphi_{15}\\ x'_1, & \cdots, & x'_{15}\end{bmatrix}$$

be the corresponding matrix of the determinant

$$VD_2\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{15}\\ x'_1, & \cdots, & x'_{15}\end{pmatrix}.$$

Then we have

$$\begin{aligned} VD_2\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{15}\\ x_1, & \cdots, & x_{15}\end{pmatrix} &= VD_2\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{15}\\ x'_1, & \cdots, & x'_{15}\end{pmatrix}\\ &= \det\Bigg(P[6,1(-b_2^2)]P\Big[6,4\Big(\frac{b_2^2}{a_2^2}\Big)\Big]P[9,2(-b_2^2)]P\Big[9,7\Big(\frac{b_2^2}{a_2^2}\Big)\Big]\\ &\qquad P[10,3(-b_2^2)]P\Big[10,8\Big(\frac{b_2^2}{a_2^2}\Big)\Big]P[13,4(-b_2^2)]P\Big[13,11\Big(\frac{b_2^2}{a_2^2}\Big)\Big]\\ &\qquad P[14,5(-b_2^2)]P\Big[14,12\Big(\frac{b_2^2}{a_2^2}\Big)\Big]P[15,6(-b_2^2)]P\Big[15,13\Big(\frac{b_2^2}{a_2^2}\Big)\Big]\\ &\qquad VD_2\begin{bmatrix}\varphi_1, & \cdots, & \varphi_{15}\\ x'_1, & \cdots, & x'_{15}\end{bmatrix}\Bigg). \end{aligned}$$

Take notice of

$$v_i'^2 = b_2^2 - \frac{b_2^2}{a_2^2}u_i'^2, \quad i = 7, \cdots, 15.$$

Interchanging the rows appropriately, we can change the above determinant to a partitioned determinant $\begin{vmatrix}F & O\\ H & G\end{vmatrix}$, where O is the 6×9 zero matrix. Since $\begin{vmatrix}F & O\\ H & G\end{vmatrix} = \det F\cdot\det G$, we have

$$VD_2\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{15}\\ x_1, & \cdots, & x_{15}\end{pmatrix} = \pm\Bigg[\prod_{i=1}^{6}\Big(v_i'^2 + \frac{b_2^2}{a_2^2}u_i'^2 - b_i^2\Big)\Bigg]$$

$$\cdot VD_1\begin{bmatrix}\varphi_1, & \cdots, & \varphi_6\\ x_1', & \cdots, & x_6'\end{bmatrix}\cdot\begin{vmatrix}1 & \cdots & 1\\ u_7' & \cdots & u_{15}'\\ v_7' & \cdots & v_{15}'\\ u_7'^2 & \cdots & u_{15}'^2\\ u_7'v_7' & \cdots & u_{15}'v_{15}'\\ u_7'^3 & \cdots & u_{15}'^3\\ u_7'^2v_7' & \cdots & u_{15}'^2v_{15}'\\ u_7'^4 & \cdots & u_{15}'^4\\ u_7'^3v_7' & \cdots & u_{15}'^3v_{15}'\end{vmatrix} \qquad (1)$$

Denote the last determinant of the above expression by D, and set

$$u_l' = a_2\cos\theta_l, \quad v_l' = b_2\sin\theta_l, \quad 7\le l\le 15.$$

Then

$$D = a_2^{16}b_2^4\cdot\begin{vmatrix}1 & \cdots & 1\\ \cos\theta_7 & \cdots & \cos\theta_{15}\\ \sin\theta_7 & \cdots & \sin\theta_{15}\\ \cos^2\theta_7 & \cdots & \cos^2\theta_{15}\\ \cos\theta_7\sin\theta_7 & \cdots & \cos\theta_{15}\sin\theta_{15}\\ \cos^3\theta_7 & \cdots & \cos^3\theta_{15}\\ \cos^2\theta_7\sin\theta_7 & \cdots & \cos^2\theta_{15}\sin\theta_{15}\\ \cos^4\theta_7 & \cdots & \cos^4\theta_{15}\\ \cos^3\theta_7\sin\theta_7 & \cdots & \cos^3\theta_{15}\sin\theta_{15}\end{vmatrix}$$

$$= \Big(\frac{1}{2^{1+2+3}}\Big)^2 a_2^{16}b_2^4\cdot\begin{vmatrix}1 & \cdots & 1\\ \cos\theta_7 & \cdots & \cos\theta_{15}\\ \sin\theta_7 & \cdots & \sin\theta_{15}\\ \cos 2\theta_7 & \cdots & \cos 2\theta_{15}\\ \sin 2\theta_7 & \cdots & \sin 2\theta_{15}\\ \cos 3\theta_7 & \cdots & \cos 3\theta_{15}\\ \sin 3\theta_7 & \cdots & \sin 3\theta_{15}\\ \cos 4\theta_7 & \cdots & \cos 4\theta_{15}\\ \sin 4\theta_7 & \cdots & \sin 4\theta_{15}\end{vmatrix}$$

$$= \Big(\frac{1}{2^{1+2+3}}\Big)^2\Big(\frac{1}{2}\Big)^4\Big(\frac{1}{2i}\Big)^4 a_2^{16}b_2^4\cdot\begin{vmatrix}1 & \cdots & 1\\ e^{i\theta_7}+e^{-i\theta_7} & \cdots & e^{i\theta_{15}}+e^{-i\theta_{15}}\\ e^{i\theta_7}-e^{-i\theta_7} & \cdots & e^{i\theta_{15}}-e^{-i\theta_{15}}\\ e^{i2\theta_7}+e^{-i2\theta_7} & \cdots & e^{i2\theta_{15}}+e^{-i2\theta_{15}}\\ e^{i2\theta_7}-e^{-i2\theta_7} & \cdots & e^{i2\theta_{15}}-e^{-i2\theta_{15}}\\ e^{i3\theta_7}+e^{-i3\theta_7} & \cdots & e^{i3\theta_{15}}+e^{-i3\theta_{15}}\\ e^{i3\theta_7}-e^{-i3\theta_7} & \cdots & e^{i3\theta_{15}}-e^{-i3\theta_{15}}\\ e^{i4\theta_7}+e^{-i4\theta_7} & \cdots & e^{i4\theta_{15}}+e^{-i4\theta_{15}}\\ e^{i4\theta_7}-e^{-i4\theta_7} & \cdots & e^{i4\theta_{15}}-e^{-i4\theta_{15}}\end{vmatrix}$$

$$= \pm\Big(\frac{1}{2^{1+2+3}}\Big)^2\Big(\frac{1}{2i}\Big)^4 a_2^{16} b_2^4 \cdot \begin{vmatrix} 1 & \cdots & 1 \\ e^{i\theta_7} & \cdots & e^{i\theta_{15}} \\ e^{-i\theta_7} & \cdots & e^{-i\theta_{15}} \\ e^{i2\theta_7} & \cdots & e^{i2\theta_{15}} \\ e^{-i2\theta_7} & \cdots & e^{-i2\theta_{15}} \\ e^{i3\theta_7} & \cdots & e^{i3\theta_{15}} \\ e^{-i3\theta_7} & \cdots & e^{-i3\theta_{15}} \\ e^{i4\theta_7} & \cdots & e^{i4\theta_{15}} \\ e^{-i4\theta_7} & \cdots & e^{-i4\theta_{15}} \end{vmatrix}$$

$$= \pm\Big(\frac{1}{2^{1+2+3}}\Big)^2\Big(\frac{1}{2i}\Big)^4 a_2^{16} b_2^4 \cdot \prod_{j=7}^{15} e^{-i4\theta_j} \cdot \begin{vmatrix} 1 & \cdots & 1 \\ e^{i\theta_7} & \cdots & e^{i\theta_{15}} \\ e^{i2\theta_7} & \cdots & e^{i2\theta_{15}} \\ e^{i3\theta_7} & \cdots & e^{i3\theta_{15}} \\ e^{i4\theta_7} & \cdots & e^{i4\theta_{15}} \\ e^{i5\theta_7} & \cdots & e^{i5\theta_{15}} \\ e^{i6\theta_7} & \cdots & e^{i6\theta_{15}} \\ e^{i7\theta_7} & \cdots & e^{i7\theta_{15}} \\ e^{i8\theta_7} & \cdots & e^{i8\theta_{15}} \end{vmatrix}$$

$$= \pm\Big(\frac{1}{2^{1+2+3}}\Big)^2\Big(\frac{1}{2i}\Big)^4 a_2^{16} b_2^4 \cdot \prod_{j=7}^{15} e^{-i4\theta_j} \cdot \prod_{6<p<q\le 15} (e^{i\theta_q} - e^{i\theta_p}).$$

Substituting this result in (1), we have

$$VD_2\begin{pmatrix} \varphi_1, & \cdots, & \varphi_{15} \\ x_1, & \cdots, & x_{15} \end{pmatrix} = \pm\frac{a_2^{(2\cdot 2)^2} b_2^{(2\cdot 2)^2}}{2^{(2\cdot 2)^2}} \cdot \Big[\prod_{s=1}^{6}\Big(\frac{u_s'^2}{a_2^2} + \frac{v_s'^2}{b_2^2} - 1\Big)\Big] \cdot \prod_{j=7}^{15} e^{-i4\theta_j}$$
$$\cdot \prod_{6<p<q\le 15} (e^{i\theta_q} - e^{i\theta_p}) \cdot VD_1\begin{pmatrix} \varphi_1, & \cdots, & \varphi_6 \\ x_1', & \cdots, & x_6' \end{pmatrix}.$$

There is one and only one ellipse γ_{2s}:

$$\frac{(u-\lambda_2)^2}{\tau_s^2 a_2^2} + \frac{(v-\mu_2)^2}{\tau_s^2 b_2^2} = 1$$

passing through the point $x_s = (u_s, v_s)$, hence

$$VD_2\begin{pmatrix} \varphi_1, & \cdots, & \varphi_{15} \\ x_1, & \cdots, & x_{15} \end{pmatrix} = \pm\frac{a_2^{(2\cdot 2)^2} b_2^{(2\cdot 2)^2}}{2^{(2\cdot 2)^2}} \cdot \Big[\prod_{s=1}^{6}(\tau_s^2 - 1)\Big] \cdot \prod_{j=7}^{15} e^{-i4\theta_j}$$
$$\cdot \prod_{6<p<q\le 15} (e^{i\theta_q} - e^{i\theta_p}) \cdot VD_1\begin{pmatrix} \varphi_1, & \cdots, & \varphi_6 \\ x_1, & \cdots, & x_6 \end{pmatrix}.$$

Starting from the center point O_2 of the ellipse X_2, we draw a ray passing through x_s, intersecting the ellipse X_2 at z_{2s}. We have

$$VD_2\begin{pmatrix} \varphi_1, & \cdots, & \varphi_{15} \\ x_1, & \cdots, & x_{15} \end{pmatrix} = \pm\frac{a_2^{(2\cdot 2)^2} b_2^{(2\cdot 2)^2}}{2^{(2\cdot 2)^2}} \cdot \Big[\prod_{s=1}^{6} \frac{(\tau_s^2 - 1)(a_2^2\cos^2\theta_s + b_2^2\sin^2\theta_s)}{a_2^2\cos^2\theta_s + b_2^2\sin^2\theta_s}\Big]$$

$$\cdot \prod_{j=7}^{15} e^{-i4\theta_j} \cdot \prod_{6<p<q\leq 15} (e^{i\theta_q} - e^{i\theta_p}) \cdot VD_1 \begin{pmatrix} \varphi_1, & \cdots, & \varphi_6 \\ x_1, & \cdots, & x_6 \end{pmatrix}$$

$$= \pm \frac{a_2^{(2\cdot 2)^2} b_2^{(2\cdot 2)^2}}{2^{(2\cdot 2)^2}} \cdot \Big[\prod_{s=1}^{6} \frac{d(x_s, z_{2s})[d(x_s, O_2) + d(z_{2s}, O_2)]}{[d(z_{2s}, O_2)]^2} \Big]$$

$$\cdot \prod_{j=7}^{15} e^{-i2\cdot 2\theta_j} \cdot \prod_{6<p<q\leq 15} (e^{i\theta_q} - e^{i\theta_p}) \cdot VD_1 \begin{pmatrix} \varphi_1, & \cdots, & \varphi_6 \\ x_1, & \cdots, & x_6 \end{pmatrix}.$$

Supposing that $\alpha, \beta, \gamma, \eta$ are real numbers and $\alpha\eta - \gamma\beta = 1$, we now consider the $(n+1) \times (n+1)$ matrix $\mathcal{L}_n = [a_{ij}^n]$, where $[a_{ij}^n]$ are the coefficients in the expansion

$$(\alpha u + \beta v)^{n+1-i} (\gamma u + \eta v)^{i-1} = \sum_{j=1}^{n+1} a_{ij}^n u^{n+1-j} v^{j-1}.$$

We have the following lemma.

Lemma 2. $\det \mathcal{L}_n = 1$ *for all* $n = 0, 1, 2, \cdots$.

It is obvious that $\det \mathcal{L}_0 = 1$ because $\mathcal{L}_0 = [\ 1\ ]$. For the proof for $n = 1, 2, \cdots$ see [1].

Take notice of $\mathcal{L}_1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \eta \end{bmatrix}$. Let $x_i^T = \mathcal{L}_1 y_i^T$, $i = 1, \cdots, L_n$. Then set $N_k = \binom{k+2}{2}$, $(k = 0, 1, \cdots)$, and $N_{-1} = 0$. From the definition of $\mathcal{L}_k$, it is easy to show

$$\begin{bmatrix} \varphi_{N_{k-1}+1}(x_1) & \cdots & \varphi_{N_{k-1}+1}(x_{L_n}) \\ \cdots & \cdots & \cdots \\ \varphi_{N_k}(x_1) & \cdots & \varphi_{N_k}(x_{L_n}) \end{bmatrix} = \mathcal{L}_k \begin{bmatrix} \varphi_{N_{k-1}+1}(y_1) & \cdots & \varphi_{N_{k-1}+1}(y_{L_n}) \\ \cdots & \cdots & \cdots \\ \varphi_{N_k}(y_1) & \cdots & \varphi_{N_k}(y_{L_n}) \end{bmatrix}$$

for $k = 1, 2, \cdots$. Thus we obtain the following lemma.

Lemma3.

$$VD_n \begin{pmatrix} \varphi_1, & \cdots, & \varphi_{L_n} \\ x_1, & \cdots, & x_{L_n} \end{pmatrix} = \det \begin{bmatrix} \mathcal{L}_0 & & & \\ & \mathcal{L}_1 & & \\ & & \ddots & \\ & & & \mathcal{L}_{2n} \end{bmatrix} \cdot VD_n \begin{pmatrix} \varphi_1, & \cdots, & \varphi_{L_n} \\ y_1, & \cdots, & y_{L_n} \end{pmatrix}.$$

3. Results

Theorem 1. *Let* $\hat{X}_n = \{x_i : i = 1, \cdots, L_n\}$ *be a set of distinct nodes in* R^2 *satisfying ETNCA. Let* O_j *be the center point of the ellipse* X_j*, and* a_j*,* b_j *be the two semi-axes of the ellipse. Let the rotation* $\mathcal{L}_{j1} = \begin{bmatrix} \alpha_j & \beta_j \\ \gamma_j & \eta_j \end{bmatrix}$ *be chosen such that the two axes of* $\mathcal{L}_{j1}^{-1} X_j$ *are parallel to the u-axis and the v-axis respectively. Set* $\mathcal{L}_{j1}^{-1} x_k^T = y_k^T = (u_k, v_k)^T$ *and*

$$u_k = \lambda_j + a_j \cos\theta_k, \quad v_k = \mu_j + b_j \sin\theta_k, \quad L_{j-1} < k \leq L_j, \quad j = 1, \cdots, n.$$

Draw a ray starting from O_j and passing through x_s $(1 \le s \le L_{j-1})$, and suppose it intersectes X_j at z_{js}, and denote the Euclidean distance between points x^* and y^* by $d(x^*, y^*)$. Then

$$VD_n\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_n}\\ x_1, & \cdots, & x_{L_n}\end{pmatrix} = c\prod_{j=1}^{n}\Big(\frac{a_j b_j}{2}\Big)^{(2j)^2}$$
$$\cdot\Big[\prod_{j=1}^{n}\prod_{s=1}^{L_{j-1}}\frac{d(x_s, z_{js})[d(x_s, O_j)+d(z_{js}, O_j)]}{[d(z_{js}, O_j)]^2}\Big]\cdot\prod_{j=1}^{n}\prod_{L_{j-1}<t\le L_j} e^{-i2j\theta_t}$$
$$\cdot\prod_{j=1}^{n}\prod_{L_{j-1}<p<q\le L_j}(e^{i\theta_q}-e^{i\theta_p}),$$

where $c = 1$ or -1.

Proof. Because α_j, β_j, γ_j, η_j are chosen such that the two axes of $\mathcal{L}_{j1}^{-1}X_j$ are parallel to the u-axis and v-axis respectively, we can write $\mathcal{L}_{n1}^{-1}X_n$ as

$$\frac{(u-\lambda_n)^2}{a_n^2}+\frac{(v-\mu_n)^2}{b_n^2} = 1.$$

Applying lemma 1, 2, 3, we have

$$VD_n\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_n}\\ x_1, & \cdots, & x_{L_n}\end{pmatrix} = c\Big(\frac{a_n b_n}{2}\Big)^{(2n)^2}\cdot\Big[\prod_{s=1}^{L_{n-1}}\frac{d(x_s, z_{ns})[d(x_s, O_n)+d(z_{ns}, O_n)]}{[d(z_{ns}, O_n)]^2}\Big]$$
$$\cdot\prod_{L_{n-1}<t\le L_n} e^{-i2n\theta_t}\cdot\prod_{L_{n-1}<p<q\le L_n}(e^{i\theta_q}-e^{i\theta_p})\cdot VD_{n-1}\begin{pmatrix}\varphi_1, & \cdots, & \varphi_{L_{n-1}}\\ x_1, & \cdots, & x_{L_{n-1}}\end{pmatrix}.$$

Hence, the proof of Theorem 1 is completed by using mathematical induction and the fact that

$$VD_0\begin{pmatrix}\varphi_1\\ x_1\end{pmatrix} = 1.$$

By the way, because $VD_n \ne 0$, we verified the known fact that ETNCA admits unique Lagrange interpolation by a new metheod.

For the point x^* lying on the ellipse X_j, set

$$y^* = \mathcal{L}_{j1}^{-1}x^* = (\lambda_j + a_j\cos\theta^*,\ \mu_j + b_j\sin\theta^*),$$

and let x be a neighbouring point of x^* and x lie on X_j, then set

$$y = \mathcal{L}_{j1}^{-1}x = (\lambda_j + a_j\cos\theta,\ \mu_j + b_j\sin\theta).$$

We define

$$D_{X_j}f(x^*) = \lim_{\theta\to\theta^*}\frac{f(x)-f(x^*)}{\theta-\theta^*}.$$

By using Theorem 1 and taking the above derivative (as a kind of limit) along the ellipses that have coincident nodes, we have

Theorem 2. *Let $\hat{X}_n = \{x_i : i = 1, \cdots, L_n\}$ satisfy ETNCB and*

$$y^*_{uv} = \mathcal{L}^{-1}_{u1} y_{uv} = (\lambda_u + a_u \cos\theta_{uv},\ \mu_u + b_u \sin\theta_{uv}), \quad v = 1, \cdots, k_u, \quad u = 1, \cdots, n.$$

Starting from O_j, draw a ray passing through y_{uv} $(1 \le u < j)$, intersecting X_j at z_{juv}. Then

$$\begin{aligned} HD_n \begin{pmatrix} \varphi_1, & \cdots, & \varphi_{L_n} \\ x_1, & \cdots, & x_{L_n} \end{pmatrix} &= c \prod_{j=1}^{n} \Big(\frac{a_j b_j}{2}\Big)^{(2j)^2} \cdot \prod_{j=1}^{n} \prod_{r=1}^{k_j} e^{-i2j\ell_{jr}\theta_{jr}} \\ &\cdot \prod_{j=1}^{n} \Big\{ \frac{d(x_1, z_{j1})[d(x_1, O_j) + d(z_{j1}, O_j)]}{[d(z_{j1}, O_j)]^2} \\ &\cdot \prod_{r=1}^{j-1} \prod_{v=1}^{k_r} \Big[\frac{d(y_{rv}, z_{jrv})[d(y_{rv}, O_j) + d(z_{jrv}, O_j)]}{[d(z_{jrv}, O_j)]^2}\Big]^{\ell_{rv}} \Big\} \\ &\cdot \Big[\prod_{j=1}^{n} \prod_{1 \le s < t \le k_j} (e^{i\theta_{jt}} - e^{i\theta_{js}})^{\ell_{js}\ell_{jt}} \prod_{u=1}^{k_j} \prod_{\rho=1}^{\ell_{ju}-1} \rho! (i e^{i\theta_{ju}})^{\rho} \Big], \end{aligned}$$

where $c = 1$ or -1. Therefore $HD_n \neq 0$ and $\hat{X}_n$ admits unique Hermite interpolation from polynomials of total degree $2n$.

Next we consider the ETNCC. For the point x^* that on the ellipse X_j, we define $N_{X_j} f(x^*)$ as the directional derivative of function f at the point x^* along the line $O_j x^*$. By using Theorem 1 and taking the limit, we have

Theorem 3. *Let $\hat{X}_n = \{x_1, \cdots, x_{L_n}\}$ be a set of distinct nodes satisfying ETNCC. Let O_r be the center point of the ellipse Y_r and a_r, b_r be the lengths of the two semi-axes respectively $(r = 1, \cdots, d)$. Then*

$$\begin{aligned} BD_n \begin{pmatrix} \varphi_1, & \cdots, & \varphi_{L_n} \\ x_1, & \cdots, & x_{L_n} \end{pmatrix} &= c \prod_{r=1}^{d} \prod_{m_1 + \cdots + m_{r-1} < p \le m_1 + \cdots + m_r} \Big(\frac{a_r b_r}{2}\Big)^{(2p)^2} \\ &\cdot \Big[\prod_{r=1}^{d} \prod_{x_s \in B_r} d(x_s, z_{rs}) \prod_{t=1}^{m_r - 1} (t!)^{4(m_1 + \cdots + m_r - t) + 1} \Big] \\ &\cdot \Big\{ \prod_{r=1}^{d} \prod_{u=1}^{m_r} \prod_{s=1}^{L_{m_1 + \cdots + m_{r-1} + u - 1}} \frac{d(x_s, O_r) + d(z_{rs}, O_r)}{[d(z_{rs}, O_r)]^2} \Big\} \\ &\cdot \prod_{j=1}^{n} \prod_{L_{j-1} < w \le L_j} e^{-i2j\theta_w} \cdot \prod_{j=1}^{n} \prod_{L_{j-1} < p < q \le L_j} (e^{i\theta_q} - e^{i\theta_p}), \end{aligned}$$

where $c = 1$ or -1 and $B_r = \{x_1\} \bigcup Y_1 \bigcup \cdots \bigcup Y_{r-1}$. Therefore $BD_n \neq 0$ and $\hat{X}_n$ admits unique Birkhoff interpolation from polynomials of total degree $2n$.

Remark 1. If every ellipse is a circle in an ETNC, we obtain the Circle Type Node Configuration (CITNC).

Remark 2. We have discussed the interpolation problems about other Curve Type Node Configurations, the Cross Type Node Configuration and the Vertical Line

Type Node Configuration in other papers. Some of these results have been extened to $R^s(s > 2)$. We have also discussed the error estimate of the interpolation and obtained an algorithm for the interpolation polynomial. We call this algorithm the cubic iteration method, which is an extension of the Aitken method in R^s.

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