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QUASI-INTERPOLATING OPERATORS AND THEIR APPLICATIONS IN HYPERSINGULAR INTEGRALS*1)

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Abstract

The purpose of this paper is to propose and study a class of quasi-interpolating operators in multivariate spline space $S_2^1(\Delta_{mn}^{2*})$ on non-uniform type-2 triangulation. Based on the operators, we construct cubature formula for two-dimensional hypersingular integrals. Some computing work have been done and the results are quite satisfactory.

Key words: Hypersingular integral, finite-part integral, quasi-interpolating operator, non-uniform type-2 triangulation.

1. Introduction

Since P. Zwart obtained an expression of bivariate B-spline[2], R.-H Wang and C.K. Chui have developed a series of results, especially, the quasi-interpolating operators of $S_2^1(\Delta_{mn}^2)$ on uniform type-2 triangulation and its approximation properties[1] which have widespread applications in Mechanics and Engineering. Furthermore, R-H Wang and C.K. Chui also obtained the function with minimum support in $S_2^1(\Delta_{mn}^{2*})$ on non-uniform type-2 triangulation and the basis of $S_2^1(\Delta_{mn}^{2*})$[4]. In this paper we introduce some quasi-interpolating operators of $S_2^1(\Delta_{mn}^{2*})$ on non-uniform type-2 triangulation and show their approximation properties. By using the operators we construct cubature formulas.which can be used to evaluate hypersingular integrals arisen from many mechanics and engineering problems.

2. Quasi-Interpolating Operators of $S(\Delta_{mn}^{2*})$

Let Δ_{mn}^{2*} be a non-uniform type-2 triangulation on the domain $\Omega[a,b]\otimes[c,d]$, and

$$x_{-2} < x_{-1} < a = x_0 < \cdots < x_m = b < x_{m+1} < x_{m+2},$$
$$y_{-2} < y_{-1} < c = y_0 < \cdots < y_n = d < y_{n+1} < y_{n+2}.$$

First we consider the linear operators

$$V_{mn} : C(\Omega) \to S_2^1(\Delta_{mn}^{2*}); \tag{2.1}$$

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$$V_{mn}(f) = \sum_{ij} f\Big(\frac{x_i + x_{i+1}}{2}, \quad \frac{y_j + y_{j+1}}{2}\Big) B_{ij}(x, y); \tag{2.2}$$

It is similar to the result in [1], we have the following results.

Theorem 2.1. *For $f \in P_1$ and $f = xy$, we have*

$$V_{mn}(f) = f. \tag{2.3}$$

Because of the theorem 2.7[4], we only need to verify the theorem for and $f(x, y) = x, y$ and xy. Since $V_{mn}(f)$ is a linear operator, we can only examine them in the domain D_{ij}:

$$D_{ij} = (x_i, x_{i+1}) \otimes (y_j, y_{j+1}); \quad (i = 0, \cdots m + 1; j = 0, \cdots n + 1).$$

By the computation of the values of $V_{mn}(f)$ at eight points

$$\begin{aligned} &(x_i, y_j), (x_i, y_{j+1}), \Big(\frac{x_i + x_{i+1}}{2}, y_j\Big), \Big(\frac{x_i + x_{i+1}}{2}, y_{j+1}\Big), \\ &(x_{i+1}, y_j), (x_{i+1}, y_{j+1}), \Big(x_i, \frac{y_j + y_{j+1}}{2}\Big), \Big(x_{i+1}, \frac{y_j + y_{j+1}}{2}\Big); \end{aligned} \tag{2.4}$$

we have $V_{mn}(f)$, at all of the eight points. Since the eight point are the adapt interpolating knot group in D_{ij}, we have $V_{mn}(f) = f$ in D_{ij}. Therefore, the theorem holds.

It is easy to prove that $V_{mn}(f) \neq f$, as $f = x^2$ or y^2, In order to make the theorem holds for all polynomials in P_2, we have to introduce another linear operator

$$W_{mn} : C(\Omega) \to S_2^1(\Delta_{mn}^{2*}), \tag{2.5}$$

$$W_{mn}(f) = \sum_{ij} \lambda_{ij}(f) B_{ij}(x, y), \tag{2.6}$$

where

$$\begin{aligned} \lambda_{ij}(f) = &2f\Big(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\Big) \\ &- \frac{1}{4}(f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})). \end{aligned} \tag{2.7}$$

It is similar to result in [4], we have the following theorem:

Theorem 2.2. *$W_{mn}(f) = f$ for any $f \in P_2$.*

By the theorem 2.1, we have $W_{mn}(f) = f$ for $f \in P_1$ and $f = xy$. Now we need to verify $W_{mn}(f) = f$ for $f(x, y) = x^2$ and y^2. Just the same as the proof of theorem 2.1, we only need to compute the values of $W_{mn}(f)$ in D_{ij} at the points

$$\begin{aligned} &(x_i, y_j), (x_i, y_{j+1}), \Big(\frac{x_i + x_{i+1}}{2}, y_j\Big), \Big(\frac{x_i + x_{i+1}}{2}, y_{j+1}\Big) \\ &(x_{i+1}, y_j),\ (x_{i+1}, y_{j+1}), \Big(x_i, \frac{y_j + y_{j+1}}{2}\Big), \Big(x_{i+1}, \frac{y_j + y_{j+1}}{2}\Big). \end{aligned} \tag{2.8}$$

By means of computation of the value of $W_{mn}(f)$, we have $W_{mn}(f) = f$ in D_{ij} for $f(x, y) = x^2$ and y^2. Therefore the theorem 2.2 holds.

3. Approximation Properties of $S(\Delta_{mn}^{2*})$

In this part we discuss how well the linear operators stated as above to approximate the function $f \in C^p(\Omega)$, $p = 0, 1, 2, 3$. Let

$$\omega_k(f,\delta) = \sup\{f(x,y) - f(u,v)| : (x,y),(y,v) \in K,\ |(x,y)-(u,v)| < \delta\}. \tag{3.1}$$

$$\begin{aligned} \delta_{mn} &= \max_{ij}[h_i, k_j], \\ \delta^*_{mn} &= \max(\sqrt{9h^2+k^2}, \sqrt{9k^2+h^2}), \\ h &= \max_i(h_i), \quad k = \max_j(k_j); \end{aligned} \tag{3.2}$$

and K be a compact set, $K \subset \Omega$.

Theorem 3.1. *Let $f \in C(K)$ and $m, n \geq N_0$ we have*

$$\|f - V_{mn}(f)\|_\Omega \leq \omega_k(f, \delta^*_{mn}), \tag{3.3}$$

if $f \in C^1(K)$ then

$$\|f - V_{mn}(f)\|_\Omega \leq \delta_{mn} \max(\omega_\Omega(f_1, \delta_{mn}/2), \omega_\Omega(f_2, \delta_{mn}/2)), \tag{3.4}$$

if $f \in C^2(K)$ then

$$\|f - V_{mn}(f)\|_\Omega \leq \delta^2_{mn}\|D^2 f\|. \tag{3.5}$$

Since δ^*_{mn} is more then "radius" of the support of B_{ij}, it is obvious that (3.3) holds.

Let $f \in C^1(K)$, F is the closure of triangle cell of B_{ij}, and make

$$\|f - V_{mn}(f)\|_\Omega = \|f - V_{mn}(f)\|_F, \tag{3.6}$$

and let (x_0, y_0) be $\Big(x_i, \dfrac{y_j + y_{j+1}}{2}\Big)$ or $\Big(\dfrac{x_i + x_{i+1}}{2}, y_j\Big)$, by means of Mean Theorem, we have

$$f(x,y) = p_1(x,y) + (f_1(u,v) - f_1(x_0,y_0))(x-x_0) + (f_2(u,v) - f_2(x_0,y_0))(y-y_0), \tag{3.7}$$

where

$$\begin{aligned} &p_1(x,y) = f(x_0,y_0) + f_1(x_0,y_0)(x-x_0) + f_1(x_0,y_0)(y-y_0), \\ &(u,v) = t(x,y) + (1-t)(x_0,y_0), \quad 0 \leq t \leq 1. \end{aligned} \tag{3.8}$$

By the theorem 3.1[4] and $\|V_{mn}\| = 1$, we have

$$\|f - V_{mn}(f)\|_F \leq \|f - p_1\|_F + \|V_{mn}(f - p_1)\|_F \leq 2\|f - p_1\|_F. \tag{3.9}$$

Therefore formula (3.4) holds.

If $f \in C^2(\Omega)$, by means of Taylor expansion, it is easy to obtain the (3.5).

We also have the following results for bivariate linear operator W_{mn}.

Theorem 3.2. *Let $f \in C^2(\Omega)$ and $m, n \geq N_0$. If $f \in C^2(K)$, then*

$$\|f - W_{mn}(f)\|_\Omega \leq \frac{1}{2}\delta_{mn}^2 \max[\omega_\Omega(f_{11}, \delta_{mn}/2), 2\omega_\Omega(f_{12}, \delta_{mn}/2),\ \omega_\Omega(f_{22}, \delta_{mn}/2)], \quad (3.10)$$

if $f \in C^3(K)$ then

$$\|f - W_{mn}(f)\|_\Omega \leq \frac{1}{12}\delta_{mn}^3 \|D^3 f\|. \quad (3.11)$$

Taking note of $\|W_{mn}\| = 3$. It is easy to prove the theorem in term of Taylor expansion.

4. Cubature Formulas

We consider integrals of the form

$$\int_\Omega K_p(v_0; v)\Phi(v)dv, \quad v_0 \in \Omega \subset R^2, \quad (4.1)$$

where the kernel K_p admit the expansion

$$K_p(v_0; v) = \sum_{l=0}^{p+1} \frac{f_{p-1}(v_0; \theta)}{r^{p+2-l}} + K_p^*(v_0, v). \quad (4.2)$$

(ρ, θ) denote the polar coordinate of v with respect to v_0. $K_p^*(v_0; v)$ may still become infinite at v_0, but with order less than 2. For simplicity we assume the functions f_{p-l}, Φ and K_p^* smooth in domain Ω. Therefore we can only consider integrals of the form

$$I = \int_\Omega \frac{f_p(v_0, \theta)}{r^p}\Phi(v)dv, \quad p = 2, 3. \quad (4.3)$$

Based on the definition of finite part integrals and the quasi interpolating operators mentioned as above, we obtain the following cubature formula for (4.3)

$$I = \int_\Omega \frac{f_p(v_0, \theta)}{r^p}\Phi(v)dv = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=i-1}^{i+1}\sum_{q=j-1}^{j+1} \lambda_{kq}(\Phi)b_{ijkq} + R_{nm}(K\Phi); \quad (4.4)$$

$$b_{ijkq} = \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} \frac{f_p(v_0; \vartheta)B_{kq}(v)}{r^p}dv. \quad (4.5)$$

Obviously, according to the definition of finite-part integrals[3], b_{ijkq} can be computed by the Gauss-Legendre or Gauss-Lobatto rule.

By the theorem 3.2, we can obtained some convergence results about the cubature formula (4.4).

Theorem 4.1. *If in (4.4), we assume $\Phi \in C^{p+2}(\Omega)$ and $f_p \in C[0, \omega]$, for the remainder term we have*

$$R_{mn}(K\Phi) \leq C\omega_\Omega(g, \delta_{mn}/2), \quad (4.6)$$

where C is a constant.

Theorem 4.2. *If we assume* $\Phi \in C^{p+3}(\Omega)$ *and* $f_p \in C^2[0,\omega]$*, the remainder in* (4.4)

$$R_{nm}(K\Phi) \le C\delta_{mn}^2 \max[\omega_\Omega(g_{11}, \delta_{mn}/2), 2\omega_\Omega(g_{12}, \delta_{mn}/2), \omega_\Omega(g_{22}, \delta_{mn}/2)], \tag{4.7}$$

where C *is a constant.*

Theorem 4.3. *If we assume* $\Phi \in C^{p+4}(\Omega)$ *and* $f_p \in C^3[0,\omega]$*, the remainder in* (4.4)

$$R_{nm}(K\Phi) \le C\delta_{mn}^3 \|D^3(g)\|, \tag{4.8}$$

where C *is a constant.*

In the boundary element methods, we often need to evaluate the CPV or hypersingular surface integrals of type

$$I = \int_S K_p(v_0, v - v_0)\Phi(v)dS_v, \quad p \text{ int } eger, \tag{4.9}$$

where $S \subset R^3$ has an analytic parametric representation $S : (x(u,w), y(u,w), z(u,w))$, the kernel K_p is homogeneous of degree $-p-2$ in the second argument and has a pole of order $p+2$ at $v = v_0$ and $\Phi(v)$ is a smooth function.

We introduce a polar coordinate system (ρ, θ) centered at η, image of singular point v_0.

$$\begin{aligned} u &= \eta_1 + \rho\cos\theta \\ w &= \eta_2 + \rho\sin\theta \end{aligned} \tag{4.10}$$

Since $K\Phi$ is singular of order $\rho^{-(p+2)}$, we have (Laurent) series expansion with respect to ρ in the form

$$K\Phi = \Big(\sum_{i=1}^{p+2} F_{-i}(\theta)/\rho^i\Big) + O(1). \tag{4.12}$$

Because of the integral

$$\int_S \Big(K\Phi - \Big(\sum_{i=1}^{p+2} F_{-p}(\theta)/\rho^i\Big)\Big)dS \tag{4.13}$$

is a regular integral, Replacing $\Big(K\Phi - \Big(\sum_{i=1}^{p+2} F_{-p}(\theta)/\rho^i\Big)\Big)$ in (4.13) by its quasi-interpolation operator (2.6) and employing Gauss-Legendre rule, we can construct the following cubatrue formula of the hypersingular integrals (4.9)

$$\begin{aligned} I &= \int_S K_p(v_0, v - v_0)\Phi(v)dS_V \\ &= \int_S \Big(K\Phi - \Big(\sum_{i=1}^{p+2} F_{-p}(\theta)/\rho^i\Big)\Big)dS - \int_S \Big(\sum_{i=1}^{p+2} F_{-p}(\theta)/\rho^i\Big)dS \end{aligned}$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=i-1}^{i+1}\sum_{q=j-1}^{j+1} \lambda_{kq}(g)b_{ijkq} - \sum_{|i|\leq p}\int_0^{\omega} F_{-i}(\theta)h_i(\theta)d\theta + R_{nm}(K\Phi); \tag{4.14}$$

where $h_{|i|}(\theta)$ is given in [3] and

$$b_{ijkq} = \int_{x_i}^{x_i+1}\int_{y_j}^{y_{j+1}} B_{kq}(v)dv.$$

$$g(v) = K\Phi(v) - \sum_{0\leq i\leq p} F_p(\theta)/\rho^i. \tag{4.15}$$

The estimate of the remainder $R_{nm}(K\Phi)$ in (4.14) is the same as the plane region's, here we do not repeat it.

5. Spline Scheme of Singular Fredholm Integral Equation of Second Kind

Without loss generality, we consider the two-dimensional second kind Fredholm integral equations of the form

$$u(x,y) = \int_{\Omega} K(x,y,s,t)u(s,t)dtds + g(x,y), \quad (x,y)\in\Omega, \tag{5.1}$$

where

$$k(x,y,s,t) = \frac{f(x,y,s,t)}{r^d}, \quad d=2,3, r = \|(x,y)-(s,t)\|_2,$$
$$f\in E, g(x,y)\in C(\Omega), E\{(x,y,s,t): a\leq x,t\leq b, c\leq y,t\leq d\}. \tag{5.2}$$

We use the quasi-interpolating function $W_{mn}(u)(s,t)$ as substitute for the integrating factor $u(s,t)$, then the integral equation (5.1) turn into following $(2nm+5m+5n+13)\times(2nm+5m+5n+13)$ linear algebraic system of equations:

$$u^{hk}(x_p,y_q) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=i-1}^{i+1}\sum_{q=j-1}^{j+1}\lambda_{kq}(u^{hk})b_{ijkq}(x_p,y_q) + g(x_p,y_q) \tag{5.3}$$

$$u^{hk}(x_p,y_q) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=i-1}^{i=1}\sum_{q=j-1}^{j+1}\lambda_{kq}(u^{hk})b_{ijkq}(x_p,y_q) + g(x_p,y_q); \tag{5.4}$$

where

$$b_{ijkq}(*,*) = \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} B_{kq}(v)\frac{f(*,*,s,t)}{r^p}dsdt. \tag{5.5}$$

$(p=-1,0,1,\cdots m+1; q=-1,0,1,\cdots n+1, p'=0,1,\cdots m+1; q'=0,1,\cdots,n+1,).$

where (x_p,y_q), $(x_{p'},y_{q'})$ are the knots of non-uniform type-2 triangulation Δ_{mn}^{2*}, (Fig.1).

Once the system is solved, the solution $u(x,y)$ of the equation (5.1) can be evaluated by the values of $u^{hk}(x_i,y_j)$, $u^{hk}(x_{i'},y_{j'})$. Where

$$(i=-1,0,1,\cdots,m+1;\ j=-1,0,1,\cdots,n+1,$$
$$i'=0,1,\cdots,m+1; j'=0,1,\cdots,n+1).$$

Fig. 1

6. Numerical Examples

Without loss of generality, the following CPV integral is considered

$$I_v=\int_v \frac{x_1-y_1}{r^3}dx_1dx_2 \tag{6.1}$$

where

$$v=[-1,1]\times[-1,1],\quad r=\|x-y\|_2.$$

The CPV integral equation (6.1) was chosen to be compared with the closed form shown in Theocaris et al.(1980).

Notice, however, that the integrand function in equation (6.1) shows all the relevant features of any CPV integral arising in the BEM. Therefore, notwithstanding the apparent simplicity, this example provides a significant test for the methods under investigation.

The coordinates of the singular point y in equation (6.1) was chosen in the three cases (Fig.2)

Case (a): $y=(0.6,0)$;

Case (b): $y=(0.6,0.5)$;

Case (c): $y=(-0.3,0.2)$;

Formula (4.4) was employed for the computation. Table1 gives the numerical results for the three cases of Fig.2 together with the exact value. At least three significant digits are exact.

Table 1 Numerical results for the CPV integral of equation (6.1) for the three cases as mentioned above

step length h	case(a) $y=(0.6,0)$	case(b) $y=(0.6,0.5)$	case(c) $y=(-0.3,0.2)$
0.1	−2.113623	−1.935197	0.878973
0.05	−2.114145	−1.935682	0.8790
Exact	−2.114175	−1.935711	0.879017

Fig.2 Plane Region (two-dimensional internal cells) singular points: (a) $y=(0.6,0)$, (b) $y=(0.6,0.5)$, (c) $y=(-0.3,0.2)$

As a more general example, let us consider the following hypersingular integral

$$I_S=\int_S \frac{-1}{4\pi r^3}\Big[3r_3\frac{\partial r}{\partial n}-n_3(x)\Big]dS, \tag{6.2}$$

where S represents a 90-deg cylindrical panel (Fig.3 radius = 1, length = 2), $r=\|x-y\|_2$, $n(x)$ is the outward unit normal vector at s, and $n_i=\dfrac{x_i-y_i}{r}$.

We also choose the singular points in the three cases as follows
Case (1): $y = (\pi/4, 1, \pi/4)$;
Case (2): $y = (\pi/4, 1.66, \pi/4)$;
Case (3): $y = (1.64\pi, 1.64, 1.64\pi/4)$;

The computation was performed according to formula (4,14). Since the evaluation is performed in the parameter plane, there is no difference in the numerical implementation between a flat and a curved one.

The results are reported in Table 2 for the three cases of Fig.3 together with the reference values obtained by M. Guiggiani[5].

Table 2 Numerical results for the hypersingular integral of equation (6.2) for the three cases mentioned as above

step length	case(1)	case(2)	case(3)
h	y_a	y_b	y_c
0.2	-0.344076	-0.498002	-0.877447
0.1	-0.344022	-0.498009	-0.877437
0.05	-0.344015	-0.498016	-0.877460
Reference value (M. Guiggiani[5])	-0.343807	-0.497099	-0.877214

Fig.3 Curved boundary element and collocation points

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