Journal of Computational Mathematics, Vol.9, No.1, 1991, 17–27.

MULTIGRID MULTI-LEVEL DOMAIN DECOMPOSITION*

Zhang Sheng   Huang Hong-ci
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

The domain decomposition method in this paper is based on PCG(Preconditioned Conjugate Gradient method). If N is the number of subdomains, the number of subproblems solved parallelly in a PCG step is $\frac{4}{3}(1-\frac{1}{4^{\log N+1}})N$. The condition number of the preconditioned system does not exceed $O(1+\log N)^3$. It is completely independent of the mesh size. The number of iterations required, to decrease the energy norm of the error by a fixed factor, is proportional to $O(1+\log N)^{\frac{3}{2}}$.

§1. Triangulation and Subdomain Selection

Let $\Omega \subset R^2$ be a bounded polygonal region, and let

$$\begin{cases} a(u,v)=(f,v), f\in H^{-1}(\Omega), v\in H_0^1(\Omega), \\ u\in H_0^1(\Omega) \end{cases} \tag{1.1}$$

be the variational form of an elliptic operator defined on it. The bilinear form satisfies

$$\begin{cases} a(u,v)=a(v,u), \\ |a(u,v)|\le M'||u||\cdot||v||, \\ a(u,u)\ge M''||u||^2, \end{cases} \tag{1.2}$$

where $||\cdot||$ is the Sobolev norm in $H^1(\Omega)$. From (1.2) the norm is equivalent to that introduced by $a(\cdot,\cdot)$ in $H_0^1(\Omega)$. In what follows, we will consider $H_0^1(\Omega)$ as a Hilbert spase with the inner product $a(\cdot,\cdot)$.

We will approximate (1.1) with the finite element method. Triangular partition and linear continuous elements will be used. The triangulation satisfies quasi-uniformity and inverse hypothesis.

1.1. Triangulation. $\mathcal{T}^0$ is a triangulation of Ω satisfying quasi-uniformity and inverse hypothesis. Divide any triangle T^0 of $\mathcal{T}^0$ into four (Fig.1), and we get a partition $\mathcal{T}^1$ with the first refinement. Continue the process with $\mathcal{T}^1$ similarly. After the m-th refinement, we get the final triangulation $\mathcal{T}^m$, which is the partition we really use for finite element approximation. The mesh size of $\mathcal{T}^m$ is h.

* Received March 10, 1990.

Let $S^m = S_0^h \subset H_0^1(\Omega)$ be the finite element space. $S^0 \subset S^1 \subset \cdots \subset S^m$ are finite element spaces corresponding to 0-level to m-level triangulations. $\hat{\Omega}^l$ represents the set of l-level finite element nodes. $\{\phi_i^l \in S^l, i \in \hat{\Omega}^l\}$ are the usual finite element basis functions.

$$A^l = (a(\phi_i^l, \phi_j^l))_{i,j\in\hat{\Omega}^l} \tag{1.3}$$

is the l-level stiffness matrix, $A^m = A, l = 0, 1, 2, \cdots, m$. $\hat{\Omega} = \hat{\Omega}^m$.

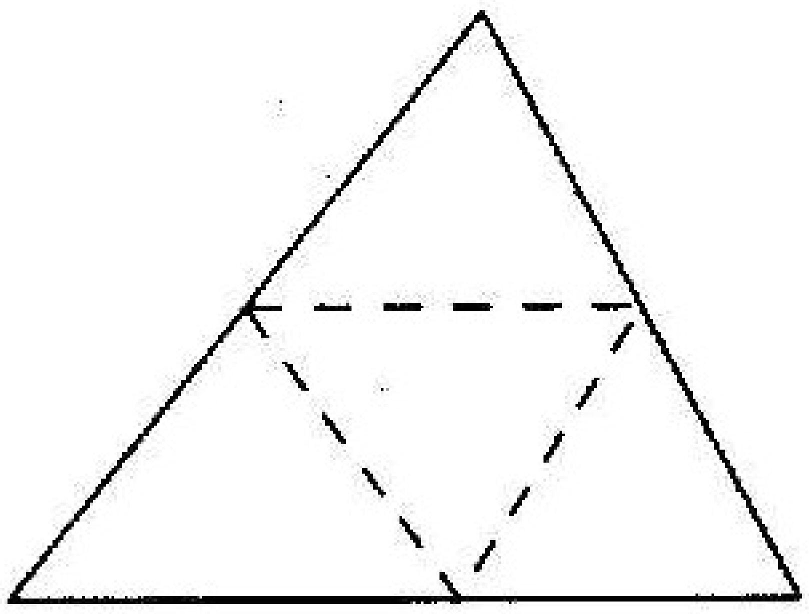

Fig. 1

1.2. Subdomain Selection. $\{\Omega_k^l, l = 0, 1, 2, \cdots, m, k = 1, 2, \cdots, N_l\}$ is a set of open subregions of Ω. For a fixed $l \in \{0, 1, 2, \cdots, m\}$, $\{\Omega_k^l, k = 1, 2, \cdots, N_l\}$ is the set of l-level subdomains.

The following requirements should be met.

A1. $\{\partial\Omega_k^l, k = 1, 2, \cdots, N_l\}$ is a part of the mesh line of the triangulation T^l, $l = 0, 1, 2, \cdots, m$.

A2. For any fixed $l \in \{0, 1, 2, \cdots, m\}$, i.e. on a given level,

$$\bigcup_{k=1}^{N_l} \Omega_k^l = \Omega, \tag{1.4}$$

$\{\Omega_k^l, k = 1, 2, \cdots, N_l\}$ satisfies the quasi-uniformity requirements. H_l is the diameter of l-level subregions, $H_m = H$.

A3. For fixed l, there is another set of subregions $\{\Omega'^l_k, k = 1, 2, \cdots, N_l\}$ so that $\Omega_k^l \subset \Omega'^l_k$ and

$$\mathrm{dist}\{\partial\Omega_k^l \setminus \partial\Omega, \partial\Omega'^l_k \setminus \partial\Omega\} \geq \alpha \cdot H,$$

where α is a fixed constant. At any point in Ω, the number of subregions in $\{\Omega'^l_k, k = 1, 2, \cdots, N_l\}$ which cover this point does not exceed a fixed number (if the coefficient of the function term in the differential operator is strictly positive, this requirement can be released.)

$\hat{\Omega}_k^l = \Omega_k^l \cap \hat{\Omega}^l$ is the set of node points in Ω_k^l. From (1.4) we get

$$\bigcup_{k=1}^{N_l} \hat{\Omega}_k^l = \hat{\Omega}^l. \tag{1.5}$$

It is well known that the relation among the numbers of nodes, triangles and edges of a triangulation is roughly $1 : 2 : 3$, and the numbers of triangles of one refined triangulation is four times that of the original one. Therefore,

$$|\hat{\Omega}^l|/|\hat{\Omega}^{l-1}| \simeq 4, \quad l = 1, 2, \cdots, m.$$

To ensure that the scales of subproblems are roughly equal to each other, we require

A4. $N_l = 4^l$, $l = 0, 1, 2, \cdots, m$.

$N = N_m$ is the number of subregions. 0-level to $(m-1)$-level subregions are considered as auxiliary subdomains. The number of levels $m = \log_4 N$ is independent of the mesh size h.

Under this assumption, we have

$$\left|\hat{\Omega}_k^l\right| \simeq O(\frac{H^2}{h^2}), \quad k = 1, 2, \cdots, N; \quad l = 0, 1, 2, \cdots, m.$$

The total number of subproblems is

$$\frac{4}{3}(1 - \frac{1}{4^{m+1}})N < \frac{4}{3}N,$$

i.e. the number of auxiliary subproblems is less than $N/3$.

Remark 1.1. We only require that the subregions of a fixed level can cover the whole region. This requirement is essentially different from that of [1] and [6].

Remark 1.2. There is no 'hierarchical' relation between subregions of neighbouring levels.

§2. Construction of the Preconditioner

2.1. Restriction and Prolongation.

2.1.1. Restriction and Prolongation between levels. For a fixed $l \in \{0, 1, 2, \cdots, m-1\}$ and $i \in \hat{\Omega}^l$, the nodes shown in Fig.2 are called one level neighbouring nodes of i; they compose the set $O_i^{l,1}$.

We define the restriction operator from a function f^{l+1} defined on $\hat{\Omega}^{l+1}$ to a function f^l on $\hat{\Omega}^l$ as

$$f^l(i) = f^{l+1}(i) + \frac{1}{2}\sum_{j \in O_i^{l,1}} f^{l+1}(j), \quad i \in \hat{\Omega}^l. \tag{2.1}$$

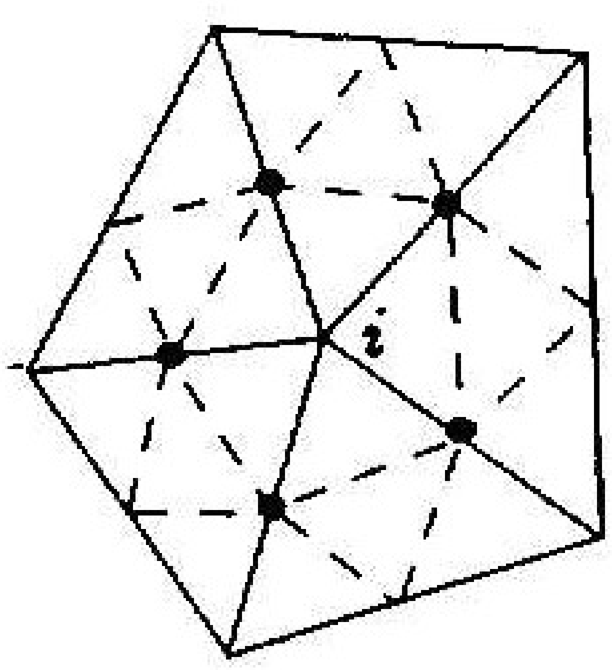

Fig. 2

This process is written as

$$f^l = r^l f^{l+1}.$$

The operator r^l can be represented with an $|\hat{\Omega}^l| \times |\hat{\Omega}^{l+1}|$ matrix, still denoted by r^l.

We define the restriction from $\hat{\Omega}$ to $\hat{\Omega}^l$ as

$$R^l = r^l \cdot r^{l+1} \cdots r^{m-1}, \quad l = 0, 1, \cdots, m-1.$$

R^l also represents an $|\hat{\Omega}^l| \times |\hat{\Omega}|$ matrix.

Let $i \in \hat{\Omega}^l$ be a node. O_i^l is its neighbouring set

$$O_i^l = (\text{Supp }\{\phi_i^l\})^0 \bigcap \hat{\Omega} - \{i\}.$$

It is easy to see that, for a function f defined on $\hat{\Omega}$, $f^l = R^l \cdot f$ is a function on $\hat{\Omega}^l$. For $i \in \hat{\Omega}^l$, $f^l(i)$ is only related with the values of f on nodes in O_i^l. From (2.1), we have the local expression of this operator

$$f^l(i) = (R^l f)(i) = f(i) + \sum_{j \in O_i^l} \phi_i^l(j) \cdot f(j). \tag{2.2}$$

For a discrete function f^l defined on $\hat{\Omega}^l$, we define its interpolation function $I^l f^l \in S^l$ as

$$I^l f^l = \sum_{i \in \hat{\Omega}^l} f(i) \cdot \phi_i^l.$$

I^m will be written briefly as I.

We define the prolongation operator using the above interpolation. From $\hat{\Omega}^l$ to $\hat{\Omega}$ we define $f = P^l f^l$,

$$f(i) = I^l f^l(i), i \in \hat{\Omega}, \quad l = 0, 1, 2, \cdots, m-1.$$

P^l represents the $|\hat{\Omega}| \times |\hat{\Omega}^l|$ matrix corresponding to the above operators as well.

Proposition 2.1. $P^l = (R^l)^T$.

This result can be easily deduced from a lemma of [3].

2.1.2. Restriction and Prolongation on Subdomains in a fixed level . For a fixed $l \in \{0, 1, 2, \cdots, m\}$ and a subdomain $\hat{\Omega}_k^l, k = 1, 2, \cdots, N_l$, let f_k^l be a function defined on $\hat{\Omega}_k^l$. We define its extention to the whole $\hat{\Omega}_k^l$ as

$$(E_k^l f_k^l)(i) = \begin{cases} f_k^l(i), & i \in \hat{\Omega}_k^l, \\ 0, & i \in \hat{\Omega}^l - \hat{\Omega}_k^l. \end{cases}$$

The restriction from $\hat{\Omega}^l$ to $\hat{\Omega}_k^l$ is defined as

$$(C_k^l f^l)(i) = f^l(i), i \in \hat{\Omega}_k^l.$$

C_k^l and E_k^l are $|\hat{\Omega}_k^l| \times |\hat{\Omega}^l|$ and $|\hat{\Omega}^l| \times |\hat{\Omega}_k^l|$ matrices respectively. It is easy to see that

$$C_k^l = (E_k^l)^T$$

with both C_1^0 are E_1^0 as identity matrices.

2.2. Preconditioner. With the above preparation, we can give the expression for the inverse of the preconditioner Q. In the PCG iteration, only Q^{-1} not Q will take part in the operation; the expression for Q is not necessary. As shown in (1.3), the l-level stiffness matrix is

$$A^l = (a(\phi_i^l, \phi_j^l))_{i,j \in \hat{\Omega}^l}.$$

We may define the stiffness matrix on a subdomain in a fixed level similarly,

$$A_k^l = (a(\phi_i^l, \phi_j^l))_{i,j \in \hat{\Omega}_k^l}, \quad l = 0, 1, 2, \cdots, m, k = 1, 2, \cdots, N_l.$$

The order of all these matrices is roughly $O(\frac{H^2}{h^2})$.

The expression for the inverse of the preconditioner is

$$Q^{-1} = \sum_{l=0}^{m} P^l \left(\sum_{k=1}^{N_l} E_k^l (A_k^l)^{-1} C_k^l\right) R^l, \tag{2.3}$$

where P^m, R^m, E_1^0, C_1^0 are identity matrices.

Remark 2.1. From the symmetry of A_k^l and Proposition 2.1 we know Q is symmetric. The first term in the outer sum symbol on the right-hand side of (2.3) is positive definite and other terms are nonnegative definite. Therefore, Q is positive definite.

Remark 2.2. For a given vector $f \in R^{|\hat{\Omega}|}$ (it is a function defined on $\hat{\Omega}$), $f^l = R^l f (l = 0, 1, 2, \cdots, m-1)$ can be computed synchronously. This process is fully parallel. The action of C_k^l does not take any time. All the subproblems (inverse action of $A_k^l, l = 0, 1, 2, \cdots, m, k = 1, 2, \cdots, N_l$) can be solved parallelly. $P^l (l = 0, 1, 2, \cdots, m-1)$ may act on each level, and even on all subdomains at the same time. Every subproblem amounts to a homogeneous boundary value Dirichlet problem. What kind of parallel computer is most suitable to this algorithm, and how to arrange subproblems on processors of a given computer need further study.

§3. Estimation of the Condition Number

Since Q is a symmetric positive definite matrix, the condition number of $Q^{-1}A$ is determined by the ratio of the maximum and minimum value of the generalized Rayleigh quotient

$$\frac{(AQ^{-1}Af, f)}{(Af, f)}. \tag{3.1}$$

3.1. An Equivalent Expression of the Rayleigh Quotient. $S^m \subset H_0^1(\Omega)$ is a Hilbert space with inner product $a(\cdot, \cdot)$, and $S_k^l = S^l \cap H_0^1(\Omega_k^l) \subset S^m (l = 0, 1, 2, \cdots, m, \quad k = 1, 2, \cdots, N_l)$ is its subspace. We use P_k^l to represent the orthogonal projection operator from S^m to S_k^l. P^l is the orthogonal projection operator from S^m to S^l.

$f \in R^{|\hat{\Omega}|}$ is an arbitrary discrete function on $\hat{\Omega}$, and $u = If = \sum_{i \in \hat{\Omega}} f(i)\phi_i^m$ is the finite element function in S^m corresponding to it. We have the following lemma:

Lemma 3.1.

$$(AQ^{-1}Af, f) = a(\sum_{l=0}^{m}\sum_{k=1}^{N_l} P_k^l u, u). \tag{3.2}$$

Proof. From the expression (2.3) of Q^{-1} we only need to prove

$$(AP^l E_k^l (A_k^l)^{-1} C_k^l R^l Af, f) = a(P_k^l u, u) \tag{3.3}$$

for all l, k, $l = 0, 1, 2, \cdots, m; \quad k = 1, 2, \cdots, N_l$. For any $i \in \hat{\Omega}_k^l$,

$$\begin{aligned} a(u - I(P^l E_k^l (A_k^l)^{-1} C_k^l R^l Af), \phi_i^l) &= a(u, \phi_i^l) - a(I^l (A_k^l)^{-1} C_k^l R^l Af, \phi_i^l) \\ &= a(u, \phi_i^l) - (R^l Af)(i) = (Af, I^{-1}\phi_i^l) - (R^l Af)(i) \\ &= 0 \ \text{(by (2.2))}. \end{aligned}$$

Thus

$$I(P^l E_k^l (A_k^l)^{-1} C_k^l R^l Af) = P_k^l u,$$

i.e. (3.3) is proved. From (3.2) we immediately get

$$\frac{(AQ^{-1}Af, f)}{(Af, f)} = \frac{a\left(\sum_{l=0}^{m}\sum_{k=1}^{N_l} P_k^l u, u\right)}{a(u, u)} \tag{3.4}$$

3.2. The Condition Number. We will prove our main theorem through several lemmas. In the following, C always represents a constant, and may have different values in different places.

Lemma 3.2. *There is a constant C independent of H and h such that for arbitrary $u \in S^m$,*

$$a\Big(\sum_{l=0}^{m}\sum_{k=1}^{N_l} P_k^l u, u\Big) \le C(m+1)a(u,u).$$

Proof.

$$a\Big(\sum_{l=0}^{m}\sum_{k=1}^{N_l} P_k^l u, u\Big) = \sum_{l=0}^{m} a\Big(\sum_{k=1}^{N_l} P_k^l u, u\Big).$$

From hypothesis A.3 and Lemma 3.2 of [6], we know there is a constant C, and for any $l \in \{0,1,2,\cdots,m\}$,

$$a\Big(\sum_{k=1}^{N_l} P_k^l u, u\Big) = a\Big(\sum_{k=1}^{N_l} P_k^l \bar{P}^l u, \bar{P}^l u\Big) \le C \cdot a\Big(\bar{P}^l u, \bar{P}^l u\Big) \le Ca(u,u),$$

where C is independent of the diameter of subdomains and mesh size. Sum it up with respect to l, and the proof will be finished. Thus we have got the upper bound of (3.4).

Lemma 3.3. *If there is a constant C, and for any $u \in S^m$ there is a set of functions $\{u_k^l \in S_k^l, l = 0,1,2,\cdots,m, k = 1,2,\cdots,N_l\}$ such that*

$$u = \sum_{l=0}^{m}\sum_{k=1}^{N_l} u_k^l \tag{3.5}$$

and

$$\sum_{l=0}^{m}\sum_{k=1}^{N_l} ||u_k^l||^2 \le C \cdot ||u||^2,$$

then

$$||u||^2 = a(u,u) \le C \sum_{l=0}^{m}\sum_{k=1}^{N_l} a(P_k^l u, u).$$

The proof of this lemma is just the same as that of Lemma I.1 of [5].

We will finish the lower bound estimation of (3.4) by looking for the constant in Lemma 3.3 and the decomposition (3.5).

Lemma 3.4. *Let $T \subset R^2$ be a triangular region, with diameter H. After j times refinements, we get the triangulation of T. Then, there exists a constant C independent of H and j, such that for any finite element function u corresponding to this triangulation on T, the following inequality holds:*

$$|u|_{0,\infty,T}^2 \le C \cdot \Big(\frac{1}{H^2}||u||_{L^2(T)}^2 + j \cdot |u|_{1,T}^2\Big). \tag{3.6}$$

This is a direct deduction of Lemma 3.3 in [4].

Let $\hat{I}$ be the linear interpolation operator on vertices of the triangle T. Then[2], for any finite element function u in the above,

$$|\hat{I}\cdot u|_{1,T}^2 \le 2\cdot |u|_{0,\infty,T}^2.$$

From (3.6) we get

$$|\hat{I}\cdot u|_{1,T}^2 \le C(\frac{1}{H^2}||u||_{L^2(T)}^2 + j|u|_{1,T}^2).$$

By adding an arbitrary constant and Poincare's inequality we get

$$|\hat{I}\cdot u|_{1,T}^2 \le C(1+j)|u|_{1,T}^2. \tag{3.7}$$

For any $u\in S^m$, we use $I^l u$ to represent its interpolation function in S^l. Then (3.7) shows

$$|I^l u|_{1,\Omega}^2 \le C\cdot(1+m-l)|u|_{1,\Omega}^2,\quad l=0,1,2,\cdots,m. \tag{3.8}$$

Since both $I^l u$ and u are zero on the boundary of Ω, from Poincare's inequality, we have

$$||I^l u||_{1,\Omega}^2 \le C(1+m-l)||u||_{1,\Omega}^2. \tag{3.9}$$

u is decomposed in the way

$$u = I^0 u + \sum_{l=1}^{m}(I^l - I^{l-1})u. \tag{3.10}$$

Let $u^0 = I^0 u$, $u^l = (I^l - I^{l-1})u$, $l=1,2,\cdots,m$. v^l represents the discrete function got from the restriction of u^l on $\hat{\Omega}^l$. $||v^l||_2$ represents the Euclidean norm of v^l.

Note that v^l vanishes on $\hat{\Omega}^{l-1}$. From Poincare's inequality and Lemma 3.4, on any triangle $T^{l-1}\in\mathcal{T}^{l-1}$, we have

$$\sum_{i\in\hat{\Omega}^l\cap T^{l-1}} |v^l(i)|^2 \le C\cdot|u^l|_{1,T^{l-1}}^2. \tag{3.11}$$

To sum on all triangles in $\mathcal{T}^{l-1}$, we see

$$||v^l||_2^2 \le \frac{1}{2}C|u^l|_{1,\Omega}^2 \le C||u^l||_{1,\Omega}^2,\quad l=1,2,\cdots,m. \tag{3.12}$$

We will define the decomposition required by Lemma 3.3 by the above $\{v^l,\ l=0,\ 1,2,\ \cdots,\ m\}$ for any function $u\in S^m$,

$$\begin{aligned} u_1^0 &= I^0 u,\\ u_k^l &= \sum_{i\in\hat{\Omega}_k^{l,1}} v^l(i)\phi_i^l,\quad l=1,2,\cdots,m;\quad k=1,2,\cdots,N_l \end{aligned}$$

where $\hat{\Omega}_k^{l,1}\subset\hat{\Omega}_k^l$. For fixed $l\in\{1,2,\cdots m\}$, $\{\hat{\Omega}_k^{l,1}, k=1,2,\cdots N_l\}$ does not intersect each other and

$$\bigcup_{k=1}^{N_l}\hat{\Omega}_k^{l,1} = \hat{\Omega}^l.$$

It is obvious that the functions constructed above satisfy $u_k^l \in S_k^l$ and

$$u = \sum_{l=0}^{m} \sum_{k=1}^{N_l} u_k^l.$$

Lemma 3.5. *There exists a constant C independent of H, h and m, such that for any $u \in S^m$, the above decomposition satisfies*

$$\sum_{l=0}^{m} \sum_{k=1}^{N_l} \|u_k^l\|_{1,\Omega_k^l}^2 \le C \cdot (m+1)^2 \|u\|_{1,\Omega}^2. \tag{3.13}$$

proof. For a fixed term $\|u_k^l\|_{1,\Omega_k^l}^2, l \ge 1$,

$$\|u_k^l\|_{1,\Omega_k^l}^2 = \|u_k^l\|_{1,\Omega}^2.$$

By the inverse inequality,

$$\|u_k^l\|_{1,\Omega}^2 \le C \cdot (2^{m-l}h)^{-2} \|u_k^l\|_{0,\Omega}^2 \le C \sum_{i \in \hat{\Omega}_k^{l,1}} |v^l(i)|^2,$$

hence

$$\sum_{k=1}^{N_l} \|u_k^l\|_{1,\Omega_k^l}^2 \le C \|v^l\|_2^2 \le C \|u^l\|_{1,\Omega}^2 \qquad \text{(by (3.12))}.$$

Therefore, the left-hand side of (3.13)

$$\begin{aligned} C \cdot (\|u_1^0\|_{1,\Omega}^2 + \sum_{l=1}^{m} \|u^l\|_{1,\Omega}^2) &= C \cdot (\|I^0 u\|_{1,\Omega}^2 + \sum_{l=1}^{m} \|(I^l - I^{l-1})u\|_{1,\Omega}^2) \\ &\le C \cdot (\|I^0 u\|_{1,\Omega}^2 + \sum_{l=1}^{m} \|I^l u\|_{1,\Omega}^2 \le C \cdot \sum_{l=0}^{m} (1+m-l) \|u^2\| 1,\Omega \quad \text{(by (3.9))} \\ &\le C \cdot (m+1)^2 \|u\|_{1,\Omega}^2. \end{aligned}$$

The lemma has been proved. The idea of this proof is from [2].

From Lemmas 3.1, 3.2, 3.3 and 3.5, we obtain

Theorem 3.1. *There exist constants C_1 and C_2 independent of H and h such that for any $u \in S^m = S_0^h$, the following estimate holds*

$$C_1 \frac{1}{(m+1)^2} \le \frac{a\Big(\sum_{l=0}^{m} \sum_{k=1}^{N_l} P_k^l u, u\Big)}{a(u,u)} \le C_2 (m+1),$$

that is, there is a constant C independent of H and h such that

$$\mathrm{Cond}(Q^{-1}A) \le C(1+m)^3 = C(1+\log_4 N)^3.$$

Remark 3.1. The hypothesis A.4 was based on the idea that capacities of processors of the imagined parallel computer are roughly the same. If one of the processors is stronger than the rest, the scale of the 0-level subproblem can be increased properly. In this way, the number of levels m will be decreased, and the condition number will be improved.

Remark 3.2. The domain of each level, except the 0-level, is decomposed into a number of subdomains, which overlap each other slightly. From the proof of the theorem we see the scales of subproblems do not affect the condition number; therefore they may be arbitrarily small or large, and should be determined by the concrete computer.

Remark 3.3. If subdomains of 1-level to m-level are selected to contain only one node in its interior. (As shown in Fig.3, the selection of Ω'^{l}_{k} will assure the hypothesis A.3. At a fixed point, the number of overlapping subdomains will be limited by a constant, which only depends on the smallest interior angle of the initial triangulation.) Then the expression for Q^{-1} will be

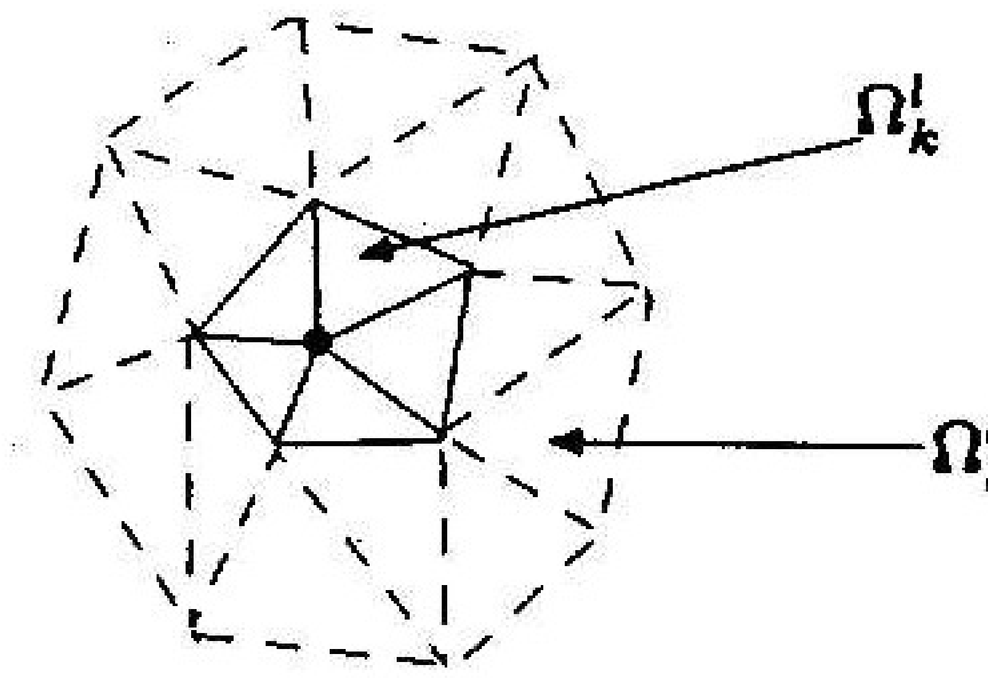

Fig. 3

$$Q^{-1} = P^0(A_1^0)^{-1}R^0 + \sum_{l=1}^{m} P^l\Big(\sum_{i\in\hat{\Omega}^l} \frac{(\vec{e}^{\,l}_i)(\vec{e}^{\,l}_i)^T}{a(\phi^l_i,\phi^l_i)}\Big)R^l$$

where $\vec{e}^{\,l}_i$ is the unit vector corresponding to node i in $R^{|\hat{\Omega}^l|}$. In a PCG step we need only to solve the subproblem on the 0-level domain. The estimate for the condition number is still true

$$\mathrm{Cond}(Q^{-1}A) \le C(1+m)^3,$$

where C is independent of H and h.

§4. Numerical Experiment

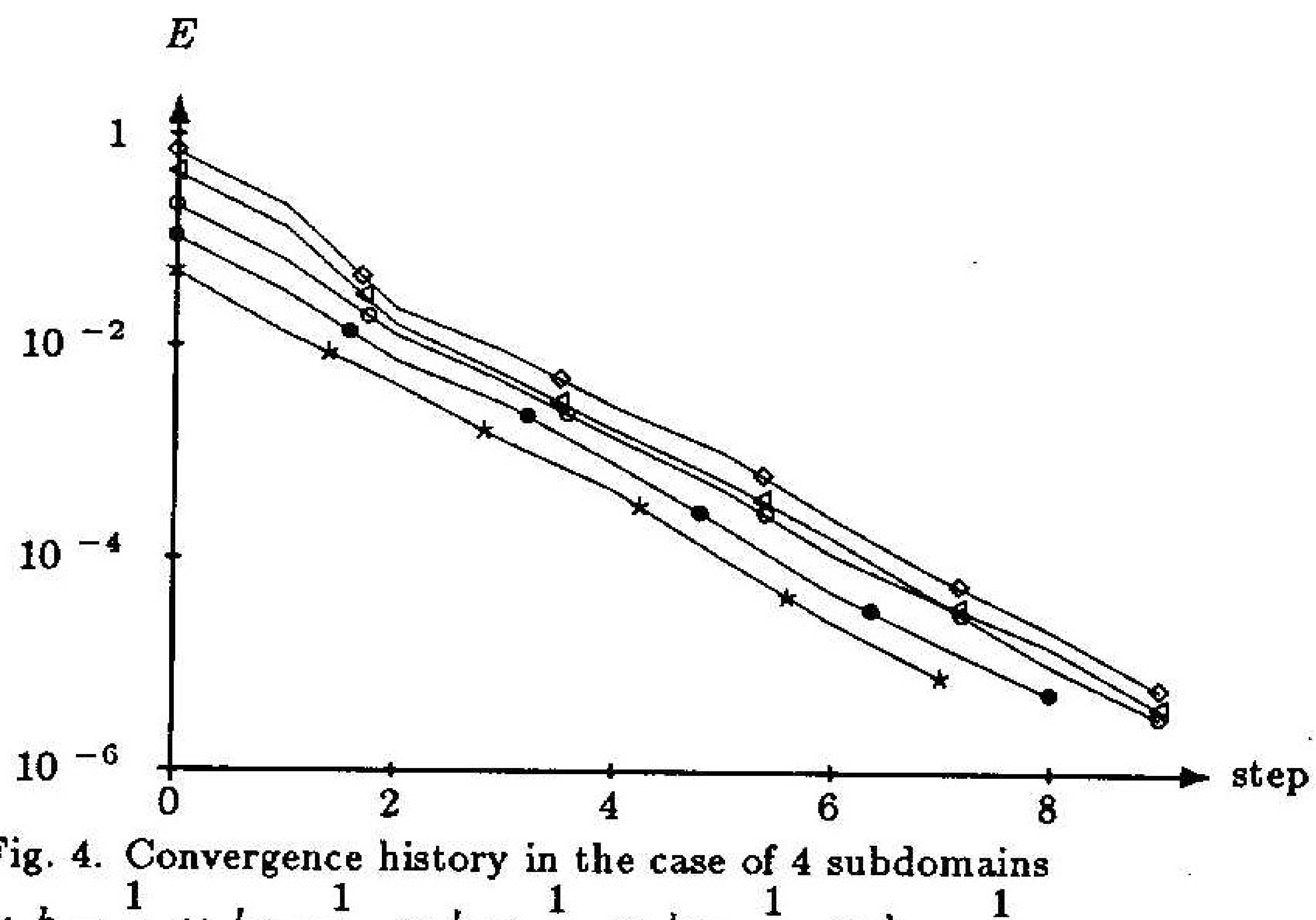

Fig. 4. Convergence history in the case of 4 subdomains
$\star$: $h=\frac{1}{8}$, $\bullet$: $h=\frac{1}{16}$, $\circ$: $h=\frac{1}{32}$, $\triangleleft$: $h=\frac{1}{64}$, $\diamond$: $h=\frac{1}{100}$

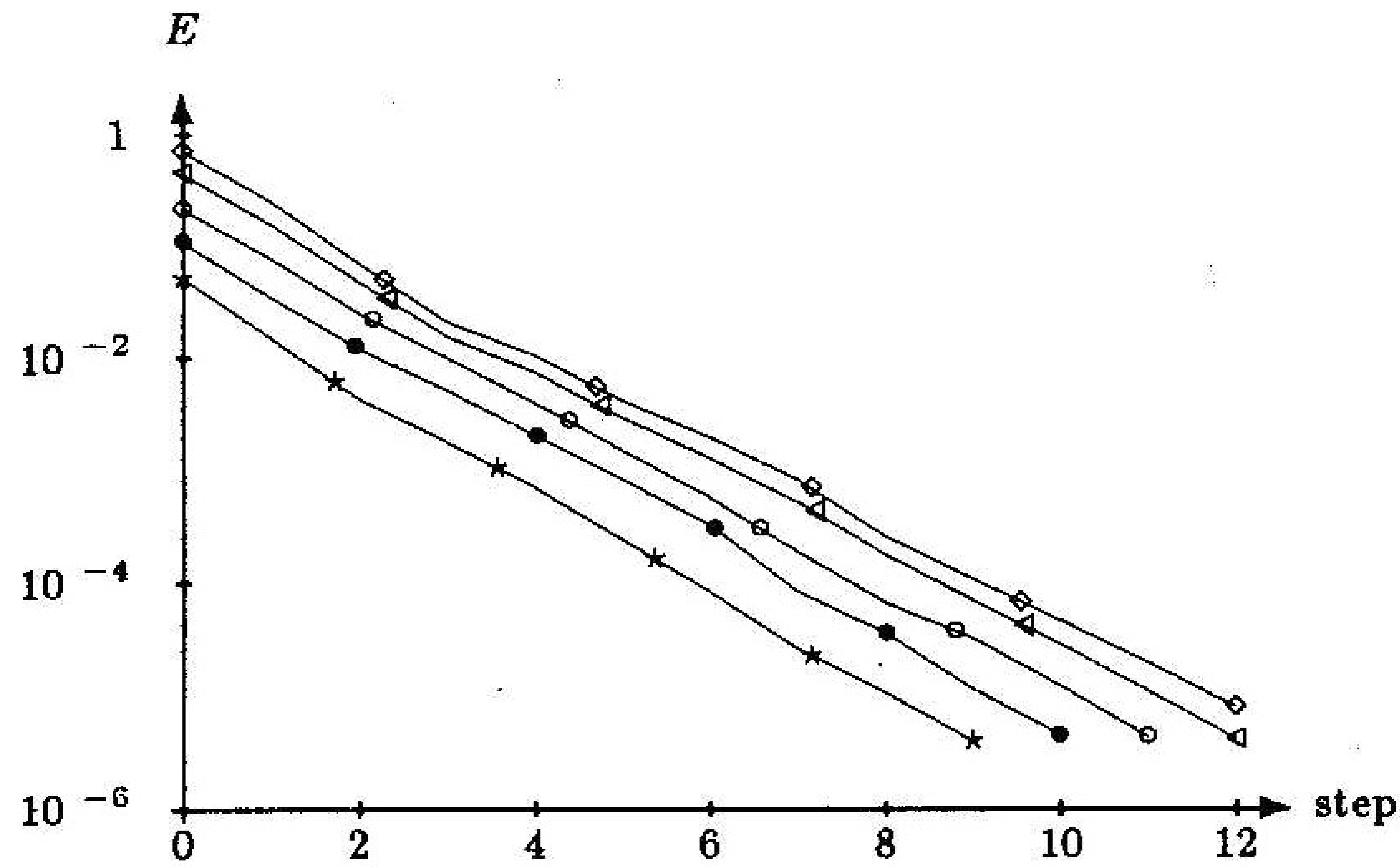

Fig. 5. Convergence history in the case of 16 subdomains
$\star$: $h = \frac{1}{8}$, $\bullet$: $h = \frac{1}{16}$, $\circ$: $h = \frac{1}{32}$, $\triangleleft$: $h = \frac{1}{64}$, $\diamond$: $h = \frac{1}{100}$

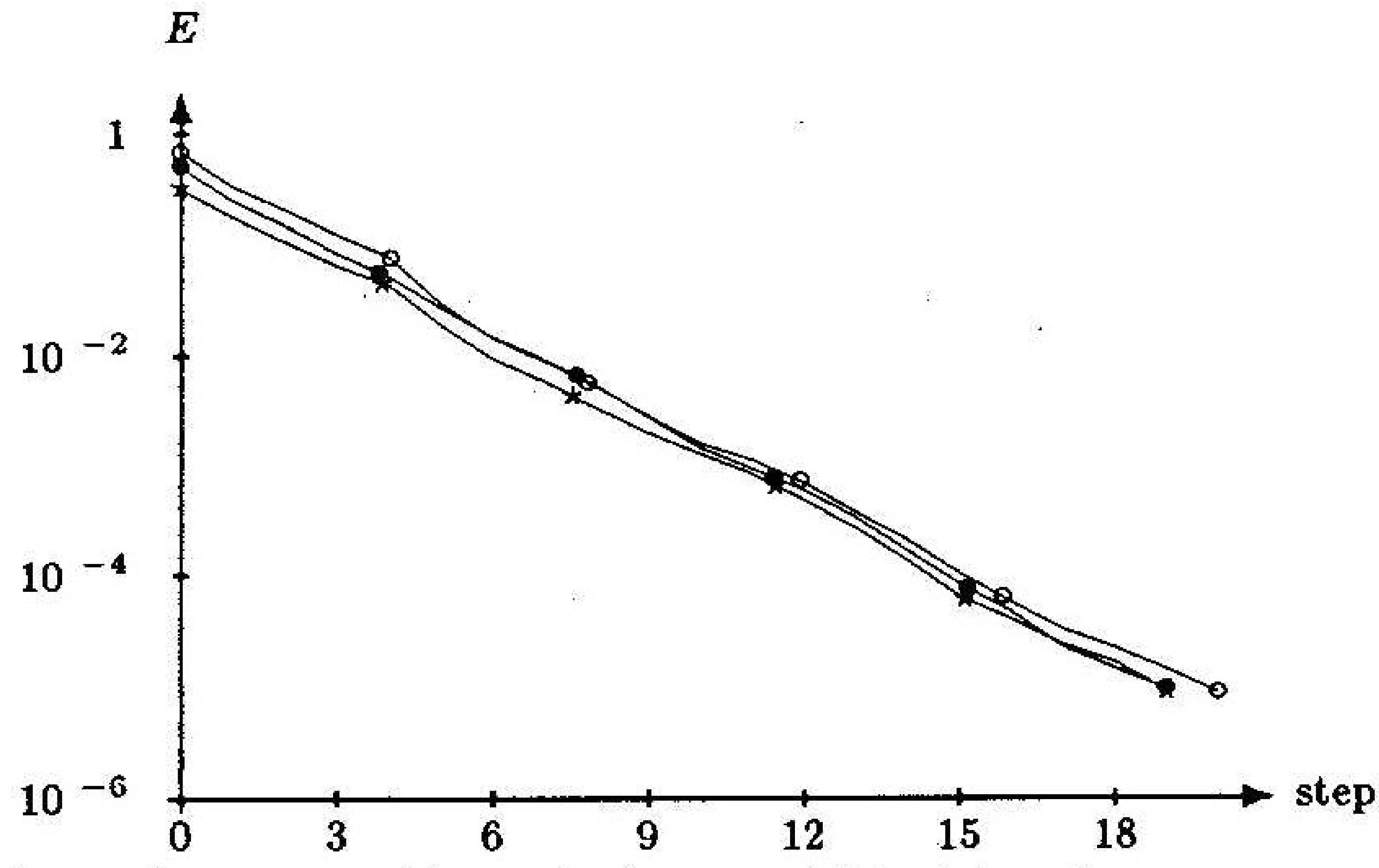

Fig. 6. Convergence history in the case of 64 subdomains.
$\star$: $h = \frac{1}{48}$, $\bullet$: $h = \frac{1}{72}$, $\circ$: $h = \frac{1}{96}$

We select the Poisson equation on the unit square region as the model problem. Uniform triangulation and linear continuous elements are used to discretize it. PCG method is used to solve the finite element equation.

We tested the preconditioner when the number of subdomains are four, sixteen and sixty-four, respectively; the number of levels is 2, 3 and 4, and the number of subproblems solved parallelly in a PCG step is 5, 21 and 85. All subdomains are rectangular regions, and are selected to ensure that the scales of subproblems are roughly the same.

The computation results are shown in Figures 4–6. In the figures h is the mesh size, E is the A–norm of error. From these figures we see that the PCG convergence rate is independent of the mesh size, and the number of subdomains is reversely proportional to the rate convergence. All these facts are in keeping with our theoretical analysis.

References

[1] M. Dryja and O. Widlund, An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions, Technical Report, No. 339, C. S. Dept. Courant, December, 1987.

[2] H. Yserentant, On the multi-level splitting of finite element spaces, *Numer. Math.*, **49**, 1986.

[3] W. Hackbush, Multigrid Methods and Applications, Springer, Berlin, 1985.

[4] J. H. Bramble, J. E. Pasciak and A. H. Schaz, The construction of preconditioners for elliptic problems by substructuring I, *Math. Comp.*, **47**, 1986.

[5] P. L. Lions, On the Schwarz Alternating Method I, in R. Glowinski, G. H. Golub, G.. Meurant, eds., Domain Decomposition Methods for Partial Differential Equations, *SIAM*, Philadelphia, 1988.

[6] Zhang Sheng, Domain Decomposition with Overlapping and PCG Method.