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A SPECTRAL–DIFFERENCE SCHEME FOR THREE–DIMENSIONAL VORTICITY EQUATIONS WITH SINGLE PERIODICAL BOUNDARY CONDITION*1)

Guo Ben-yu
(Shanghai University of Science and Technology, Shanghai, China)

Abstract

We develop a spectral-difference scheme to solve three-dimensional vorticity equation with single periodical boundary condition. We prove the conservation, generalized stability and convergence. The numerical experiments show that this scheme gives much better results than usual difference schemes.

§1. Introduction

Let $x = (x_1, x_2, x_3)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ and $\Omega = Q \times I$, where $Q = \{(x_1, x_2), 0 < x_1, x_2 < 1\}$ and $I = \{x_3, 0 < x_3 < 2\pi\}$. Let $\Gamma_p^- = \{x \in \bar{\Omega}, x_p = 0\}$, $\Gamma_p^+ = \{x \in \bar{\Omega},\ x_p = 1\}$ $(p = 1,2)$, $\Gamma_p = \Gamma_p^- \cup P_p^+$ and $\Gamma = \Gamma_1 \cup \Gamma_2$.

Let $\xi(x,t)$ and $\psi(x,t)$ denote the vorticity vector and the stream vector respectively. $\xi_0(x)$ and $f_l(x,t)(l = 1,2)$ are given functions. Their components are denoted by $\xi^{(p)}(x,t)$, $\psi^{(p)}(x,t)$, $\xi_0^{(p)}(x,t)$ and $f_l^{(p)}(x,t)$, $p = 1,2,3$. We consider the three-dimensional vorticity equation as follows:

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + [(\nabla \times \psi)\cdot]\xi - (\xi \cdot \nabla)(\nabla \times \psi) - \nu \nabla^2 \xi = f_1, & (x,t) \in \Omega \times (0,T], \\ -\nabla^2 \psi = \xi + f_2, & (x,t) \in \Omega \times [0,T], \\ \xi(x,0) = \xi_0(x), & x \in \bar{\Omega} \end{cases} \tag{1.1}$$

where ν is a positive constant and $f_l(x,t)$ and $\xi_0(x)$ have the period 2π for the variable x_3.

There are many papers concerning the finite difference methods for solving (1.1) [1,2]. The author and others[3,4] proposed spectral and pseudospectral methods to solve the periodical problem. Because problem (1.1) is only periodical for the variable x_3, we cannot use the method in [3, 4]. Recently, the author[5,6] developed a spectral-difference method to solve such partially periodical problems. In this paper, we propose a spectral-difference method for (1.1).

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1)The computation was done by Mr. Xiong Yue-shan.

§2. Notations and the Scheme

Let h and τ be the mesh sizes of the variables $x_p (p = 1, 2)$ and $t, Mh = 1$. Define

$$
\begin{aligned}
&Q_h = \{(x_1, x_2) = (j_1 h, j_2 h)/1 \le j_1, j_2 \le M-1\}, && \bar{Q}_h = \{(j_1 h, j_2 h)/0 \le j_1, j_2 \le M\},\\
&\Omega_h = Q_h \times I, && \bar{\Omega}_h = \bar{Q}_h \times I,\\
&\Gamma_{h,p}^- = \{x \in \bar{\Omega}_h / x_p = 0\}, && \Gamma_{h,p}^+ = \{x \in \bar{\Omega}_h / x_p = 1\}, \quad p = 1, 2,\\
&\Gamma_{h,p} = \Gamma_{h,p}^- \cup \Gamma_{h,p}^+, && \Gamma_h = \Gamma_{h,1} \cup \Gamma_{h,2},\\
&\Omega_{h,p}^{*,-} = \{x \in \Omega_h / x_p = h\}, && \Omega_{h,p}^{*,+} = \{x \in \Omega_h / x_p = 1 - h\}, \quad p = 1, 2,\\
&\Omega_{h,p}^* = \Omega_{h,p}^{*,-} \cup \Omega_{h,P}^{*,+}, && \Omega_h^* = \Omega_{h,1}^* \cup \Omega_{h,2}^*,\\
&S_\tau = \{t = k\tau / k = 0, 1, 2, \cdots, \}.
\end{aligned}
$$

Let $u(x, t)$ and $v(x, t)$ be the three-dimensional vectors. We define

$$
u_{x_p}(x,t) = \frac{1}{h}(u(x + he_p, t) - u(x,t)), \quad u_{\bar{x}_p}(x,t) = u_{x_p}(x - he_p, t), \quad p = 1, 2,
$$

$$
u_{\hat{x}_p}(x,t) = \frac{1}{2}(u_{x_p}(x,t) + u_{\bar{x}_p}(x,t)), \quad u_n(x,t) = \begin{cases} -u_{x_p}(x,t), & \text{for } x \in \Gamma_{h,p}^-,\\ u_{\bar{x}_p}(x,t), & \text{for } x \in \Gamma_{h,p}^+, \end{cases}
$$

$$
\Delta u(x,t) = u_{x_1 \bar{x}_1}(x,t) + u_{x_2 \bar{x}_2}(x,t) + \frac{\partial^2 u}{\partial x_3^2}(x,t), \quad u_t(x,t) = \frac{1}{\tau}(u(x, t+\tau) - u(x,t)).
$$

We define the following scalar products and norms:

$$
\begin{aligned}
&(u(x_1, x_2, t), v(x_1, x_2, t))_I = \frac{1}{2\pi}\int_0^{2\pi} u(x,t)\bar{v}(x,t)dx_3,\\
&\|u(x_1, x_2, t)\|_I^2 = (u(x_1, x_2, t), u(x_1, x_2, t))_I,\\
&(u(x_3, t), v(x_3, t))_{Q_h} = h^2 \sum_{(x_1, x_2) \in Q_h} u(x,t)\bar{v}(x,t),\\
&\|u(x_3, t)\|_{Q_h}^2 = (u(x_3, t), u(x_3, t))_{Q_h},\\
&(u(t), v(t)) = h^2 \sum_{(x_1, x_2) \in Q_h} (u(x_1, x_2, t), v(x_1, x_2, t))_I,\\
&\|u(t)\|^2 = (u(t), u(t)), \quad \|u(t)\|_{l^4}^4 = \||u(t)|^2\|^2,\\
&|u(t)|_1^2 = \frac{1}{2}\sum_{p=1,2}(\|u_{x_p}(t)\|^2 + \|u_{\bar{x}_p}(t)\|^2) + \left\|\frac{\partial u}{\partial x_3}(t)\right\|^2,\\
&\|u(t)\|_1^2 = \|u(t)\|^2 + |u(t)|_1^2,\\
&|u(t)|_2^2 = \frac{1}{2}\sum_{p=1,2}(|u_{x_p}(t)|_{1,\Omega_h/\Omega_h^*} + |u_{\bar{x}_p}(t)|_{1,\Omega_h/\Omega_h^*}) + \left|\frac{\partial u}{\partial x_3}(t)\right|_1^2,\\
&\|u(t)\|_2^2 = \|u(t)\|_1^2 + |u(t)|_2^2,
\end{aligned}
$$

where the definition of $|u(t)|_{1,\Omega_h/\Omega_h^*}$ is similar to $|u(t)|_1^2$, but the summation is only for all $x \in \Omega_h/\Omega_h^*$. We shall use also the following norms:

$$\|u(t)\|_{\Gamma_h}^2 = h \sum_{j=1}^{M-1} (\|u(o, jh, t)\|_I^2 + \|u(1, jh, t)\|_I^2 + \|u(jh, 0, t)\|_I^2 + \|u(jh, 1, t)\|_I^2),$$

$$\|u(t)\|_{\Omega_h^*}^2 = h \sum_{j=1}^{M-1} (\|u(h, jh, t)\|_I^2 + \|u(1-h, jh, t)\|_I^2 + \|u(jh, h, t)\|_I^2 + \|u(jh, 1-h, t)\|_I^2).$$

Let N be any positive integer and

$$V_N = \text{ span } \{e^{ilx_3}/|l| \le N\}.$$

P_N is the orthogonal projection, i.e.,

$$\int_0^{2\pi} (P_N u - u)\bar{v} dx_3 = 0, \quad \forall \quad v \in V_N.$$

One of the key problems is to construct a scheme which has the property similar to that of (1.1). Indeed, the solution of (1.1) satisfies

$$\|\xi(t)\|_{L^2(\Omega)}^2 + \int_0^t L(y)dy = \|\xi_0\|_{L^2(\Omega)}^2 + 2\int_0^t (f_1(y), \xi(y))_{L^2(\Omega)} dy \tag{2.1}$$

where

$$\begin{aligned} L(y) = {} & 2\nu |\xi(y)|_{H^1(\Omega)}^2 - 2\nu \int_\Gamma \xi(x, y) \frac{\partial \xi}{\partial n}(x, y) ds \\ & - \int_{\Gamma_1^-} \xi^2(x, y) \Big(\frac{\partial}{\partial x_2} \psi^{(3)}(x, y) - \frac{\partial}{\partial x_3} \psi^{(2)}(x, y) \Big) ds \\ & + \int_{\Gamma_1^+} \xi^2(x, y) \Big(\frac{\partial}{\partial x_2} \psi^{(3)}(x, y) - \frac{\partial}{\partial x_3} \psi^{(2)}(x, y) \Big) ds \\ & + \int_{\Gamma_2^-} \xi^2(x, y) \Big(\frac{\partial}{\partial x_1} \psi^{(3)}(x, y) - \frac{\partial}{\partial x_3} \psi^{(1)}(x, y) \Big) ds \\ & + \int_{\Gamma_2^+} \xi^2(x, y) \Big(\frac{\partial}{\partial x_1} \psi^{(3)}(x, y) - \frac{\partial}{\partial x_3} \psi^{(1)}(x, y) \Big) ds \\ & -2((\xi(y) \cdot \nabla)(\nabla \times \psi(y)), \xi(y))_{L^2(\Omega)}. \end{aligned}$$

In order to simulate (2.1), we define the following operators:

$$R_{(w)}^{(1)} = w_{\hat{x}_2}^{(3)} - \frac{\partial w^{(2)}}{\partial x_3}, \quad R_{(w)}^{(2)} = \frac{\partial w^{(1)}}{\partial x_3} - W_{\hat{x}_1}^{(3)}, \quad R_{(w)}^{(3)} = w_{\hat{x}_1}^{(2)} - w_{\hat{x}_2}^{(1)},$$

$$R(w) = (R_{(w)}^{(1)}, R_{(w)}^{(2)}, R_{(w)}^{(3)})^*, \quad J_1(u, w) = \sum_{p=1}^{2} R^{(p)}(w) u_{\hat{x}_p} + R^3(w) \frac{\partial u}{\partial x_3},$$

$$J_2(u,w)=\sum_{p=1}^{2}(R^{(p)}(w)u)_{\hat{x}_p}+\frac{\partial}{\partial x_3}(R^{(3)}_{(w,u)}),\quad J(u,w)=\frac{1}{2}J_1(u,w)+\frac{1}{2}J_2(u,w),$$

$$H(u,w)=\sum_{p=1}^{2}u^{(p)}(R^{(p)}(w))_{\hat{x}_p}+u^{(3)}\frac{\partial}{\partial x_3}R^{(3)}(w).$$

It can be shown that

$$(u(t),J(v(t),w(t)))+(v(t),J(u(t),w(t)))=\frac{1}{2}A(u(t),v(t),w(t))+\frac{1}{2}A(v(t),u(t),w(t)) \tag{2.2}$$

where

$$\begin{aligned}A(u\ (t),v(t),w(t)),=\frac{h}{2}\sum_{j=1}^{M-1}&\Big[(u(1,jh,t),R^{(1)}(w(1-h,jh,t))v(1-h,jh,t))_I\\&+(u(1-h,jh,t),R^{(1)}(w(1,jh,t))v(1,jh,t))_I+(u(jh,1,t),\\&R^{(2)}(w(jh,1-h,t))v(jh,1-h,t))_I+(u(jh,1-h,t),R^{(2)}(w(jh,1,t))v(jh,1,t))_I\\&-(u(h,jh,t),R^{(1)}(w(0,jh,t))v(0,jh,t))_I-(u(0,jh,t),R^{(1)}(w(h,jh,t))v(h,jh,t))_I\\&-(u(jh,h,t),R^{(2)}(w(jh,0,t))v(jh,0,t))_I-(u(jh,0,t),R^{(2)}(w(jh,h,t))v(jh,h,t))_I\Big].\end{aligned}$$

In particular,

$$(u(t),J(u(t),w(t)))=\frac{1}{2}A(u(t),u(t),w(t)). \tag{2.3}$$

We can also prove that

$$(u(t),\Delta v(t))+\frac{1}{2}\sum_{p=1}^{2}[(u_{x_p}(t),v_{\bar{x}_p}(t))+(u_{\bar{x}_p}(t),v_{x_p}(t))]+\Big(\frac{\partial u}{\partial x_3}(t),\frac{\partial v}{\partial x_3}(t)\Big)=B(u(t),v(t)) \tag{2.4}$$

where

$$\begin{aligned}B(u(t),\ v(t))=\frac{h}{2}\sum_{j=1}^{M-1}&\Big[(u(1,jh,t)+u(1-h,jh,t),u_n(1,jh,t))_I+(u(jh,1,t)\\&+u(jh,1-h,t),u_n(jh,1,t))_I+(u(0,jh,t)+u(h,jh,t),\\&u_n(0,jh,t))_I+(u(jh,0,t)+u(jh,h,t),u_n(jh,1,t))_I\Big].\end{aligned}$$

In particular,

$$(u(t),\Delta u(t))+|u(t)|_1^2=B(u(t),u(t)). \tag{2.5}$$

If in addition $u(x,t)=0$ on Γ_h, then

$$(u(t),\Delta u(t))+|u(t)|_1^2+S(u(t))=0 \tag{2.6}$$

where

$$S(u(t)) = \frac{1}{2}\sum_{j=1}^{M-1}[\|u(h,jh,t)\|_I^2 + \|u(1-h,jh,t)\|_I^2 + \|u(jh,h,t)\|_I^2 + \|u(jh,1-h,t)\|_I^2].$$

Now let $\eta^{(N)}(x,t)$ and $\psi^{(N)}(x,t)$ be the approximations to $\xi(x,t)$ and $\psi(x,t)$ where $\eta^{(N)}(x,t), \psi^{(N)}(x,t) \in (V_N)^3$ for all $(x_1,x_2) \in Q_h$ and $t \in S_\tau$. The spectral-difference scheme for (1.1) is the following

$$\begin{cases} \eta_t^{(N)}(x,t) + P_N J(\eta^{(N)}(x,t) + \delta\tau\eta_t^{(N)}(x,t), \varphi^{(N)}(x,t)) - P_N H(\eta^{(N)}(x,t), \varphi^{(N)}(x,t)) \\ \qquad -\nu\Delta(\eta^{(N)}(x,t) + \sigma\tau\eta_t^{(N)}(x,t)) = P_N f_1(x_1,t), \qquad (x,t) \in \Omega_h \times S_\tau, \\ -\Delta\varphi^{(N)}(x,t) = \eta^{(N)}(x,t) + P_N f_2(x,t), \qquad (x,t) \in \Omega_h \times S_\tau, \\ \eta^{(N)}(x,0) = \eta_0^{(N)}(x) = P_N\xi_0(x), \qquad x \in \bar{\Omega}_h \end{cases} \tag{2.7}$$

where δ and σ are parameters, $0 \le \delta, \sigma \le 1$.

We next check the conservation. Let $\delta = \sigma = \dfrac{1}{2}$. We obtain from (2.3) and (2.5)

$$\begin{aligned} &\|\eta^{(N)}(t)\|^2 + \tau \sum_{\substack{y \in S_\tau \\ y \le t-\tau}} \Big[\frac{\nu}{2}\Big|\eta^{(N)}(y) + \eta^{(N)}(y+\tau)\Big|_1^2 - \frac{\nu}{2}B(\eta^{(N)}(y) + \eta^{(N)}(y+\tau), \\ &\quad \eta^{(N)}(y) + \eta^{(N)}(y+\tau)) + \frac{1}{4}A(\eta^{(N)}(y) + \eta^{(N)}(y+\tau), \eta^{(N)}(y) \\ &\quad + \eta^{(N)}(y+\tau), \varphi^{(N)}(y)) - (H(\eta^{(N)}(y), \varphi^{(N)}(y)), \eta^{(N)}(y) + \eta^{(N)}(y+\tau))\Big] \\ &= \|\eta_0^{(N)}\|^2 + \tau \sum_{\substack{y \in S_\tau \\ y \le t-\tau}} (f_1(y), \eta^{(N)}(y) + \eta^{(N)}(y+\tau)), \end{aligned}$$

which is a reasonable analogy of (2.1).

§3. Theoritical Results

We consider the following boundary condition:

$$\begin{cases} \psi(x,t) = \psi_0(x,t), \qquad (x,t) \in \Gamma \times [0,T], \\ a\dfrac{\partial\xi}{\partial n}(x,t) + b\xi(x,t) = g(x,t), \qquad (x,t) \in \Gamma \times [0,T], \end{cases} \tag{3.1}$$

where a and b are nonnegative constants, and ψ and g are given functions. The corresponding approximation is

$$\begin{cases} \varphi^{(N)}(x,t) = P_N\psi_0(x,t), \qquad (x,t) \in \Gamma_H \times S_\tau, \\ a\eta_n^{(N)}(x,t) + b\bar{\eta}^{(N)}(x,t) = P_N g(x,t), \qquad (x,t) \in \Gamma_h \times S_\tau,\ a > 0, \\ \eta^{(N)}(x,t) = P_N g(x,t), \qquad (x,t) \in \Gamma_h \times S_\tau, a = 0,\ b = 1. \end{cases} \tag{3.2}$$

where

$$\bar{\eta}^{(N)}(x,t)=\begin{cases}\frac{1}{2}(\eta^{(N)}(x+he_p,t)+\eta^{(N)}(x,t)), & \text{for } x\in\Gamma_{h,p}^{-},\ p=1,2,\\ \frac{1}{2}(\eta^{(N)}(x-he_p,t)+\eta^{(N)}(x,t)), & \text{for } x\in\Gamma_{h,p}^{+},\ p=1,2.\end{cases}$$

Let $\tilde{\eta}_0^{(N)}(t)$, $\tilde{f}_l(x,t)$ and $\tilde{g}(x,t)$ be the errors of $\eta_0(x)$, $f_l(x,t)$ and $g(x,t)$, which induce the errors of $\eta^{(N)}(x,t)$ and $\varphi^{(N)}(x,t)$ denoted by $\tilde{\eta}^{(N)}(x,t)$ and $\tilde{\varphi}^{(N)}(x,t)$. For simplicity, assume $\tilde{\varphi}^{(N)}(x,t)=0$ on $\Gamma_h\times S_\tau$. The errors satisfy the following equation:

$$\begin{cases}\eta_t^{(N)}(x,t)+P_NJ(\tilde{\eta}^{(N)}(x,t)+\delta\tau\tilde{\eta}_t^{(N)}(x,t),\varphi^{(N)}(x,t)+\tilde{\varphi}^{(N)}(x,t))+P_NJ(\eta^{(N)}(x,t)\\ \quad+\delta\tau\eta^{(N)}(x,t),\tilde{\varphi}^{(N)}(x,t))-P_NH(\tilde{\eta}^{(N)}(x,t),\varphi^{(N)}(x,t)+\tilde{\varphi}^{(N)}(x,t))\\ \quad-P_NH(\eta^{(N)}(x,t),\tilde{\varphi}^{(N)}(x,t))-\nu\Delta(\tilde{\eta}^{(N)}(x,t)+\sigma\tau\hat{\eta}_t^{(N)}(x,t))\\ \quad=P_N\tilde{f}_1(x,t), \qquad (x,t)\in\Omega_h\times S_\tau,\\ -\Delta\tilde{\varphi}^{(N)}(x,t)=\tilde{\eta}^{(N)}(x,t)+P_N\tilde{f}_1(x,t), \qquad (x,t)\in\Omega_h\times S_\tau,\\ a\tilde{\eta}_n^{(N)}(x,t)+b\tilde{\bar{\eta}}^{(N)}(x,t)=P_N\tilde{g}(x,t), \qquad (x,t)\in\Gamma_h\times S_\tau,\ a>0,\\ \tilde{\eta}^{(N)}(x,t)=P_N\tilde{g}(x,t), \qquad (x,t)\in\Gamma_h\times S_\tau,\ a=0,\ b=1,\\ \tilde{\eta}^{(N)}(x,0)=\tilde{\eta}_0^{(N)}(x), \qquad x\in\bar{\Omega}_h.\end{cases}\tag{3.3}$$

Define

$$\|u(t)\|_{q,\infty}=\max_{r_0+r_1+r_2+r_3+r_4\le q}\max_{x\in\Omega_h}\left|\left(\frac{\partial^{r_0}u}{\partial x_3^{r_0}}\right)\underbrace{x_1\cdots x_1}_{r_1}\underbrace{\bar{x}_1\cdots\bar{x}_1}_{r_2}\underbrace{x_2\cdots x_2}_{r_3}\underbrace{\bar{x}_2\cdots\bar{x}_2}_{r_4}\right|,$$

$$|||u|||_{q,\infty}=\max_{\substack{t\in S_t\\ t\le T}}\|u(t)\|_{q,\infty}.$$

Let P_0 be a suitably small positive constant and

$$E_1(u;t)=\|u(t)\|^2+\nu\tau(|u(t)|_1^2+S(u(t)))+\tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}(P_0\tau\|u_t(y)\|^2+\nu|u(y)|_1^2+\nu S(u(y))),$$

$$E_2(u;t)=\|u(t)\|^2+\nu\tau|u(t)|_1^2+\tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}(P_0\tau\|u_t(y)\|^2+\nu|u(y)|_1^2),$$

$$E_3(u;t)=\|u(t)\|^2+\nu\tau(|u(t)|_1^2+S^*(u(t)))+\tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}(P_0\tau\|u_t(y)\|^2+\nu|u(y)|_1^2+\nu S^*(u(y))),$$

$$\rho(v_0,v_1,v_2,v_3;t)=\|v_0\|^2+\tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}(\|v_1(y)\|^2+\|v_2(y)\|^2+\|v_3(y)\|_{\Gamma_h}^2+\|v_3(y+\tau)\|_{\Gamma_h}^2)$$

where

$$S^*(x(t))=S(u(t)+u(t+\tau)).$$

Theorem 1. *If the following conditions are fulfilled:*

(i) $a=0, b=1, \tau=O(h^2), \tau=O(\frac{1}{N^2})$,

(ii) $\sigma \geq \frac{1}{2}$ *or* $\tau < \frac{4h^2}{\nu(1-2\sigma)(17+2h^2N^2)}$,

(iii) *for all* $t \leq T$, $\|\tilde{f}_2(t)\|^2 \leq b_1 h$, $\|\tilde{g}(t)\|^2_{\Gamma_h} \leq \frac{b_2 h}{N}$, $\rho(\tilde{\eta}_0, \tilde{f}_1, \tilde{f}_2, h^{-\frac{1}{2}}\tilde{g}; t) \leq b_3 h^2$,

where b_α *are suitably small positive constants depending only on* $\||\eta^{(N)}\||_{1,\infty}, \||\varphi^{(N)}\||_{2,\infty}$ *and* ν, *then for all* $t \leq T$, *we have*

$$E_1(\tilde{\eta}^{(N)}; t) \leq b_5 e^{b_6 t} \rho(\tilde{\eta}_0, \tilde{f}_1, \tilde{f}_2, h^{-\frac{1}{2}}\tilde{g}; t).$$

Theorem 2. *If the following conditions are fulfilled:*

(i) $a=1,\ b=0,\ \delta=0,\ \tau=O(h^2),\ \tau=O(\frac{1}{N^2})$,

(ii) $\sigma \geq \frac{1}{2}$ *or* $\tau < \frac{2h^2}{\nu(1-2\sigma)(8+h^2N^2)}$,

(iii) *for all* $t \leq T$, $\|\tilde{f}_2(t)\|^2 \leq b_7 h$, $\rho(\tilde{\eta}_0, \tilde{f}_1, \tilde{f}_2, \tilde{g}; t) \leq b_8 h^2$,

then for all $t \leq T$, *we have*

$$E_2(\tilde{\eta}^{(N)}; t) \leq b_9 e^{b_{10} t} \rho(\tilde{\eta}_0, \tilde{f}_1, \tilde{f}_2, \tilde{g}; t).$$

Theorem 3. *If the following conditions are fulfilled:*

(i) $a=1,\ b>0,\ \delta=0,\ \tau=O(h^2),\ \tau=O(\frac{1}{N^2})$,

(ii) $\sigma \geq \frac{1}{2}$ *or* $\tau < \frac{2h^2}{\nu(1-2\sigma)(8+b+h^2N^2)}$,

(iii) *for all* $t \leq T$, $\|\tilde{f}_2(t)\|^2 \leq b_{11} h$, $\rho(\tilde{\eta}_0, \tilde{f}_1, \tilde{f}_2, \tilde{g}; t) \leq b_{12} h^2$,

then for all $t \leq T$, *we have*

$$E_3(\tilde{\eta}^{(N)}; t) \leq b_{13} e^{b_{14} t} \rho(\tilde{\eta}_0, \tilde{f}_1, \tilde{f}_2, \tilde{g}; t).$$

We next deal with the convergence. By putting

$$\xi^{(N)}(x,t) = P_N \xi(x,t),\quad \tilde{\xi}^{(N)}(x,t) = \eta^{(N)}(x,t) - \xi^{(N)}(x,t),$$
$$\psi^{(N)}(x,t) = P_N \psi(x,t),\quad \tilde{\psi}^{(N)}(x,t) = \varphi^{(N)}(x,t) - \psi^{(N)}(x,t),$$

we have

$$\begin{aligned}
&\tilde{\xi}_t^{(N)}(x,t) + P_N J(\tilde{\xi}^{(N)}(x,t) + \delta\tau\tilde{\xi}_t^{(N)}(x,t), \psi^{(N)}(x,t) + \tilde{\psi}^{(N)}(x,t)) + P_N J(\xi^{(N)}(x,t) \\
&\quad + \delta\tau\xi_t^{(N)}(x,t), \tilde{\psi}^{(N)}(x,t)) - P_N H(\tilde{\xi}^{(N)}(x,t), \psi^{(N)}(x,t) + \tilde{\psi}^{(N)}(x,t)) \\
&\quad - P_N H(\xi^{(N)}(x,t), \tilde{\psi}^{(N)}(x,t)) - \nu\Delta(\tilde{\xi}^{(N)}(x,t) + \sigma\tau\tilde{\xi}_t^{(N)}(x,t)) \\
&\quad = -\sum_{q=1}^{6} M_q^{(N)}(x,t), \qquad (x,t) \in \Omega_h \times S_\tau, \\
&\quad -\tilde{\psi}^{(N)}(x,t) = \tilde{\xi}^{(N)}(x,t) - M_7^{(N)}(x,t), \qquad (x,t) \in \Omega_h \times S_\tau, \\
&\quad a\tilde{\xi}_n^{(N)}(x,t) + b\tilde{\xi}^{(N)}(x,t) = M_8^{(N)}(x,t), \qquad (x,t) \in \Gamma_h \times S_\tau,\ a>0,
\end{aligned}$$

$$\tilde{x}^{(N)}(x,t)=0, \qquad (x,t)\in\Gamma_h\times S_\tau,\ a=0, b=1,$$
$$\tilde{\varphi}^{(N)}(x,t)=0, \qquad (x,t)\in\Gamma_h\times S_\tau,$$
$$\tilde{\xi}^{(N)}(x,0)=0, \qquad (x,t)\in\bar{\Omega}_h, \tag{3.4}$$

where

$$M_1^{(N)}=\xi_t^{(N)}-\frac{\partial\xi^{(N)}}{\partial t}, \qquad M_2^{(N)}=P_N J(\xi^{(N)}(x,t),\psi^{(N)}(x,t))-P_N((\nabla\times\psi)\cdot\nabla)\xi,$$
$$M_3^{(N)}=\delta\tau P_N J(\xi_t^{(N)},\psi^{(N)}), \qquad M_4^{(N)}=P_N((\xi\cdot\nabla)(\nabla\times\psi))-P_N H(\xi^{(N)},\psi^{(N)}),$$
$$M_5^{(N)}=\nu\sum_{p=1}^{2}\Big(\frac{\partial^2\xi^{(N)}}{\partial x_p^2}-\xi_{x_p x_p}^{(N)}\Big), \qquad M_6^{(N)}=-\nu\sigma\tau\Delta\xi_t^{(N)},$$
$$M_7^{(N)}=\nu\sum_{p=1}^{2}\Big(\frac{\partial^2\psi^{(N)}}{\partial^2 x_p}-\psi_{x_p x_p}^{(N)}\Big), \qquad M_8^{(N)}=a\xi_n^{(N)}-a\frac{\partial\xi^{(N)}}{\partial n}.$$

Let $B_1(I), B_2(Q)$ be Banach spaces and $B_2(B_1)=B_2(Q,B_1(I))$.

Theorem 4. *If condition* (i), (ii) *of Theo.* 1 *hold and* $\beta>1$, $r>0$, $\xi\in C(0,T;H^{\frac{5}{2}+r}(\Omega)\cap C^4(L^2)\cap H^{1+r}(H^{1+\beta})\cap H^{2+r}(H^\beta)\cap H^{4+r}(L^2)$, $\frac{\partial\xi}{\partial t}=C(0,T;C(H^2)\cap C^2(L^2)\cap H^{1+r}(H^2)\cap H^{2+r}(L^2)$, $\frac{\partial^2\xi}{\partial t^2}\in C(0,T;C(L^2));\psi\in(0,T;H^{\frac{5}{2}+r}(\Omega)\cap C^2(L^2)\cap H^{1+r}(H^{2+\beta})\cap H^{3+r}(H^3)\cap H^{5+r}(L^2))$, *then for all* $t\le T$, *we have*

$$\|\xi(t)-\eta^{(N)}(t)\|^2\le b_1^*(\tau^2+h^4+N^{-2\beta}),$$

where b_a^* *is a positive constant depending only on the norms of* ξ *and* ψ *in the mentioned spaces.*

Theorem 5. *If condition* (i), (ii) *of Theorem* 2 *hold and* ξ,ψ *satisfy the conditions in Theorem* 4, *then for all* $t\le T$, *we have*

$$\|\xi(t)-\eta^{(N)}(t)\|^2\le b_2^*(\tau^2+h^2+N^{-2\beta}).$$

Theorem 6. *If condition* (i), (ii) *of Theorem* 3 *hold and* ξ,ψ *satisfy the conditions in Theorem* 4, *then for all* $t\le T$, *we have*

$$\|\xi(t)-\eta^{(N)}(t)\|^2\le b_3^*(\tau^2+h^2+N^{-2\beta}).$$

§4. Numerical Results

We take the test functions

$$\xi^{(p)}=A_p\exp(B_p\sin(C_p x_1+D_p x_2+2\pi x_3)+\omega_p t),$$
$$\psi^{(p)}=A_p\sin C_p x_1\sin D_p x_2\sin 2\pi x_3 e^{\omega_p t},$$

where

$$A_p = \omega_p = 0.1\ (p = 1,2,3),\ B_1 = C_2 = 0.2,\ B_2 = B_3 = C_1 = C_3 = 0.1,$$
$$D_1 = 0.1,\ D_2 = 0.2,\ D_3 = 0.3.$$

To measure the computational errors, we introduce the following norms:

$$R_\infty(z;t) = \max_{1\le j_1,j_2\le M-1} \max_{1\le j_3\le 2N} \max_{1\le p\le 3} \left| z^{(p)}(j_1 h, j_2 h, \frac{j_3}{2N}, t)\right|,$$

$$R_2(z;t) = \left(\frac{h^2}{2N} \sum_{p=1}^{3} \sum_{j_1,j_2=1}^{M-1} \sum_{j_3=1}^{2N} |z^{(p)}(j_1 h, j_2 h, \frac{j_3}{2N}, t)|^2 \right)^{\frac{1}{2}}.$$

We first use scheme (2.7), (3.2) to solve (1.1), (3.1) with $\nu = 1$, $a = 0$, $b = 1$, $\delta = \sigma = 0$, $h = 0.125$, $N = 2$, $\tau = 0.002$. The errors are shown in Table I.

On the other hand, we define the difference operator

$$R_h^{(1)}(w) = w_{\hat{x}_2}^{(3)} - w_{\hat{x}_3}^{(2)}, \quad R_h^{(2)}(w) = w_{\hat{x}_3}^{(1)} - w_{\hat{x}_1}^{(3)}, \quad R_h(w) = w_{\hat{x}_1}^{(2)} - w_{\hat{x}_2}^{(1)},$$

$$J_h(u,w) = \frac{1}{2}\sum_{p=1}^{3}\left(R_h^{(p)}(w)u_{\hat{x}_p} + (R_h^{(p)}(w)u)_{\hat{x}_p}\right), \quad H_h(u,w) = \sum_{p=1}^{3} u^{(p)} R_{\hat{x}_p}^{(p)}(w),$$

where $u_{x_p}, u_{\bar{x}_p}, u_{\hat{x}_p} (p = 1,2)$ are the same as before and

$$u_{x_3}(x,t) = \frac{1}{h^*}(u(x + h^* e_3, t) - u(x,t)), \quad u_{\bar{x}_3}(x,t) = u_{x_3}(x - h^* e_3, t),$$

$$u_{\hat{x}_3}(x,t) = \frac{1}{2}(u_{x_3}(x,t) + u_{\bar{x}_3}(x,t)), \quad \Delta_h u(x,t) = \sum_{p=1}^{3} u_{x_p \bar{x}_p}(x,t).$$

Table I. The Error of Scheme (2.7), (3.2)

t	$R_\infty(\xi - \eta^{(N)};t)$	$R_2(\xi - \eta^{(N)};t)$	$\frac{R_2(\xi - \eta^{(N)};t)}{R_2(\xi;t)}$
0.1	0.2419E-03	0.1117E-03	0.7228E-03
0.2	0.2534E-03	0.1192E-03	0.7634E-03
0.3	0.2562E-03	0.1209E-03	0.7666E-03
0.4	0.2582E-03	0.1220E-03	0.7659E-03
0.5	0.2601E-03	0.1231E-03	0.7650E-03
0.6	0.2620E-03	0.1242E-03	0.7638E-03
0.7	0.2638E-03	0.1252E-03	0.7628E-03
0.8	0.2657E-03	0.1263E-03	0.7618E-03
0.9	0.2675E-03	0.1274E-03	0.7607E-03
1.0	0.2694E-03	0.1285E-03	0.7597E-03

The explicite difference scheme in [2] is the following

$$\begin{cases} \eta_t^{(h)}(x,t) + J_h\left(\eta^{(h)}(x,t), \varphi^{(h)}(x,t)\right) - H_h\left(\eta^{(h)}(x,t), \varphi^{(h)}(x,t)\right) - \nu\Delta_h \eta^{(h)}(x,t) \\ \qquad = f_1(x,t), \\ -\Delta_h \varphi^{(h)}(x,t) = \eta^{(h)}(x,t) + f_2(x,t). \end{cases}$$

We use (4.1), (3.2) to solve (1.1), (3.1) with $\nu = 1,\ a = 0,\ b = 1,\ h = 0.125,\ h^* = 0.25,\ \tau = 0.002$.

The errors are shown in Table II.

The computational time for the above two schemes are nearly the same. But Tables I and II show that the spectral–difference scheme gives much better results than the difference scheme.

Table II. The Error of Scheme (4.1), (3.2)

t	$R_\infty(\xi - \eta^{(h)}; t)$	$R_2(\xi - \eta^{(h)}; t)$	$\dfrac{R_2(\xi - \eta^{(h)}; t)}{R_2(\xi; t)}$
0.1	0.5238E-02	0.2314E-02	0.1497E-01
0.2	0.5314E-02	0.2345E-02	0.1502E-01
0.3	0.5367E-02	0.2369E-02	0.1502E-01
0.4	0.5420E-02	0.2392E-02	0.1501E-01
0.5	0.5473E-02	0.2416E-02	0.1501E-01
0.6	0.5527E-02	0.2440E-02	0.1501E-01
0.7	0.5581E-02	0.2464E-02	0.1501E-01
0.8	0.5636E-02	0.2489E-02	0.1501E-01
0.9	0.5691E-02	0.2531E-02	0.1501E-01
1.0	0.5747E-02	0.2538E-02	0.1501E-01

§5. Some Lemmas.

Let C denote a positive constant independent of any function. Let $H^{(\beta)}(I)$ be the usual Sobolev space and $H_p^\beta(I) = H^{(\beta)}(I) \cup \{v(x_3), v(x_3 + 2\pi) = v(x_3)\}$.

Lemma 1[7]. *If $v(x_3) \in (H_p^\beta(I))^3$ and $0 \le \alpha \le \beta$, then*

$$\|P_N v - v\|_{H^\alpha(I)} \le CN^{\alpha-\beta}\|v\|_{H^\beta(I)}, \qquad \|P_N v\|_{H^\alpha(I)} \le C\|v\|_{H^\alpha(I)}.$$

Lemma 2. *If $v(x) \in (V_N)^3$ for all $(x_1, x_2) \in Q_h$, then*

$$|v|_1^2 \le \left(N^2 + \frac{8}{h^2}\right)\|v\|^2 + \min(h\|v_n\|_{\Gamma_h}^2, \frac{2}{h}\|v\|_{\Gamma_h}^2).$$

Lemma 3. *If $v(x) \in (V_N)^3$ for all $(x_1, x_2) \in Q_h$ and $v(x) = 0$ on Γ_h, then*

$$\|v\|^2 \le C\left(|v|_1^2 + S(v)\right).$$

Lemma 4. *If $v(x) \in (V_N)^3$ for all $(x_1, x_2) \in Q_h$ and $v(x) = 0$ on Γ_h, then*

$$|v|_2^2 \le \|\Delta v\|^2.$$

Proof. Suppose that $u(x)$ has the same property as $v(x)$. By using (2.4) and Laplace's formula twice and the boundary conditions, we have

$$4(\Delta u, \Delta v) = \sum_{p=1}^{2}\left[2(u_{x_p\bar{x}_p}, v_{x_p\bar{x}_p}) + (u_{x_px_p}, v_{x_px_p})_{\Omega_h/\Omega_{h,p}^{*,-}} + (u_{\bar{x}_p\bar{x}_p}, v_{\bar{x}_p\bar{x}_p})_{\Omega_h/\Omega_{h,p}^{*,+}}\right]$$

$$+4\left(\frac{\partial^2 u}{\partial x_3^2}, \frac{\partial^2 v}{\partial x_3^2}\right) + 2(u_{x_1x_2}, v_{x_1x_2}) + 2(u_{\bar{x}_1\bar{x}_2}, v_{\bar{x}_1\bar{x}_2}) + 2(u_{\bar{x}_1x_2}, v_{\bar{x}_1x_2})$$

$$+2(u_{x_1x_2},v_{x_1x_2})+4\sum_{p=1}^{2}\left[\left(\frac{\partial u_{x_p}}{\partial x_3},\frac{\partial v_{x_p}}{\partial x_3}\right)+\left(\frac{\partial u_{x_p}}{\partial x_3},\frac{\partial v_{x_p}}{\partial x_3}\right)\right]+4S_2(u_{x_1},v_{x_1})$$
$$+4S_2(u_{x_1},v_{x_1})+4S_1(u_{x_2},v_{x_2})+4S_1(u_{x_2},v_{x_2})+4S\left(\frac{\partial u}{\partial x_3},\frac{\partial v}{\partial x_3}\right)$$
$$+h^2\sum_{j=1}^{M-1}\Big[(u_{x_1x_1}(h,jh),v_{x_1x_1}(h,jh))_I+(u_{x_1x_1}(1-h,jh),v_{x_1x_1}(1-h,jh))_I$$
$$+(u_{x_2x_2}(jh,h),v_{x_2x_2}(jh,h))_I+(u_{x_2x_2}(jh,1-h),v_{x_2x_2}(jh,1-h))_I\Big]$$
$$+\frac{1}{h^2}\Big[(u(h,h),v(h,h))_I+(u(h,1-h),v(h,1-h))_I+(u(1-h,h),v(1-h,h))_I$$
$$+(u(1-h,1-h),v(1-h,1-h))_I\Big], \tag{5.1}$$

where

$$S(u,v)=S_1(u,v)+S_2(u,v)$$

and

$$S_1(u,v)=\frac{1}{2}\sum_{j=1}^{M-1}\left[(u(h,jh),v(h,jh))_I+(u(1-h,jh),v(1-h,jh))_I\right],$$
$$S_2(u,v)=\frac{1}{2}\sum_{j=1}^{M-1}\left[(u(jh,h),v(jh,h))_I+(u(jh,1-h),v(jh,1-h))_I\right].$$

Clearly, $S(u,u)=S(u)$. By putting $u=v$ in (5.1), we complete the proof.

Lemma 5. *If $u(x),v(x)\in(V_N)^3$ for all $(x_1,x_2)\in Q_h$, then*

$$\|u(x_1,x_2)v(x_1,x_2)\|_I^2\le(2N+1)\|u(x_1,x_2)\|_I^2\|v(x_1,x_2)\|_I^2,$$
$$\|u(x_3)v(x_3)\|_{Q_h}^2\le\frac{1}{h^2}\|u(x_3)\|_{Q_h}^2\|v(x_3)\|_{Q_h}^2,\quad \|uv\|^2\le\frac{2N+1}{h^2}\|u\|^2\|v\|^2,$$
$$\|uv\|_{\Gamma_h}^2\le\frac{2N+1}{h}\|u\|_{\Gamma_h}^2\|v\|_{\Gamma_h}^2,\quad \|uv\|_{\Omega_h^*}^2\le\frac{2N+1}{h}\|u\|_{\Omega_h^*}^2\|v\|_{\Omega_h^*}^2.$$

Lemma 6. *If $v(x)\in(V_N)^3$ for all $(x_1,x_2)\in Q_h$, then*

$$\|v\|_{l^4}^4\le\frac{C}{h^2}\|v\|^3\|v\|_1.$$

Proof. Let

$$v^{(p)}(x)=\sum_{|l|=0}^{\infty}v_l^{(p)}(x_1,x_2)e^{ilx_3}.$$

From [8], we have

$$\frac{1}{2\pi}\int_0^{2\pi}|v^{(p)}(x)|^4dx_3\le\left(\sum_{|l|=0}^{\infty}|v_l^{(p)}(x_1,x_2)|^{\frac{4}{3}}\right)^3\equiv D_0(x_1,x_2).$$

From Holder's inequality,

$$\sum_{|l|=0}^{\infty} |v_l^{(p)}(x_1,x_2)|^{\frac{4}{3}} \le D_1^{\frac{2}{3}}(x_1,x_2) D_2^{\frac{1}{3}}(x_1,x_2),$$

where

$$D_1(x_1,x_2) = \sum_{|l|=0}^{\infty} |v_l^{(p)}(x_1,x_2)|^2 \left(1 + \frac{l^2 \|v^{(p)}(x_1,x_2)\|_I^2}{\|v^{(p)}(x_1,x_2)\|_{H^1(I)}^2}\right),$$

$$D_2(x_1,x_2) = \sum_{|l|=0}^{\infty} \left(1 + \frac{l^2 \|v^{(p)}(x_1,x_2)\|_I^2}{\|v^{(p)}(x_1,x_2)\|_{H^1(I)}}\right)^{-2}.$$

Moreover,

$$D_1(x_1,x_2) = 2\|v^{(p)}(x_1,x_2)\|_I^2$$

and

$$D_2(x_1,x_2) \le 1 + \frac{2\|v^{(p)}(x_1,x_2)\|_{H^1(I)}}{\|v^{(p)}(x_1,x_2)\|_I^2} \int_0^{\infty} \frac{dr}{(1+r^2)^2} \le C \frac{\|v^{(p)}(x_1,x_2)\|_{H^1(I)}}{\|v^{(p)}(x_1,x_2)\|_I}.$$

Therefore,

$$D_0(x_1,x_2) \le C\|v^{(p)}(x_1,x_2)\|_I^3 \|v^{(p)}(x_1,x_2)\|_{H^1(I)}$$

and so

$$\frac{h^2}{2\pi} \sum_{j_1,j_2=1}^{M-1} \int_0^{2\pi} |v^{(p)}(j_1 h, j_2 h, x_3)|^4 dx_3 \le \frac{C}{h^2} \|v^{(p)}\|^3 \|v^{(p)}\|_1$$

from that the conclusion follows.

Lemma 7. *If $u(x), v(x) \in (V_N)^3$ for all $(x_1,x_2) \in Q_h$, then*

$$\|H(u,v)\|^2 \le \frac{CN}{h^2} \|u\|^2 |v|_2^2.$$

Lemma 8. *If $h < 2\varepsilon$, and $v(x) \in (V_N)^3$ for all $(x_1,x_2) \in Q_h$, then*

$$\|v\|_{\Gamma_h}^2, \|u\|_{\Omega_h^*}^2 \le \varepsilon |u|_1^2 + C(\varepsilon)\|u\|^2.$$

Lemma 9. *For any function $u(x,t)$ we have*

$$2(u(t), u_t(t))_I = (\|u(t)\|_I^2)_t - 2\|u(t)\|_I^2, \quad 2(u(t), u_t(t)) = (\|u(t)\|^2)_t - 2\|u(t)\|^2.$$

Lemma 10. *If $u(x,t) \in (V_N)^3$ for all $(x_1,x_2) \in Q_h$, then*

$$2(u_t(t), \Delta u(t)) + (|u(t)|_1^2)_t - \tau |u_t(t)|_1^2 = 2B(u_t(t), u(t)),$$
$$2(u(t), \Delta u_t(t)) + (|u(t)|_1^2)_t - \tau |u_t(t)|_1^2 = 2B(u(t), u_t(t)).$$

Lemma 11. *If the following conditions are fulfilled:*

(i) *$Z_1(t)$ and $Z_2(t)$ are nonnegative functions; a, b, ρ and M_l are nonnegative constants,*

(ii) *$F(Z)$ is such a function that $Z \le M_3$ implies $F(Z) \le 0$,*

(iii) $Z_1(0) \le \rho$ *and for all* $t \in S_\tau$,

$$Z_1(t) \le \rho + \sum_{\substack{y \le S_\tau \\ y \le t-\tau}} \left[M_1 Z_1(y) + M_2 N^a h^{-b} Z_1^2(y) + F(Z_2(y)) \right],$$

(iv) $\rho e^{(M_1+M_2)T} \le \min\left(M_3, \dfrac{h^b}{N^a} \right)$,

then for all $t \in S_\tau$ *and* $t \le T$, *we have* $Z_1(t) \le \rho e^{(M_1+M_2)t}$.

§6. The proofs of Theorems

We consider Theorem 1. By taking the scalar product of the first formula of (3.3) with $2\tilde{\eta}^{(N)}(x,t)$, we have from (2.2) - (2.5) and Lemmas 9 - 10

$$\begin{aligned}
&\|\tilde{\eta}^{(N)}(t)\|_t^2 - \tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + 2\nu|\tilde{\eta}^{(N)}(t)|_1^2 + \nu\sigma\tau(|\tilde{\eta}^{(N)}(t)|_1^2)_t - \nu\sigma\tau^2|\tilde{\eta}_t^{(N)}(t)|_1^2 \\
&+2(\tilde{\eta}^{(N)}(t), J(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \tilde{\varphi}^{(N)}(t))) - 2\delta\tau(\tilde{\eta}_t^{(N)}(t), J(\tilde{\eta}^{(N)}(t), \varphi^{(N)}(t) \\
&+\tilde{\varphi}^{(N)}(t))) - 2(\tilde{\eta}^{(N)}(t), H(\tilde{\eta}^{(N)}(t), \varphi^{(N)}(t) + \tilde{\varphi}^{(N)}(t)) + H(\eta^{(N)}(t), \tilde{\varphi}^{(N)}(t))) \\
&+\sum_{\alpha=1}^{6} D_\alpha(t) + B_1(t) + B_2(t) = 2(\tilde{\eta}^{(N)}(t), \tilde{f}_1(t))
\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}
&D_1(t) = A(\tilde{\eta}^{(N)}(t), \tilde{\eta}^{(N)}(t), \tilde{\varphi}^{(N)}(t)), \quad D_2(t) = A(\tilde{\eta}^{(N)}(t), \tilde{\eta}^{(N)}(t), \varphi^{(N)}(t)), \\
&D_3(t) = \delta\tau A(\tilde{\eta}^{(N)}(t), \tilde{\eta}_t^{(N)}(t), \tilde{\varphi}^{(N)}(t)), \quad D_4(t) = \delta\tau A(\tilde{\eta}^{(N)}(t), \tilde{\eta}_t^{(N)}(t), \varphi^{(N)}(t)), \\
&D_5(t) = \delta\tau A(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}^{(N)}(t), \tilde{\varphi}^{(N)}(t)), \quad D_6(t) = \delta\tau A(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}^{(N)}(t), \varphi^{(N)}(t)), \\
&B_1(t) = -2\nu B(\tilde{\eta}^{(N)}(t), \tilde{\eta}^{(N)}(t)), \quad B_2(t) = -2\nu\sigma\tau B(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}_t^{(N)}(t)).
\end{aligned}$$

Let m be an undetermined positive constant. By taking the scalar product of the first formula of (3.3) with $m\tau\tilde{\eta}_t^{(N)}(x,t)$, we obtain

$$\begin{aligned}
&m\tau\|\tilde{\eta}^{(N)}(t)\|^2 + \frac{m\nu\tau}{2}(|\tilde{\eta}^{(N)}(t)|_1^2)_t - \frac{m\nu\tau^2}{2}|\tilde{\eta}_t^{(N)}(t)|_1^2 + m\nu\sigma\tau^2|\tilde{\eta}_t^{(N)}(t)|_1^2 \\
&+m\tau(\tilde{\eta}_t^{(N)}(t), J(\tilde{\eta}^{(N)}(t), \varphi^{(N)}(t) + \tilde{\varphi}^{(N)}(t)) + J(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \tilde{\varphi}^{(N)}(t))) \\
&+D_7(t) + D_8(t) + B_3(t) + B_4(t) = m\tau(\tilde{\eta}_t^{(N)}(t), \tilde{f}_1(t))
\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
&D_7(t) = \frac{1}{2}m\delta\tau^2 A(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}_t^{(N)}(t), \tilde{\varphi}^{(N)}(t)), \quad D_8(t) = \frac{1}{2}m\delta\tau^2 A(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}_t^{(N)}(t), \varphi^{(N)}(t)), \\
&B_3(t) = -m\nu\tau B(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}^{(N)}(t)), \quad B_4(t) = -m\nu\sigma\tau^2 B(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}_t^{(N)}(t)).
\end{aligned}$$

Letting $\varepsilon > 0$ and putting (6.1), (6.2) together, we obtain

$$\begin{aligned} &\|\tilde{\eta}^{(N)}(t)\|_t^2 + \tau(m-1-\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2 + 2\nu|\tilde{\eta}^{(N)}(t)|_1^2 + \nu\tau(\sigma + \frac{m}{2})(|\tilde{\eta}^{(N)}(t)|_1^2)_t \\ &+\nu\tau^2(m\sigma - \sigma - \frac{m}{2})|\tilde{\eta}_t^{(N)}(t)|_1^2 + \sum_{\alpha=1}^{5} G_\alpha(t) + \sum_{\alpha=1}^{8} D_\alpha(t) + \sum_{\alpha=1}^{4} B_\alpha(t) \\ &\le \|\tilde{\eta}^{(N)}(t)\|^2 + (1 + \frac{m^2\tau}{4\varepsilon})\|\tilde{f}_1(t)\|^2, \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} G_1(t) &= \left(2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t), J(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \tilde{\varphi}^{(N)}(t))\right), \\ G_2(t) &= \tau(m-2\delta)\left(\tilde{\eta}_t^{(N)}(t), J(\tilde{\eta}^{(N)}(t), \varphi^{(N)}(t))\right), \\ G_3(t) &= \tau(m-2\delta)\left(\tilde{\eta}_t^{(N)}(t), J(\tilde{\eta}^{(N)}(t), \tilde{\varphi}^{(N)}(t))\right), \\ G_4(t) &= -\left(2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t), P_N H(\tilde{\eta}^{(N)}(t), \tilde{\varphi}^{(N)}(t))\right), \\ G_5(t) &= -\left(2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t), P_N H(\tilde{\eta}^{(N)}(t), \tilde{\varphi}^{(N)}(t)) + P_N H(\tilde{\eta}^{(N)}(t), \varphi^{(N)}(t))\right). \end{aligned}$$

On the other hand, by taking the scalar product of the second formula of (3.3) with $\tilde{\varphi}^{(N)}(x,t)$ we have from (2.6)

$$|\tilde{\varphi}^{(N)}(t)|_1^2 + S(\tilde{\varphi}^{(N)}(t)) \le \frac{1}{2C}\|\tilde{\varphi}^{(N)}(t)\|^2 + C(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2).$$

Thus, Lemma 3 leads to

$$|\tilde{\varphi}^{(N)}(t)|_1^2 + S(\tilde{\varphi}^{(N)}(t)) \le C(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{6.4}$$

We are going to estimate $|G_\alpha(t)|$. We have from (6.4)

$$|G_1(t)| \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C}{\varepsilon}\|\|\eta^{(N)}(t)\|\|_{1,\infty}^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2), \tag{6.5}$$

$$|G_2(t)| \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C(m-2\delta)^2}{\varepsilon}\|\|\varphi^{(N)}(t)\|\|_{1,\infty}^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \frac{\tau}{h}\|\tilde{g}(t)\|_{\Gamma_h}^2), \tag{6.6}$$

$$|G_3(t)| \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C\tau N(m-2\delta)^2}{\varepsilon h^2}|\tilde{\eta}^{(N)}(t)|_1^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{6.7}$$

By Lemma 4,

$$|\tilde{\varphi}^{(N)}(t)|_2^2 \le \|\Delta\tilde{\varphi}^{(N)}(t)\|^2 \le 2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2) \tag{6.8}$$

from that and Lemma 7,

$$\begin{aligned} &|m\tau(\tilde{\eta}_t^{(N)}(t),\ P_N H(\tilde{\eta}^{(N)}(t), \tilde{\varphi}^{(N)}(t)))| \\ &\qquad \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C\tau N}{h^2}\|\tilde{\eta}^{(N)}(t)\|^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \end{aligned}$$

On the other hand, Lemma 6 gives

$$\begin{aligned}&|(\tilde{\eta}^{(N)}(t), P_N H(\tilde{\eta}^{(N)}(t), \tilde{\varphi}^{(N)}(t)))| \le C\left(\|\tilde{\eta}^{(N)}(t)\|_{l^4}^2 + |\tilde{\varphi}^{(N)}(t)|_2^2\right)\\ &\le C\left(\frac{1}{h^2}\|\tilde{\eta}^{(N)}(t)\|^4 + \frac{1}{h^2}\|\tilde{\eta}^{(N)}(t)\|^2|\tilde{\eta}^{(N)}(t)|_1^2 + \|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2\right).\end{aligned}$$

Thus, the combination of the above two estimates gives

$$\begin{aligned}|G_4(t)| &\le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C}{h^2}(1+\frac{\tau N}{\varepsilon})\|\tilde{\eta}^{(N)}(t)\|^4 + \frac{C}{h^2}\|\tilde{\eta}^{(N)}(t)\|^2|\tilde{\eta}^{(N)}(t)|_1^2\\ &+C\left(\frac{\tau N}{h^2}\|\tilde{f}_2(t)\|^2+1\right)\|\tilde{\eta}^{(N)}(t)\|^2 + C\|\tilde{f}_2(t)\|^2.\end{aligned} \tag{6.9}$$

Similarly, we have

$$|G_5(t)| \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + C(1+\frac{\tau}{\varepsilon})\left(|||\eta^{(N)}(t)|||_{0,\infty}^2 + |||\varphi^{(N)}(t)|||_{2,\infty}^2\right)(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{6.10}$$

We next estimate $|D_\alpha(t)|$. From Lemma 5 and (6.4), we have

$$\begin{aligned}|D_1(t)| &\le \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \frac{CN}{\varepsilon h}\|\tilde{g}(t)\|_{\Gamma_h}^2|\tilde{\varphi}^{(N)}(t)|_1^2\\ &\le \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \frac{CN}{\varepsilon h}\|\tilde{g}(t)\|_{\Gamma_h}^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2).\end{aligned} \tag{6.11}$$

Similarly,

$$\begin{aligned}|D_3(t)| + |D_5(t)| &\le \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) + \frac{CN}{\varepsilon h}(\|\tilde{g}(t)\|_{\Gamma_h}^2\\ &+\tau^2\|\tilde{g}_t(t)\|_{\Gamma_h}^2)(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2),\end{aligned} \tag{6.12}$$

$$|D_7(t)| \le \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) + \frac{CN\tau^2}{\varepsilon h}\|\tilde{g}_t(t)\|_{\Gamma_h}^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{6.13}$$

On the other hand,

$$|D_2(t)| \le \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \frac{Ch}{\varepsilon}|||\varphi^{(N)}(t)|||_{1,\infty}^2\|\tilde{g}(t)\|_{\Gamma_h}^2, \tag{6.14}$$

$$\begin{aligned}|D_4(t)| + |D_6(t)| &\le \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t))\\ &+\frac{Ch}{\varepsilon}|||\varphi^{(N)}(t)|||_{1,\infty}^2(\|\tilde{g}(t)\|_{\Gamma_h}^2 + \tau^2\|\tilde{g}_t(t)\|_{\Gamma_h}^2),\end{aligned} \tag{6.15}$$

$$|D_8(t)| \le \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) + \frac{Ch\tau^2}{\varepsilon}|||\varphi^{(N)}(t)|||_{1,\infty}^2\|\tilde{g}_t(t)\|_{\Gamma_h}^2. \tag{6.16}$$

We now estimate $|B_\alpha(t)|$. We have

$$|B_1(t)| \ge 2\nu S(\tilde{\eta}^{(N)}(t)) - \frac{C}{\varepsilon h}\|\tilde{g}(t)\|_{\Gamma_h}^2, \tag{6.17}$$

$$\begin{aligned}|B_2(t)| + |B_3(t)| &\ge \nu\tau(\sigma+\frac{m}{2})S_t(\tilde{\eta}^{(N)}(t)) - \nu\tau^2(\sigma+\frac{m}{2})S(\tilde{\eta}_t^{(N)}(t))\\ &-\varepsilon\nu S(\tilde{\eta}^{(N)}(t)) - \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) - \frac{C}{\varepsilon h}(\|\tilde{g}(t)\|_{\Gamma_h}^2 + \tau h^2\|\tilde{g}_t(t)\|_{\Gamma_h}^2),\end{aligned} \tag{6.18}$$

$$|B_4(t)| \ge m\nu\sigma\tau^2 S(\tilde{\eta}_t^{(N)}(t)) - \frac{C\tau h}{\varepsilon}\|\tilde{g}_t(t)\|_{\Gamma_h}^2. \tag{6.19}$$

By substituting (6.5) – (6.9) into (6.3), we obtain

$$
\begin{aligned}
&\|\tilde{\eta}^{(N)}(t)\|_t^2 + \tau(m-1-6\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2 + \nu|\tilde{\eta}^{(N)}(t)|_1^2 \\
&\quad + \nu(2-6\varepsilon)S(\tilde{\eta}^{(N)}(t)) + \nu\tau(\sigma + \frac{m}{2})(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t)))_t \\
&\quad + \nu\tau^2(m\sigma - \sigma - \frac{m}{2} - 6\varepsilon)(|\tilde{\eta}_t^{(N)}(t)|_1^2 + S(\tilde{\eta}_t^{(N)}(t))) \\
&\quad \le F_0(t)\|\tilde{\eta}^{(N)}(t)\|^2 + F_1(t)\|\tilde{\eta}^{(N)}(t)\|^4 + F_2(t)|\tilde{\eta}^{(N)}(t)|_1^2 + R(t),
\end{aligned}
\tag{6.20}
$$

where

$$
\begin{aligned}
F_0(t) &= C(1+\frac{1}{\varepsilon})(|||\eta^{(N)}|||_{1,\infty}^2 + |||\varphi^{(N)}(t)|||_{2,\infty}^2) + C(1+\frac{\tau N}{\varepsilon})\|\tilde{f}_2(t)\|^2 \\
&\quad + \frac{CN}{\varepsilon h}(\|\tilde{g}(t)\|_{\Gamma_h}^2 + \tau^2\|\tilde{g}_t(t)\|_{\Gamma_h}^2), \\
F_1(t) &= \frac{C}{h^2}(1+\frac{\tau N}{\varepsilon}), \\
F_2(t) &= -\nu + \frac{C}{h^2}(1+\frac{\tau N(m-2\delta)^2}{\varepsilon})(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2), \\
R(t) &= C(1+\frac{\tau}{\varepsilon})\|\tilde{f}_1(t)\|^2 + C(1+\frac{1}{\varepsilon}|||\eta^{(N)}|||_{1,\infty}^2 + \frac{1}{\varepsilon}|||\varphi^{(N)}|||_{2,\infty}^2 + \frac{N}{h}\|\tilde{g}(t)\|_{\Gamma_h}^2 \\
&\quad + \frac{N\tau^2}{h}\|\tilde{g}_t(t)\|^2)\|\tilde{f}_2(t)\|^2 + \frac{C}{\varepsilon h}(1+\tau|||\varphi^{(N)}|||_{2,\infty}^2)\|\tilde{g}(t)\|_{\Gamma_h}^2 \\
&\quad + \frac{Ch\tau^2}{\varepsilon}(1+|||\varphi^{(N)}|||_{2,\infty}^2)\|\tilde{g}_t^{(N)}(t)\|_{\Gamma_h}^2.
\end{aligned}
$$

Let ε be suitably small and choose the value of m as follows:

Case I. $\sigma > \frac{1}{2}$. We take

$$m > m_1 = \max(\frac{2\sigma + 12\varepsilon}{2\sigma - 1}, 1 + 6\varepsilon + p_0).$$

Then (6.20) becomes

$$
\begin{aligned}
&\|\tilde{\eta}^{(N)}(t)\|_t^2 + p_0\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \nu|\tilde{\eta}^{(N)}(t)|_1^2 + \nu S(\tilde{\eta}^{(N)}(t)) + \nu\tau(\sigma + \frac{m}{2})(|\tilde{\eta}^{(N)}(t)|_1^2 \\
&\quad + S(\tilde{\eta}^{(N)}(t)))_t \le F_0(t)\|\tilde{\eta}^{(N)}(t)\|^2 + F_1(t)\|\tilde{\eta}^{(N)}(t)\|^4 + F_2(t)|\tilde{\eta}^{(N)}(t)|_1^2 + R(t).
\end{aligned}
\tag{6.21}
$$

Case II. $\sigma = \frac{1}{2}$. By Lemma 2 and the definition of $S(u(t))$, we get

$$|\tilde{\eta}_t^{(N)}(t)|_1^2 \le (N^2 + \frac{8}{h^2})\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{2}{h}\|\tilde{g}_t(t)\|_{\Gamma_h}^2, \quad S(\tilde{\eta}_t^{(N)}(t)) \le \frac{1}{2h^2}\|\tilde{\eta}_t^{(N)}(t)\|^2.$$

Thus,

$$
\begin{aligned}
&\tau(m-1-6\varepsilon)\|\tilde{\eta}^{(N)}(t)\|^2 - \nu\tau^2(\frac{1}{2}+6\varepsilon)(|\tilde{\eta}_t^{(N)}(t)|_1^2 + S(\tilde{\eta}_t^{(N)}(t))) \\
&\quad \ge \tau(m-1-6\varepsilon - \nu\tau(\frac{1}{2}+6\varepsilon)(N^2 + \frac{17}{2h^2}))\|\tilde{\eta}_t^{(N)}(t)\|^2.
\end{aligned}
$$

So (6.21) holds provided

$$m > m_2 = 1 + 6\varepsilon + p_0 + \frac{\nu\tau}{4}(1 + 12\varepsilon)(2N^2 + \frac{17}{h^2}).$$

Case III. $\sigma < \frac{1}{2}$ and $\tau < \frac{4h^2}{\nu(1-2\sigma)(17+2h^2N^2)}$. We take

$$m > m_3 = (4 + 24\varepsilon + 4p_0 + \nu\tau(\sigma + 6\varepsilon)(4N^2 + \frac{34}{h^2})(4 + \nu\tau(2\sigma - 1)(2N^2 + \frac{17}{h^2}))^{-1}.$$

Then (6.21) holds too.

By summing (6.21) for $t \in S_\tau$ we obtain

$$E_1(\tilde{\eta}^{(N)};t) \le \rho(\tilde{\eta}_0^{(N)}, \tilde{f}_1, \tilde{f}_2, h^{-\frac{1}{2}}\tilde{g}; t) + \tau \sum_{\substack{y\in S_\tau \\ y\in t-\tau}} (F_0(y)E_1(\tilde{\eta}^{(N)};y)$$
$$+F_1(y)E_1^2(\tilde{\eta}^{(N)};y) + F_2(y)|\tilde{\eta}^{(N)}(y)|_1^2).$$

Finally, we apply Lemma 11 to finish the proof of Theorem 1.

We can prove other theorems similarly.

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