Journal of Computational Mathematics, Vol.8, No.3, 1990, 307–320.

THE NUMERICAL SOLUTION OF SECOND–ORDER WEAKLY SINGULAR VOLTERRA INTEGRO–DIFFERENTIAL EQUATIONS*

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Abstract

In this paper we investigate the attainable order of (global) convergence of collocation approximations in certain polynomial spline spaces for solutions of a class of second-order volterra integro-differential equations with weakly singular kernels. While the use of quasi-uniform meshes leads, due to the nonsmooth nature of these solutions, to convergence of order less than one, regardless of the degree of the approximating spline function, collocation on suitably graded meshes will be shown to yield optimal convergence rates.

§1. Introduction

In this paper we present an analysis of certain numerical methods for solving the second-order Volterra integro-differential equation (VIDE)

$$y''(t) = f(t, y(t)) + \int_0^t (t-s)^{-\alpha} k(t, s, y(s))ds, \quad t \in I := [0, T], \tag{1.1}$$

with initial conditions $y(0) = y_0, y'(0) = z_0$. Here, the given functions $f : I \times \mathbb{R} \to \mathbb{R}$ and $k : S \times \mathbb{R} \to \mathbb{R}$ (with $S := \{(t, s) : 0 \le s \le t \le T\}$) denote given smooth functions, and constant α satisfies $0 < \alpha < 1$. In practical applications one very frequently encounters the linear counterpart of (1.1)

$$y''(t) = p(t)y(t) + q(t) + \int_0^t (t-s)^{-\alpha} K(t, s)y(s)ds, \quad t \in I (0 < \alpha < 1). \tag{1.2}$$

In the subsequent analysis we shall, for ease of exposition, usually utilize the linear version of (1.1) to display the principal ideas.

Equations of type (1.1) (in practical applications one occasionally encounters second-order VIDEs whose right-hand sides contain also terms involving y'; see e.g., [8, 9]; we shall consider this general case in a subsequent paper) arise in many areas of physics and engineering. But the literature on the numerical solution of (1.1) or its general case is comparatively small. Very little convergence analysis has been given so far. Moreover, as far as high-order Volterra integro-differential equations are concerned, Aguilar & Brunner [1] have presented a study of collocation techniques for Eq. (1.1) with $\alpha = 0$, and Tang [10]

*Received December 9, 1987.

for high-order Volterra integro-differential equations without singularity. Prosperetti [8, 9] introduced methods based on piecewise cubic Hermite interpolation for a class of second-order integro-differential equations, where care is taken that on a suitable initial interval the nonsmooth solution is approximated accurately. No convergence analysis has been given to this method.

The numerical methods to be analyzed will be collocation methods in the polynomial spline space,

$$S_{m+1}^{(1)}(Z_N) := \{u : u|_{\sigma_n} =: u_n \in P_{m+1}, 0 \le n \le N-1,$$
$$u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \text{ for } t_n \in Z_N \text{ and } j = 0, 1\}, \tag{1.3}$$

associated with a given partition (or : mesh) Π_N of the interval I,

$$\Pi_N : 0 = t_0^{(N)} < t_1^{(N)} < \cdots < t_N^{(N)} = T$$

(the index indicating the dependence of the mesh points on N will, for ease of notation, subsequently be suppressed). Here, P_{m+1} denotes the space of real polynomials of degree not exceeding $m+1$, and we have set $\sigma_0 := [t_0, t_1]$, $\sigma_n := (t_n, t_{n+1}]$ $(1 \le n \le N-1)$; the set $Z_N := \{t_n : 1 \le n \le N-1\}$ (i.e., the interior mesh points) will be referred to as the knots of these polynomial splines. In addition, we define

$$h := \max\{h_n : 0 \le n \le N-1\}, \quad h' := \min\{h_n : 0 \le n \le N-1\}, \tag{1.4}$$

where $h_n := t_{n+1} - t_n$; the quantity h is often called the diameter of the mesh Π_N (note that, according to the above remark on our notation, both h and h' will depend on N).

In order to describe these collocation methods we rewrite (1.1), for $t \in \sigma_n$, in "one-step form",

$$y''(t) = F_n(y;t) + \int_{t_n}^{t} (t-s)^{-\alpha} k(t, s, y(s)) ds, \tag{1.5}$$

where

$$F_n(y;t) := f(t, y(t)) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t-s)^{-\alpha} k(t, s, y(s)) ds. \tag{1.6}$$

For given parameters $\{c_j\}$ with $0 \le c_1 < \cdots < c_m \le 1$, we introduce the sets

$$X_n := \{t_{nj} := t_n + c_j h_n; 1 \le j \le m\}, \quad 0 \le n \le N-1, \tag{1.7}$$

and we define

$$X(N) := \bigcup_{n=0}^{N-1} X_n;$$

the set $X(N)$ will be referred to as the set of collocation points, while $\{c_j\}$ will be called collocation parameters. A numerical approximation to the exact solution y of (1.1) (or (1.2)) is an element of $S_{m+1}^{(1)}(Z_N)$ satisfying the given equation on $X(N)$, i.e., by (1.5), this approximation u is computed recursively from

$$u_n''(t_{nj}) = F_n(u; t_{nj}) + h_n^{1-\alpha} \int_0^{c_j} (c_j - s)^{-\alpha} k(t_{nj}, t_n + sh_n, u_n(t_n + sh_n)) ds,$$
$$j = 1, \cdots, m, \tag{1.8}$$

where

$$F_n(u;t_{nj}) = f(t_{nj}, u_n(t_{nj})) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{nj}-s)^{-\alpha} k(t_{nj}, s, u_i(s))ds,$$
$$n = 0, \cdots, N-1. \tag{1.9}$$

Note that if the collocation parameters are chosen so that $c_1 = 0$ and $c_m = 1$, then the approximation u lies in the smoother spline space

$$S_{m+1}^{(1)}(Z_N) \cap C^2(I) = S_{m+1}^{(2)}(Z_N).$$

We now rewrite (1.8) in a form which is more amenable to numerical computations. Since u_n'' is a polynomial of degree at most $m-1$, we may write

$$u_n''(t_n + sh_n) = \sum_{j=1}^{m} L_j(s) Y_{nj}, \tag{1.10a}$$

where

$$Y_{nj} := u_n''(t_{nj}), \quad L_j(s) := \prod_{k \neq j}^{m} (s-c_k)/(c_j - c_k).$$

It follows that

$$u_n'(t_n + sh_n) = z_n + h_n \sum_{j=1}^{m} a_j(s) Y_{nj}, \tag{1.10b}$$

with

$$z_n := u_n'(t_n), \quad a_j(s) := \int_0^s L_j(z)dz;$$

and

$$u_n(t_n + sh_n) = y_n + h_n s z_n + h_n^2 \sum_{j=1}^{M} b_j(s) Y_{nj}, \tag{1.10c}$$

with

$$y_n := u_n(t_n), \quad b_j(s) := \int_0^s (s-z) L_j(z)dz.$$

With this notation the collocation equation (1.8) assumes the form

$$Y_{nj} = F(u;t_{nj}) + h_n^{1-\alpha} \int_0^{c_j} (c_j - s)^{-\alpha} k(t_{nj}, t_n + sh_n, y_n + h_n s z_n + h_n^2 \sum_{k=1}^{m} b_k(s) Y_{nk}) ds, \tag{1.11a}$$

with

$$F(u;t_{nj}) := f(t_{nj}; y_n + h_n c_j z_n + h_n^2 \sum_{k=1}^{m} b_k(c_j) Y_{nk})$$
$$+ \sum_{i=0}^{n-1} h_i^{1-\alpha} \int_0^1 \left[\frac{t_{nj} - t_i}{h_i} - s\right]^{-\alpha} k(t_{nj}, t_i + sh_i, y_i + h_i s z_i + h_i^2 \sum_{k=1}^{m} b_k(s) Y_{ik}) ds. \tag{1.11b}$$

At $t = t_{n+1}$ the values of u and its derivatives are given by

$$y_{n+1} := u_n(t_{n+1}) = y_n + h_n z_n + h_n^2 \sum_{k=1}^{m} b_k(1) Y_{nk}, \tag{1.12a}$$

$$z_{n+1} := u_n'(t_{n+1}) = z_n + h_n \sum_{k=1}^{m} a_k(1) Y_{nk}, \tag{1.12b}$$

and

$$u_n''(t_{n+1}) = \sum_{k=1}^{m} L_k(1) Y_{nk}. \tag{1.12c}$$

For each $n = 0, \cdots, N-1$, equation (1.11) is a system of m nonlinear equations for $Y_n := (Y_{n1}, \cdots, Y_{nm})^T$. Once Y_n has been computed, the approximation spline $u \in S_{m+1}^{(1)}(Z_N)$ and its derivatives u' and u'' are completely determined on the subinterval σ_n by (1.10).

We note in passing that Brunner has presented in [6] an analysis of the convergence properties of collocation approximations in $S_{m-1}^{(-1)}(Z_N)$ to the solution of the Abel-type Volterra integral equations of the second kind, and, in [12], an analysis of those in $S_m^{(0)}(Z_N)$ to the Abel-type Volterra integro-differential equations of the first order. A survey of collocation methods for Volterra integral and integro-differential equations with weakly singular kernels, as well as additional references, may also be found in [3], [5] and [7].

In this paper we carry out an study of the convergence properties of collocation approximations in $S_{m+1}^{(1)}(Z_N)$ to (1.1) and (1.2), both for quasi-uniform sequences of meshes and for graded meshes. Moreover, we extend this analysis to the fully discretized version of the collocation equation (1.8) in which the integrals have been approximated by appropriate quadrature processes.

§2. Preliminary Results

Consider the scalar second-order linear differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = r(t), \quad t \in I, \tag{2.1}$$

where p, q and r are continuous on I. We have the following results[2].

Lemma 2.1. *Let $f_1(t), f_2(t)$ be linearly independent solutions of*

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0, \quad t \in I. \tag{2.2}$$

Then the function

$$u_1(t) = \int_0^t Q_1(t,s) r(s) ds, \quad t \in I, \tag{2.3a}$$

is the (unique) solution of (2.1) satisfying the initial conditions $y(0) = 0, y'(0) = 0$, where

$$Q_1(t,s) := W_1(t,s)/W_2(s), \tag{2.3b}$$

with

$$W_1(t,s) := f_1(s)f_2(t) - f_2(s)f_1(t), \tag{2.3c}$$

$$W_2(s) := f_1(s)f_2'(s) - f_2(s)f_1'(s). \tag{2.3d}$$

Moreover, every solution of (2.1) *with* $y(0) = y_0, y'(0) = z_0$ *has the form*

$$y(t) = y_1(t) + u_1(t), \quad t \in I. \tag{2.4}$$

Here $y_1(t)$ *is the unique solution of* (2.2) *satisfying* $y_1(0) = y_0$ *and* $y_1'(0) = z_0$.

Note that f_1 and f_2 are linearly independent solutions of (2.2). We have $W_2(t) \neq 0$ for all $t \in I$.

Now we turn our attention to the linear counterpart of (1.1)

$$y''(t) = p(t)y(t) + q(t) + \int_0^t (t-s)^{-\alpha}k(t,s)y(s)ds, \quad 0 < \alpha < 1, \quad t \in I, \tag{2.5}$$

with initial conditions $y(0) = y_0$ and $y'(0) = z_0$.

Lemma 2.2.[4]. *Consider second kind Abel-type Volterra equation*

$$y(t) = g(t) + \int_0^t (t-s)^{-\alpha}K(t,s)y(s)ds, \quad 0 < \alpha < 1, \quad t \in I. \tag{2.6}$$

If $K \in C^m(\hat{S}), g \in C^m(I)$, *with* $m \geq 1$, *then the unique solution of* (2.6) *has the form*

$$y(t) = g(t) + \sum_{n=1}^{\infty} \psi_n(t)t^{n(1-\alpha)}, \quad t \in I. \tag{2.7}$$

The functions $\{\psi_n\}$ *satisfy* $\psi_n \in C^m(I)$.

The proof of the above result can be found in [7].

Lemma 2.3. *If function* $f \in C^m(I) \cap C^{m+2}(0,T]$, *then* f *can be expressed in the form* $f = f_1 + f_2$ *such that* $f_1 \in C^{m+2}(I)$ *and* $t^{3-\alpha}f_2 \in C^{m+2}(I)$.

Proof. If $m = 0$, letting $g_1 = t^{2-\alpha}f$, we have $g_1'(0) = 0$ from the definition of differentiation. Furthermore, since $g_1 \in C^2(0,T]$, we have $g_1 \in C(I)$. Considering the function $g_2 := tg_1$ we have

$$g_2'(t) = g_1(t) + tg_1'(t).$$

Again from the definition of differentiation we have $g_2''(0) = 2g_1'(0) = 0$. Hence, $t^{3-\alpha}f \in C^2(I)$. In the case of $m \geq 1$, function f can be expressed in the form $f(t) = f_1(t) + f_2(t)$ with

$$f_1(t) = \sum_{j=0}^{m-1} f^{(j)}(0)t^j/j!, \tag{2.8a}$$

$$f_2(t) = \int_0^t (t-z)^{m-1}f^{(m)}(z)dz/(m-1)! \tag{2.8b}$$

By use of an analogous discussion to the case $m = 0$, we can show that $t^{3-\alpha}f_2$ is $m+2$ times differentiable at $t = 0$. Thus $t^{3-\alpha}f \in C^{m+2}(I)$. The proof of the Lemma is thereby complete.

Theorem 2.4. *Let p, q, K in (2.5) satisfy $p, q \in C^m(I), K \in C^m(S)$, with $m \geq 1$. Then the (unique) solution of the initial-value problem (2.5) has the form*

$$y(t) = v_0(t) + \sum_{n=1}^{\infty} v_n(t) t^{n(3-\alpha)} \tag{2.9}$$

(uniformly on I), where the function v_n satisfies $v_n \in C^{m+2}(I)$ for all $n \geq 0$ and $\alpha \in (0,1)$.

Proof. Let f_1, f_2 be linearly independent solutions of

$$y''(t) = p(t)y, \quad t \in I, \tag{2.10}$$

and y_1 be the solution of (2.10) with initial conditions $y_1(0) = y_0, y_1'(0) = z_0$.

Consider the following equation

$$y''(t) = p(t)y(t) + q(t) + r(t), \quad t \in I, \tag{2.11a}$$

with $y(0) = y_0$, $y'(0) = z_0$, where $r(t)$ is of the form

$$r(t) := \int_0^t (t-s)^{-\alpha} K(t,s) y(s) ds. \tag{2.11b}$$

Then using Lemma 2.1, we have

$$y(t) = y_1(t) + \int_0^t Q_1(t,s) q(s) ds + \int_0^t Q_1(t,s) r(s) ds, \quad t \in I, \tag{2.12}$$

where the function $Q_1(t,s)$ is defined in (2.3b), (2.3c) and (2.3d). By observing $Q_1(t,t) = 0$, we have $Q_1(t,s) = (t-s)Q_2(t,s)$, where

$$Q_2(t,s) = \int_0^1 \frac{\partial Q_1(u,s)}{\partial^u}\bigg|_{u=s+z(t-s)} dz. \tag{2.13}$$

By applying the Dirichlet lemma to double integrals, we have

$$y(t) = q_1(t) + \int_0^t Q_3(t,s) y(s) ds, \quad t \in I, \tag{2.14}$$

where

$$q_1(t) = y_1(t) + \int_0^t Q_1(t,s) q(s) ds, \tag{2.15a}$$

$$Q_3(t,s) = \int_0^t (u-s)^{-\alpha} Q_1(t,u) K(u,s) du. \tag{2.15b}$$

Let

$$Q_4(t,s) = \int_0^1 z^{-\alpha} Q_2(t, s+z(t-s)) K(s+z(t-s), s) dz, \tag{2.16}$$

where Q_2 is defined by (2.13). Hence we obtain

$$Q_3(t,s) = (t-s)^{2-\alpha} Q_4(t,s), \tag{2.17}$$

and

$$y(t) = q_1(t) + \int_0^t (t-s)^{2-\alpha} Q_4(t,s) y(s) ds. \tag{2.18}$$

It follows from Lemma 2.2 that the solution of (2.5) with $y(0) = y_0$, $y'(0) = z_0$ can be expressed in the form

$$y(t) = q_1(t) + \sum_{n=1}^{\infty} \psi_n(t) t^{n(3-\alpha)}, \quad t \in I. \tag{2.19}$$

By the hypotheses for p, q and K, we have $q_1 \in C^{m+2}(I)$ and $Q_4 \in C^m(I)$. Hence we may obtain that $\psi_n \in C^m(I)$. Now we shall prove that $\psi_n \in C^{m+2}(0,T], n \geq 1$. Let

$$\bar{y}_j := q_1(t) + \sum_{n=1}^{j} \psi_n(t) t^{n(3-\alpha)}, \quad j \geq 1, \quad t \in I.$$

Then from the proof of Lemma 2.2 (cf. [7] pp. 11-12, pp. 29–31), we have for $j \geq 1$

$$\bar{y}_j = q_1(t) + \int_0^t (t-s)^{2-\alpha} Q_4(t,s) \bar{y}_{j-1}(s) ds, \quad \bar{y}_0(t) := q_1(t), \quad t \in I. \tag{2.20}$$

From (2.20), it is easy to find that $\bar{y}_j$ satisfies

$$\bar{y}_j''(t) = p(t)\bar{y}_j(t) + q(t) + \int_0^t (t-s)^{-\alpha} K(t,s) \bar{y}_{j-1}(s) ds, \quad t \in I. \tag{2.21}$$

It now follows from $p, q \in C^m(I), K \in C^m(S)$ and $\bar{y}_0 \in C^{m+2}(I)$ that $\bar{y}_j \in C^{m+2}(0,T]$, $j = 1, 2, \cdots$, for all values of $\alpha \in (0,1)$. Then we have $\psi_n \in C^{m+2}(0,T]$, $n = 1, 2, \cdots$. An application of Lemma 2.3 to (2.19) leads to the expression

$$y(t) = v_0(t) + \sum_{n=1}^{\infty} v_n(t) t^{n(3-\alpha)}, \tag{2.22}$$

where the function v_n satisfies $v_n \in C^{m+2}(I)$ for all $n \geq 0$. The proof of Theorem 2.4 is thereby complete.

The result of Theorem 2.4 shows that for Eq. (2.5) smooth, p, q and K lead, for $0 < \alpha < 1$, to an exact solution y which behaves like $y(t) = O(t^{3-\alpha})$ near $t = 0$; it thus has unbounded derivatives $y^{(n)}(t) (n \geq 3)$ at $t = 0$.

It is an easy matter to establish the following result (cf. [4], [7]).

Corollary 2.5. Let the assumptions of Theorem 2.3 hold, and assume that $\alpha = p/q$ (with p and q coprime). Then the solution of (2.5) can be expressed in the form

$$y(t) = w_0(t) + \sum_{n=1}^{q-1} w_n(t) t^{n(3-\alpha)}, \quad t \in I, \tag{2.23}$$

with $w_n \in C^{m+2}(I), n = 1, \cdots, q-1$.

§3. The Attainable Order of Convergence

In this section we state the results on the attainable order of convergence of the collocation approximation $u \in S_{m+1}^{(1)}(Z_N)$ with respect to quasiuniform sequences of meshes and graded meshes, assuming that the integrals occurring in (1.8) and (1.9) are known exactly. The fully discretized collocation equation will be investigated in Section 5.

A sequence of meshes for the interval I is said to be quasi-uniform if there exists a finite constant C_1 such that, for all N,

$$h/h' \leq C_1 \tag{3.1}$$

holds (recall the notation introduced in (1.4)). It is easily seen that such a mesh sequence has the property

$$h_n \leq h \leq C_1 T N^{-1}, \quad 0 \leq n \leq N-1; \tag{3.2}$$

hence $h = O(N^{-1})$ for any compact interval I.

Theorem 3.1. *Let the functions p, q and K in (1.2) satisfy $p, q \in C^m(I)$, $K \in C^m(S)$, with $m \geq 1$. If $u \in S_{m+1}^{(1)}(Z_N)$ is the collocation approximation defined by (1.8), and if y denotes the exact solution of (1.2), then*

$$\|y^{(k)} - u^{(k)}\|_\infty = O(N^{-(1-\alpha)}), \quad k = 0, 1, 2 \tag{3.3}$$

for any quasi-uniform mesh sequence and for all collocation parameters $\{c_j\}$ with $0 \leq c_1 < \cdots < c_m \leq 1$.

We observe that, for quasi-uniform meshes, the order of convergence of u is governed by the degree of the smoothness of the derivatives of the exact solution, not by that of the solution itself. This is, of course, not surprising since the collocation equation defining u involves also u'' and u'.

Consider now graded meshes of the form

$$t_n := (n/N)^r T, \quad 0 \leq n \leq N-1, \quad N \geq 2, \tag{3.4}$$

where the grading exponent r satisfies $r \geq 1$. For any such mesh we have $0 \leq h_0 = h' < h_1 < \cdots < h_{N-1} = h$, and, in analogy to (3.2),

$$h_n \leq h \leq rTN^{-1}, \quad 0 \leq n \leq N-1. \tag{3.5}$$

Thus the mesh diameters of a sequence of graded meshes of the form (3.4) behave like $h = O(N^{-1})$ on compact intervals.

Theorem 3.2. *Let the functions p, q and K in (1.2) satisfy the assumptions stated in Theorem 2.4. If $u \in S_{m+1}^{(1)}(Z_N)$ is the collocation approximation defined by (1.8), and if y denotes the exact solution of (1.2), then*

$$\|y^{(k)} - u^{(k)}\|_\infty = O(N^{-m}), \quad k = 0, 1, 2, \tag{3.6}$$

provided we employ the sequence of graded meshes (3.4) corresponding to the grading exponent

$$r = m/(1-\alpha). \tag{3.7}$$

This holds for all collocation parameters $\{c_j\}$ with $0 \leq c_1 < \cdots < c_m \leq 1$.

This result tells us that the choice (3.7) for the grading exponent leads to optimal (globle) convergence, in the sense that the exponent m in (3.6) cannot be replaced by $m+1$.

§4. Proofs of Theorem 3.1 and 3.2

We shall follow the techniques proposed in [7] and [10] to give our proofs of Theorems 3.1 and 3.2. Let

$$Y(t) := (y(t), y'(t))^T, \tag{4.1a}$$

$$P(t) := \begin{bmatrix} 0 & 1 \\ p(t) & 0 \end{bmatrix}, \tag{4.1b}$$

$$Q(t) := (0, q(t))^T, \tag{4.1c}$$

and

$$H(t,s) := \begin{bmatrix} 0 & 0 \\ 0 & K(t,s) \end{bmatrix}, \tag{4.1d}$$

where p, q and K occurring in (1.2) satisfy the conditions stated in Theorem 2.4. Then Eq. (2.1) is equivalent to

$$Y'(t) = P(t)Y(t) + Q(t) + \int_0^t (t-s)^{-\alpha} H(t,s)Y(s)ds, \quad t \in I, \quad 0 < \alpha < 1, \tag{4.2}$$

with initial condition $Y(0) = (y_0, z_0)^T$. Let $U(t) = (u(t), u'(t))^T$. It follows from (4.2) (with $t = t_{nj}$) and from the linear counterpart of (1.8) that

$$\begin{aligned} e_n'(t_{nj}) &= P(t_{nj})e_n(t_{nj}) + h_n^{1-\alpha} \int_0^{c_j} (c_j - s)^{-\alpha} H(t_{nj}, t_n + sh_n) e_n(t_n + sh_n) ds \\ &+ \sum_{i=0}^{n-1} h_i^{1-\alpha} \int_0^1 \left[\frac{t_{nj} - t_i}{h_i} - s\right]^{-\alpha} H(t_{nj}, t_i + sh_i) e_i(t_i + sh_i) ds, \\ &\qquad j = 1, \cdots, m; \quad n = 0, \cdots, N-1, \end{aligned} \tag{4.3}$$

with $e_0(t_0) = (0,0)^T$, where $e(t) := Y(t) - U(t)$, and e_n denotes again the restriction of e to the subinterval σ_n.

For ease of notation we assume, without loss of generality, that α is rational : $\alpha = p/q$, with p and q coprime. Hence, according to Corollary 2.5, the solution of (1.2) has the form

$$y(t) = \sum_{n=0}^{q-1} w_n(t) t^{n(3-\alpha)}, \quad w_n \in C^{m+2}(I), \quad 0 \le n \le q-1. \tag{4.4}$$

On the first subinterval σ_0 we have, in complete analogy to [6] and [7],

$$y(t_0 + sh_0) = \sum_{k=0}^{m+1} c_{0k} s^k + h_0^{3-\alpha} C_0(s) + h_0^{m+2} R_0(s), \tag{4.5}$$

where we have

$$c_{0k} := \sum_{r=0}^{q-1} h_0^{(3-\alpha)} c_{0k}^{(r)}, \quad R_0(s) := \sum_{r=0}^{q-1} h_0^{(3-\alpha)} R_{0r}(s) s^{r(3-\alpha)},$$

and

$$C_0(s) := \sum_{r=1}^{q-1} h_0^{(r-1)(3-\alpha)}\left(s^{r(3-\alpha)} - 1\right) \sum_{k=0}^{m+1} c_{0k}^{(r)} s^k.$$

This follows from the application of Taylor's formula to the functions $w_r(t_0 + sh_0)$ in the expression for $y(t_0 + sh_0)$:

$$w_r(t_0 + sh_0) = \sum_{k=0}^{m+1} c_{0k}^{(r)} s^k + h_0^{m+2} R_{0r}(s),$$

with

$$c_{0k}^{(r)} := w_r^{(k)}(t_0) h_0^k / k!,$$

and

$$R_{0r}(s) := \int_0^s (s-z)^{m+1} w_r^{(m+2)}(t_0 + zh_0)\, dz/(m+1)!, \quad r = 0, \cdots, q-1.$$

If $n \geq 1$, then $t_n > 0$, and we may write

$$y(t_n + sh_n) = \sum_{k=0}^{m+1} c_{nk} s^k + h_n^{m+2} R_n(s), \tag{4.6}$$

where

$$c_{nk} := y^{(k)}(t_n) h_n^k / k!$$

and

$$R_n(s) := \int_0^s (s-z)^{m+1} y^{(m+2)}(t_n + zh_n)\, dz/(m+1)!$$

The corresponding expressions for the first and second derivatives of the exact solution on the subinterval σ_0 and σ_n, $n = 1, \cdots, N-1$, are, respectively,

$$y'(t_0 + sh_0) = h_0^{-1}\Big(\sum_{k=1}^{m+1} kC_{0k} s^{k-1} + h_0^{3-\alpha} C_0'(s) + h_0^{m+2} R_0'(s) \Big) \tag{4.7}$$

$$y''(t_0 + sh_0) = h_0^{-2}\Big(\sum_{k=2}^{m+1} k(k-1) C_{0k} s^{k-2} + h_0^{3-\alpha} C_0''(s) + h_0^{m+2} R_0''(s) \Big), \tag{4.8}$$

and

$$y'(t_n + sh_n) = h_n^{-1}\Big(\sum_{k=1}^{m+1} kc_{nk} s^{k-1} + h_n^{m+1} R_n'(s) \Big), \tag{4.9}$$

$$y''(t_n + sh_n) = h_n^{-2}\Big(\sum_{k=2}^{m+1} k(k-1) c_{nk} s^{k-2} + h_n^{m+2} R_n''(s) \Big), \quad n = 1, \cdots, N-1. \tag{4.10}$$

Thus, setting

$$u_n(t_n + sh_n) = \sum_{k=0}^{m+1} \alpha_{nk} s^k, \quad s \in [0,1], \quad n = 0, \cdots, N-1,$$

we obtain

$$e_n(t_n+sh_n)=\begin{cases}\displaystyle\sum_{k=0}^{m+1}\bar{d}_{0k}s^k+h_0^{2-\alpha}\bar{C}_0(s)+h_0^{m+1}\bar{R}_0(s), & \text{if } n=0,\\ \displaystyle\sum_{k=0}^{m+1}\bar{d}_{nk}s^k+h_n^{m+1}\bar{R}_n(s), & \text{if } 1\le n\le N-1,\end{cases}\tag{4.11}$$

and

$$e_n'(t_n+sh_n)=\begin{cases}\displaystyle h_0^{-1}\Big(\sum_{k=1}^{m+1}k\bar{d}_{0k}s^{k-1}+h_0^{2-\alpha}\bar{C}_0'(s)+h_0^{m+1}\bar{R}_0'(s)\Big), & \text{if } n=0,\\ \displaystyle h_n^{-1}\Big(\sum_{k=1}^{m+1}k\bar{d}_{nk}s^{k-1}+h_n^{m+1}\bar{R}_n'(s)\Big), & \text{if } 1\le n\le N-1,\end{cases}\tag{4.12}$$

where

$$\bar{C}_0(s):=(h_0C_0(s),C_0'(s))^T,$$
$$\bar{R}_n(s):=(h_nR_n(s),R_n'(s))^T,$$
$$\bar{d}_{nk}:=\begin{cases}(c_{nk}-\alpha_{nk};(k+1)(c_{n,k+1}-\alpha_{n,k+1})/h_n)^T, & 0\le k\le m,\\ (c_{nk}-\alpha_{nk};0)^T, & k=m+1,\end{cases}$$

with $n=0,1,\cdots,N-1$. If we now employ (4.11) and (4.12) in the error equation (4.3), then we find

$$\begin{aligned}&\sum_{k=1}^{m+1}\Big\{kc_j^{k-1}I_2-h_nP(t_{nj})c_j^k-h_n^{2-\alpha}\int_0^{c_j}(c_j-s)^{-\alpha}H(t_{nj},t_n+sh_n)s^kds\Big\}\bar{d}_{nk}\\ &=h_n\Big\{P(t_{nj})+h_n^{1-\alpha}\int_0^{c_j}(c_j-s)^{-\alpha}H(t_{nj},t_n+sh_n)ds\Big\}\bar{d}_{n0}\\ &+h_n\sum_{i=0}^{n-1}h_i^{1-\alpha}\int_0^1\Big[\frac{t_{nj}-t_i}{h_i}-s\Big]^{-\alpha}H(t_{nj},t_i+sh_i)ds\bar{d}_{i0}\\ &+h_n\sum_{i=0}^{n-1}h_i^{1-\alpha}\sum_{k=1}^{m+1}\int_0^1\Big[\frac{t_{nj}-t_i}{h_i}-s\Big]^{-\alpha}H(t_{nj},t_i+sh_i)s^kds\bar{d}_{ik}+h_n\bar{r}_{nj},\end{aligned}\tag{4.13}$$

with

$$\begin{aligned}\bar{r}_{nj}:=&-h_n^m\bar{R}_n'(c_j)+h_n^{m+1}P(t_{nj})\bar{R}_n(c_j)\\ &+h_n^{1-\alpha}\int_0^{c_j}(c_j-s)^{-\alpha}H(t_{nj},t_n+sh_n)(h_n^{m+1}\bar{R}_n(s))ds\\ &+\sum_{i=1}^{n-1}h_i^{1-\alpha}\int_0^1\Big[\frac{t_{nj}-t_i}{h_i}-s\Big]^{-\alpha}H(t_{nj},t_i+sh_i)(h_i^{m+1}\bar{R}_i(s))ds\\ &+h_0^{1-\alpha}\int_0^1\Big[\frac{t_{nj}-t_0}{h_0}-s\Big]^{-\alpha}H(t_{nj},t_0+sh_0)(h_0^{2-\alpha}\bar{C}_0(s)+h_0^{m+1}\bar{R}_0(s))ds.\end{aligned}\tag{4.14}$$

On the first subinterval, (4.13) and (4.14) become, respectively,

$$\sum_{k=1}^{m}\left\{kc_i^{k-1}I_2 - h_0P(t_{0j})c_j^k - h_0^{2-\alpha}\int_0^{c_j}(c_j-s)^{-\alpha}H(t_{0j},t_0+sh_0)s^k ds\right\}\bar{d}_{0k}$$
$$= \left\{h_0P(t_{0j}) + h_0^{2-\alpha}\int_0^{c_j}(c_j-s)^{-\alpha}H(t_{0j},t_0+sh_0)ds\right\}\bar{d}_{00} + h_0\bar{r}_{0j}, \tag{4.15}$$

and

$$\bar{r}_{0j} := h_0^{1-\alpha}\Big\{-\bar{C}_0'(c_j) - h_0^{m+\alpha-1}\bar{R}_0'(c_j) + h_0P(t_{0j})(\bar{C}_0(c_j) + h_0^{m+\alpha-1}\bar{R}_0(c_j))$$
$$+\int_0^{c_j}(c_j-s)^{-\alpha}H(t_{0j},t_0+sh_0)(h_0^{2-\alpha}\bar{C}_0(s) + h_0^{m+1}\bar{R}_0(s))ds\Big\}, \tag{4.16}$$

where I_2 on the left-hand sides of (4.13) and (4.15) is the 2×2 identity matrix. We first note that the matrix defined by the coefficients of $\{\bar{d}_{nk}\}$ on the left-hand side of (4.13) (or (4.15)) is invertible whenever $h>0$ is sufficiently small; this follows from the fact $P(t)$ and $H(t,s)$ are continuous, and the observation that, for $h=0$ (recall (1.4) for h), the determinant of this matrix is $(m!\prod_{i>j}(c_i-c_j))^2$. Furthermore, in complete analogy to the technique used by [7, pp. 381–387] and [10], the results stated in Theorems 3.1 and 3.2 can be immediately established. The proofs of Theorems 3.1 and 3.2 are thereby complete.

So far it has been assumed that α is rational. If α is irrational, then, the solution of (1.2) corresponding to functions $p.q$ and K satisfying the hypotheses of Theorem 2.4 is of the form (2.9), where the infinite series converges absolutely and uniformly on I; this also holds for (1.1), provided that the given functions f and k are subject to appropriate smoothness and boundedness conditions (cf. [11]). Hence, the above proofs are readily adapted to deal with this general situation; the key observation is the uniform convergence of the infinite series in (2.9) which implies, for example, that $y^{(m+2)}(t)(t>0)$ can be obtained by termwise differentiation of the right-hand side in (2.9).

§5. Discretization of the Collocation Equation

Until now it has been assumed that the integrals

$$\Phi_{ni}^{(j)}[u_i] := \begin{cases}\displaystyle\int_0^1\left[\frac{t_{nj}-t_i}{h_i}-s\right]^{-\alpha}k(t_{nj},t_i+sh_i,u_i(t_i+sh_i))ds, & 0\le i\le n-1,\\ \displaystyle\int_0^{c_j}(c_i-s)^{-\alpha}k(t_{nj},t_n+sh_n,u_n(t_n+sh_n))ds, & i=n,\quad j=0,\cdots,m\end{cases} \tag{5.1}$$

occurring in the collocation equation (1.8) are known exactly, i.e., the collocation approximation $u\in S_{m+1}^{(1)}(Z_N)$ is obtained from what we shall refer to as the exact collocation equation

$$u_n''(t_{nj}) = f(t_{nj},u_n(t_{nj})) + h_n^{1-\alpha}\Phi_{nn}^{(j)}[u_n] + \sum_{i=0}^{n-1}h_i^{1-\alpha}\Phi_{ni}^{(j)}[u_i],\quad 1\le j\le m,\ 0\le n\le N-1. \tag{5.2}$$

In most cases the integrals $\Phi_{ni}^{(j)}[u_i]$ cannot be found analytically but have to be approximated by suitable quadrature formulas. Denoting these quadrature approximations by $\hat{\Phi}_{ni}^{(j)}[u_i]$, the resulting discrete version of (1.8),

$$\hat{u}_n''(t_{nj}) = f(t_{nj}, \hat{u}_n(t_{nj})) + h_n^{1-\alpha}\hat{\Phi}_{nn}^{(j)}[\hat{u}_n] + \sum_{i=0}^{n-1} h_i^{1-\alpha}\hat{\Phi}_{ni}^{(j)}[\hat{u}_i], \quad 1 \le j \le m, \quad 0 \le n \le N-1, \tag{5.3}$$

with $\hat{u}_0(t_0) = y_0, \hat{u}_0'(t_0) = z_0$, defines for all sufficiently small mesh diameters h a polynomial spline $\hat{u} \in S_{m+1}^{(1)}(Z_N)$, which, due to the errors induced by numerical integration, will generally be different from the exact collocation approximation $u \in S_{m+1}^{(1)}(Z_n)$.

In the following we restrict our analysis to the case where the approximations $\hat{\Phi}_{ni}^{(j)}[\hat{u}_i]$ are of the form

$$\hat{\Phi}_{ni}^{(j)}[u_i] := \begin{cases} \sum_{k=1}^{m} w_{jk}^{(ni)} k(t_{nj}, t_{ik}, u_i(t_{ik})), & \text{if } i \ne n, \\ \sum_{k=1}^{m} w_{jk} k(t_{nj}, t_n + c_j c_k h_n, u_n(t_n + c_j c_k h_n)), & \text{if } i = n, \end{cases} \tag{5.4}$$

where

$$w_{jk}^{(ni)} := \int_0^1 \left[\frac{t_{nj} - t_i}{h_i} - s\right]^{-\alpha} L_k(s)ds, \quad \text{if } i \ne n, \tag{5.5a}$$

and

$$w_{jk} := c_j^{1-\alpha} \int_0^1 (1-s)^{-\alpha} L_k(s)ds, \quad j, k = 1, \cdots, m, \tag{5.5b}$$

with

$$L_k(s) := \prod_{\substack{j=1 \\ j \ne k}}^{m} (s - c_j)/(c_k - c_j). \tag{5.5c}$$

Setting

$$a_k(s) := \int_0^s L_k(z)dz,$$

$$b_k(s) := \int_0^s (s-z)L_k(z)dz,$$

we have, as in (1.10b) and (1.10c),

$$\hat{u}_n'(t_n + sh_n) = \hat{z}_n + h_n \sum_{k=1}^{m} a_k(s)\hat{Y}_{nk}, \quad s \in [0,1], \tag{5.6}$$

$$\hat{u}_n(t_n + sh_n) = \hat{y}_n + h_n s \hat{z}_n + h_n^2 \sum_{k=1}^{m} b_k(s)\hat{Y}_{nk}, \quad s \in [0,1], \tag{5.7}$$

with $\hat{z}_n := \hat{u}_n'(t_n), \hat{y}_n := \hat{u}_n(t_n)$ and with $\hat{Y}_{nk} := \hat{u}_n''(t_{nk})$. Thus, (5.3) represents, for each $n = 0, \cdots, N-1$, a system of m nonlinear equations for $\hat{Y}_n := (\hat{Y}_{n1}, \cdots, \hat{Y}_{nm})^T$. Once these values have been found, the approximating spline $\hat{u} \in S_{m+1}^{(1)}(Z_N)$ on σ_n is given by (5.6) and (5.7).

Theorem 5.1. *Let p, q, and K in (1.2) satisfy the assumptions stated in Theorem 2.4. Moreover, let $\hat{u} \in S_{m+1}^{(1)}(Z_N)$ denote the solution of the fully discretized collocation equation (5.3), with quadrature approximations $\hat{\Phi}_{ni}^{(j)}[u_i]$ given by (5.4) and (5.5). Then*

(i) If the underlying mesh sequence $\{\Pi_N\}$ is quasi-uniform, then the error $\hat{e} := y - \hat{u}$ satisfies

$$\|y^{(k)} - \hat{u}^{(k)}\|_\infty = O(N^{-(1-\alpha)}), \quad k = 0, 1, 2. \tag{5.8}$$

(ii) If $\{\Pi_N^{(r)}\}$ is the sequence of graded meshes (3.4), and if the grading exponent is given by $r = m/(1-\alpha)$, then we have

$$\|y^{(k)} - \hat{u}^{(k)}\|_\infty = O(N^{-m}), \quad k = 0, 1, 2. \tag{5.9}$$

These convergence results hold for all collocation parameters $\{c_j\}$ with $0 \le c_1 < \cdots < c_m \le 1$.

The above results are derived without difficulty along the lines of [7].

References

[1] M. Aguilar and H. Brunner, Collocation methods for second-order Volterra integro-differential equations, Private communication and to appear in *Appl. Numer. Math.*, 1986.

[2] R. Brauer and J.A. Nohel, Ordinary Differential Equations, A First Course, 2nd, ed. Benjamin, New York, 1973.

[3] H. Brunner, The application of the variation of constants formulas in the numerical analysis of integral and integro-differential equations, *Utilitas Math.*, **19** (1981), 255–290.

[4] H. Brunner, Nonpolynomial spline collocation for Volterra equations with weakly singular kernels, *SIAM J. Numer. Anal.*, **20** (1983), 1106–1119.

[5] H. Brunner, The numerical solution of integral equations with weakly singular kernels. In: Numerical Analysis (D.F. Griffiths, ed.), *Lecture Notes in Math.*, Springer Berlin, Heidelberg, New York, 1984, 1066.

[6] H. Brunner, The Numerical solution of weakly singular Volterra equations by collocation on graded meshes, *Math. Comp.*, **45** (1985), 417–437.

[7] H. Brunner and P.J. Houwen van der, The Numerical Solution of Volterra Equations, North Holland, Amsterdam, New York, 1986.

[8] A. Prosperetti, A numerical method for the solution of certain class of nonlinear Volterra integro-differential and integral equations, *Int. J. Numer. Methods Engrg.*, **11** (1977), 431–438.

[9] A. Prosperetti, A method for the solution of singular Volterra integro-differential equations, *J. Comput. Phys.*, **46** (1982), 462–468.

[10] Tang Tao, On the collocation methods for high-order Volterra integro-differential equations, Preprint, 1987.

[11] Ch. Lubich, Runge-Kutta theory for Volterra and Abel integral equations of the second kind, *Math. Comp.*, **41** (1983), 87–102.

[12] H. Brunner, Polynomial spline collocation methods for Volterra integro-differential equations with weakly singular kernels, *IMA J. Numer. Anal.*, **6** (1986), 221–239.