A SPECTRAL METHOD FOR A CLASS OF SYSTEM OF MULTI–DIMENSIONAL NONLINEAR WAVE EQUATIONS *

XIANG XIN–MIN
(*Heilongjiang University, Harbin, China*)

In [1,2], the problem of three–dimensional soliton of a class of system for three–dimensional nonlinear wave equations was investigated, and the existence and stability of three–dimensional soliton was proved. In [3] the system discussed in [1,2] was generalized and a more general class of system of multi–dimensional nonlinear wave equations were studied. It was proved that the solution of its initial–boundary value problem was well posed under some conditions. This system has been studied by the finite difference method and the finite element method [4,5]. In this paper, we take the trigonometric functions as a basis to derive a spectral method for the system and give a strict error analysis in theory.

1. Notations and Statement of the Problem

We consider the periodic initial value problem of a system of nonlinear wave equations

$$\Box\varphi + \mu^2\varphi + \nu^2\chi^2\varphi + f(|\varphi|^2)\varphi = 0, \quad (x,t)\in\Omega\times(0,T], \tag{1.1}$$

$$\Box\chi + \delta^2\chi + \nu^2\chi|\varphi|^2 + h(\chi) = 0, \quad (x,t)\in\Omega\times(0,T], \tag{1.2}$$

with initial conditions

$$\varphi|_{t=0} = \varphi_0(x), \frac{\partial\varphi}{\partial t}|_{t=0} = \varphi_1(x), \quad \chi|_{t=0} = \chi_0(x), \frac{\partial\chi}{\partial t}|_{t=0} = \chi_1(x), x\in\Omega, \tag{1.3}$$

where $\Omega = [-\pi,\pi]^n, \Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$, and $\varphi(x,t), \chi(x,t)$ are unknown complex and real periodic functions respectively. $\varphi_o(x), \varphi_1(x), \chi_0(x), \chi_1(x)$ are known complex and real valued periodic functions respectively. All of them have the period 2π for $x_s, 1\le s\le n, n\le 3$. $f(s)$ and $h(s)$ are known real functions. μ,ν,δ are real constants.

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Let I denote the integers. $l=(l_1,\cdots,l_n)\in I^n, |\,l\,|=\max\limits_{1\le i\le n}|l_i|$ and

$$S_N=\mathrm{Span}\Big\{\psi_l=e^{il\cdot x}|l\in I^n, |\,l\,|\le N\Big\}.$$

Put $u(x,t)=\sum\limits_{l\in I^n}u_l(t)e^{il\cdot x}$ and $u^{(N)}(x,t)=\sum\limits_{|l|\le N}u_l(t)e^{ix\cdot l}, R^{(n)}(u(x,t))=u(x,t)-u^{(N)}(x,t)$. Define $(u,v)=\int_\Omega u\bar{v}dx$. Let $H_p^s(\Omega)$ denote the Sth-order Sobolev spaces of real or complex valued periodic functions with the norm $\|\cdot\|_s$. For $r\ge1$, we denote $\|u\|_{L^r}=(\int_\Omega|u|^rdx)^{1/r}$ and $\|u\|_{L^2}=\|u\|$.

Let τ be the mesh spacing in time and

$$u_t(x,t)=\frac{1}{\tau}(u(x,t+\tau)-u(x,t)),\quad u_{\bar{t}}(x,t)=\frac{1}{\tau}(u(x,t)-u(x,t-\tau)).$$

In this paper, we assume

(1) $f,h\in C^2$, and $|f(s)|\le A_0s, |f'(s)|\le A_1, s\ge0; |h(s)|\le B_0|s|^3, |h'(s)|\le B_1s^2$, where $A_i,B_i,\ i=0,1$, are positive constants;

(2) $F(s)=\int_0^s f(z)dz,\ \ H(s)=\int_0^s h(z)dz,\ \ F(s)\ge0, s\ge0,\ \ H(s)\ge0, s\in(-\infty,+\infty)$;

(3) system (1.1)–(1.3) has solutions, which and the initial data are properly smooth. For (1.1)–(1.3) we construct the following conservative fully discrete spectral scheme

$$\begin{aligned}&(\Phi_{t\bar{t}}^{(N)}(t),\psi_j)+\tfrac{1}{2}(\nabla(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\nabla\psi_j)+\tfrac{\mu^2}{2}(\Phi^{(N)}(t+\tau)\\&\quad+\Phi^{(N)}(t-\tau),\psi_j)+\tfrac{\nu^2}{2}((\Sigma^{(N)}(t))^2(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\psi_j)\\&\quad+\tfrac{1}{2}(F(|\Phi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\psi_j)=0,\end{aligned}\tag{1.4}$$

$$(\Phi^{(N)}(0),\psi_j)=(\varphi_0(x),\psi_j),\tag{1.5}$$

$$\begin{aligned}&(\Sigma_{t\bar{t}}^{(N)}(t),\psi_j)+\tfrac{1}{2}(\nabla(\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau)),\nabla\psi_j)+\tfrac{\delta^2}{2}(\Sigma^{(N)}(t+\tau)\\&\quad+\Sigma^{(N)}(t-\tau),\psi_j)+\tfrac{\nu^2}{2}(|\Phi^{(N)}(t)|^2(\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau)),\psi_j)\\&\quad+(H(\Sigma^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau)),\psi_j)=0,\end{aligned}\tag{1.6}$$

$$\Big(\Sigma^{(N)}(0),\psi_j\Big)=(\chi_0(x),\psi_j),\quad |j|\le N,\tag{1.7}$$

where

$$F(z_1,z_2)=\begin{cases}\dfrac{1}{z_2-z_1}\displaystyle\int_{z_1}^{z_2}f(s)ds, & \text{if } z_1\ne z_2,\\ f(z_1), & \text{if } z_1=z_2,\end{cases}$$

$$H(z_1,z_2)=\begin{cases}\dfrac{1}{z_2-z_1}\displaystyle\int_{z_1}^{z_2}h(s)ds, & \text{if } z_1\neq z_2,\\ h(z_1), & \text{if } z_1=z_2.\end{cases}$$

Besides, we take $\Phi^{(N)}(\tau),\Sigma^{(N)}(\tau)$ properly in the following .

2. The Priori Estimations for the Solution of the Spectral Method

We estimate the solution of the spectral method. Put $\Phi^{(N)}(x,t)=\sum\limits_{|l|\le N}\Phi_l(t)\psi_l(x)$, $\Sigma^{(N)}(x,t)=\sum\limits_{|l|\le N}\sigma_l(t)\psi_l(x)$. Multiplying $\frac{1}{2\tau}[\Phi_l(t+\tau)-\Phi_l(t-\tau)]$ on both sides of equality (1.4) and summing for l, we have

$$\begin{aligned}&\tfrac{1}{2}(\Phi^{(N)}_{tt}(t),\Phi^{(N)}_t(t)+\Phi^{(N)}_t(t-\tau))+\tfrac{1}{4\tau}(\nabla(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\nabla(\Phi^{(N)}(t+\tau)\\&\quad-\Phi^{(N)}(t-\tau)))+\tfrac{\mu^2}{4\tau}(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau),\Phi^{(N)}(t+\tau)-\Phi^{(N)}(t-\tau))\\&\quad+\tfrac{\nu^2}{4\tau}((\Sigma^{(N)}(t))^2(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\Phi^{(N)}(t+\tau)-\Phi^{(N)}(t-\tau))\\&\quad+\tfrac{1}{4\tau}(F(|\Phi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\Phi^{(N)}(t+\tau)\\&\quad-\Phi^{(N)}(t-\tau))=0.\end{aligned}\tag{2.1}$$

Since

$$\begin{aligned}&\mathrm{Re}\tfrac{1}{2}(\Phi^{(N)}_{tt}(t),\Phi^{(N)}_t(t)+\Phi^{(N)}_t(t-\tau))=\tfrac{1}{2\tau}[\|\Phi^{(N)}_t(t)\|^2-\|\Phi^{(N)}_t(t-\tau)\|^2],\\&\mathrm{Re}\tfrac{1}{4\tau}(\nabla(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\nabla(\Phi^{(N)}(t+\tau)-\Phi^{(N)}(t-\tau)))\\&\quad=\tfrac{1}{4\tau}[(\|\nabla\Phi^{(N)}(t+\tau)\|^2+\|\nabla\Phi^{(N)}(t)\|^2)-(\|\nabla\Phi^{(N)}(t)\|^2+\|\nabla\Phi^{(N)}(t-\tau)\|^2)],\\&\mathrm{Re}\tfrac{\mu^2}{4\tau}(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau),\Phi^{(N)}(t+\tau)-\Phi^{(N)}(t-\tau))\\&\quad=\tfrac{\mu^2}{4\tau}[(\|\Phi^{(N)}(t+\tau)\|^2+\|\Phi^{(N)}(t)\|^2)-(\|\Phi^{(N)}(t)\|^2+\|\Phi^{(N)}(t-\tau)\|^2)],\\&\mathrm{Re}\tfrac{\nu^2}{4\tau}((\Sigma^{(N)}(t))^2(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\Phi^{(N)}(t+\tau)-\Phi^{(N)}(t-\tau))\\&\quad=\tfrac{\nu^2}{4\tau}\int_\Omega(\Sigma^{(N)}(t))^2[|\Phi^{(N)}(t+\tau)|^2-|\Phi^{(N)}(t-\tau)|^2]dx,\\&\mathrm{Re}\tfrac{1}{4\tau}(F(|\Phi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\\&\quad\Phi^{(N)}(t+\tau)-\Phi^{(N)}(t-\tau))=\tfrac{1}{4\tau}\int_\Omega[(F(|\Phi^{(N)}(t+\tau)|^2)+F(|\Phi^{(N)}(t)|^2))\\&\quad-(F(|\Phi^{(N)}(t)|^2)+F(|\Phi^{(N)}(t-\tau)|^2))]dx.\end{aligned}$$

Multiplying $\frac{1}{2\tau}[\sigma_l(t+\tau)-\sigma_l(t-\tau)]$ on both sides of equality (1.6) and summing for l, we have

$$\frac{1}{2}(\Sigma_{tt}^{(N)}(t),\Sigma_t^{(N)}(t)+\Sigma_t^{(N)}(t-\tau))+\frac{1}{4\tau}(\nabla(\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau)),\nabla(\Sigma^{(N)}(t+\tau)$$
$$-\Sigma^{(N)}(t-\tau)))+\frac{\delta^2}{4\tau}(\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau),\Sigma^{(N)}(t+\tau)-\Sigma^{(N)}(t-\tau))$$
$$+\frac{\nu^2}{4\tau}(|\Phi^{(N)}(t)|^2(\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau)),\Sigma^{(N)}(t+\tau)-\Sigma^{(N)}(t-\tau))$$
$$+\frac{1}{2\tau}(H(\Sigma^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau)),\Sigma^{(N)}(t+\tau)-\Sigma^{(N)}(t-\tau))=0. \tag{2.2}$$

Similarly,

$$\frac{1}{2}(\Sigma_{tt}^{(N)}(t),\Sigma_t^{(N)}(t)+\Sigma_t^{(N)}(t-\tau))=\frac{1}{2\tau}[\|\Sigma_t^{(N)}(t)\|^2-\|\Sigma_t^{(N)}(t-\tau)\|^2],$$
$$\frac{1}{4\tau}(\nabla(\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau)),\nabla(\Sigma^{(N)}(t+\tau)-\Sigma^{(N)}(t-\tau)))$$
$$=\frac{1}{4\tau}[(\|\nabla\Sigma^{(N)}(t+\tau)\|^2+\|\nabla\Sigma^{(N)}(t)\|^2)-(\|\nabla\Sigma^{(N)}(t)\|^2+\|\nabla\Sigma^{(N)}(t-\tau)\|^2)],$$
$$\frac{\delta^2}{4\tau}(\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau),\Sigma^{(N)}(t+\tau)-\Sigma^{(N)}(t-\tau))$$
$$=\frac{\delta^2}{4\tau}[(\|\Sigma^{(N)}(t+\tau)\|^2+\|\Sigma^{(N)}(t)\|^2)-(\|\Sigma^{(N)}(t)\|^2+\|\Sigma^{(N)}(t-\tau)\|^2)],$$
$$\frac{\nu^2}{4\tau}(|\Phi^{(N)}(t)|^2(\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau)),\Sigma^{(N)}(t+\tau)-\Sigma^{(N)}(t-\tau))$$
$$=\frac{\nu^2}{4\tau}\int_\Omega|\Phi^{(N)}(t)|^2[(\Sigma^{(N)}(t+\tau))^2-(\Sigma^{(N)}(t-\tau))^2]dx,$$
$$\frac{1}{2\tau}(H(\Sigma^{(N)}(t+\tau)),\Sigma^{(N)}(t-\tau),\Sigma^{(N)}(t+\tau)-\Sigma^{(N)}(t-\tau))$$
$$=\frac{1}{2\tau}\int_\Omega[(H(\Sigma^{(N)}(t+\tau))+H(\Sigma^{(N)}(t)))-(H(\Sigma^{(N)}(t))+H(\Sigma^{(N)}(t-\tau)))]dx.$$

Taking the real part in (2.1), substituting (2.1) and (2.2) with the above results and summing up , we get

$$\frac{1}{2\tau}(\|\Phi_t^{(N)}(t)\|^2+\|\Sigma_t^{(N)}(t)\|^2)+\frac{1}{4\tau}(\|\nabla\Phi^{(N)}(t+\tau)\|^2$$
$$+\|\nabla\Phi_{(t)}^{(N)}\|^2+\|\nabla\Sigma^{(N)}(t+\tau)\|^2+\|\nabla\Sigma^{(N)}(t)\|^2)$$
$$+\frac{\mu^2}{4\tau}(\|\Phi^{(N)}(t+\tau)\|^2+\|\Phi^{(N)}(t)\|^2)+\frac{\delta^2}{4\tau}(\|\Sigma^{(N)}(t+\tau)\|^2$$
$$+\|\Sigma^{(N)}(t)\|^2)+\frac{\nu^2}{4\tau}\int_\Omega[(\Sigma^{(N)}(t))^2|\Phi^{(N)}(t+\tau)|^2$$
$$+|\Phi^{(N)}(t)|^2(\Sigma^{(N)}(t+\tau))^2]dx+\frac{1}{4\tau}\int_\Omega[F(|\Phi^{(N)}(t+\tau)|^2)$$
$$+F(|\Phi^{(N)}(t)|^2)]dx+\frac{1}{2\tau}\int_\Omega[H(\Sigma^{(N)}(t+\tau))+H(\Sigma^{(N)}(t))]dx$$
$$=\frac{1}{2\tau}(\|\Phi_t^{(N)}(t-\tau)\|^2+\|\Sigma_t^{(N)}(t-\tau)\|^2)+\frac{1}{4\tau}(\|\nabla\Phi^{(N)}(t)\|^2$$
$$+\|\nabla\Phi^{(N)}(t-\tau)\|^2+\|\nabla\Sigma^{(N)}(t)\|^2+\|\nabla\Sigma^{(N)}(t-\tau)\|^2)$$
$$+\frac{\mu^2}{4\tau}(\|\Phi^{(N)}(t)\|^2+\|\Phi^{(N)}(t-\tau)\|^2)+\frac{\delta^2}{4\tau}(\|\Sigma^{(N)}(t)\|^2$$
$$+\|\Sigma^{(N)}(t-\tau)\|^2)+\frac{\nu^2}{4\tau}\int_\Omega[(\Sigma^{(N)}(t-\tau))^2|\Phi^{(N)}(t)|^2$$

$$+|\Phi^{(N)}(t-\tau)|^2(\Sigma^{(N)}(t))^2]dx + \frac{1}{4\tau}\int_\Omega [F(|\Phi^{(N)}(t)|^2)$$
$$+F(|\Phi^{(N)}(t-\tau)|^2)]dx + \frac{1}{2\tau}\int_\Omega [H(\Sigma^{(N)}(t)) + H(\Sigma^{(N)}(t-\tau))]dx$$
$$= \cdots = \frac{1}{2\tau}(\|\Phi_t^{(N)}(0)\|^2 + \|\Sigma_t^{(N)}(0)\|^2) + \frac{1}{4\tau}(\|\nabla\Phi^{(N)}(\tau)\|^2$$
$$+\|\nabla\Phi^{(N)}(0)\|^2 + \|\nabla\Sigma^{(N)}(\tau)\|^2 + \|\nabla\Sigma^{(N)}(0)\|^2)$$
$$+\frac{\mu^2}{4\tau}(\|\Phi^{(N)}(\tau)\|^2 + \|\Phi^{(N)}(0)\|^2) + \frac{\delta^2}{4\tau}(\|\Sigma^{(N)}(\tau)\|^2$$
$$+\|\Sigma^{(N)}(0)\|^2) + \frac{\nu^2}{4\tau}\int_\Omega [(\Sigma^{(N)}(0))^2|\Phi^{(N)}(\tau)|^2 + |\Phi^{(N)}(0)|^2$$
$$(\Sigma^{(N)}(\tau))^2]dx + \frac{1}{4\tau}\int_\Omega [F(|\Phi^{(N)}(\tau)|^2) + F(|\Phi^{(N)}(0)|^2)]dx$$
$$+\frac{1}{2\tau}\int_\Omega [H(\Sigma^{(N)}(\tau)) + H(\Sigma^{(N)}(0))]dx.$$

Lemma 1 *(Sobolev's imbedding theorem). Let the domain Ω satisfy the cone property; then we have $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), p \le q \le \frac{np}{n-mp}$. Especially $W^{2,2}(\Omega) \hookrightarrow C^0(\Omega), W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)(n \le 3); W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)(n \le 4)$.*

Since $F(s) \ge 0$, and $H(s) \ge 0$ and by Lemma 1, if we take $\Phi^{(N)}(\tau)$ and $\Sigma^{(N)}(\tau)$, such that $\|\Phi^{(N)}(\tau)\|_1, \|\Sigma^{(N)}(\tau)\|_1 \le c$, then we have

$$\int_\Omega \left[(\Sigma^{(N)}(0))^2|\Phi^{(N)}(\tau)|^2 + |\Phi^{(N)}(0)|^2(\Sigma^{(N)}(\tau))^2\right]dx$$
$$\le \|\Sigma^{(N)}(0)\|_{L^4}^2\|\Phi^{(N)}(\tau)\|_{L^4}^2 + \|\Phi^{(N)}(0)\|_{L^4}^2\|\Sigma^{(N)}(\tau)\|_{L^4}^2$$
$$\le C(\|\Sigma^{(N)}(0)\|_1^2 \ \ \|\Phi^{(N)}(\tau)\|_1^2 + \|\Phi^{(N)}(0)\|_1^2\|\Sigma^{(N)}(\tau)\|_1^2)$$
$$\le C.$$

Here and hereafter we use C to denote constants independent of N, τ, which take different values in different places. In addition ,

$$\left|\int_\Omega \left[F(|\Phi^{(N)}(\tau)|^2) + F(|\Phi^{(N)}(0)|^2)\right]dx\right.$$
$$= \left|\int_\Omega (\int_0^{|\Phi^{(N)}(\tau)|^2} + \int_0^{|\Phi^{(N)}(0)|^2} f(s)ds)dx\right|$$
$$\le A_0\int_\Omega (\int_0^{|\Phi^{(N)}(\tau)|^2} + \int_0^{|\Phi^{(N)}(0)|^2} sds)dx$$
$$\le \frac{A_0}{2}(\|\Phi^{(N)}(\tau)\|_{L^4}^4 + \|\Phi^{(N)}(0)\|_{L^4}^4) \le C.$$

Similarly, $\left|\int_\Omega \left[H(\Sigma^{(N)}(\tau)) + H(\Sigma^{(N)}(0))\right]dx \le C\right.$. Hence obtain

Lemma 2. *For the solution of (1.4)-(1.7), if we take $\Phi^{(N)}(\tau)$ and $\Sigma^{(N)}(\tau)$ properly, then we have the estimate*

$$\|\Phi_t^{(N)}(t)\| + \|\Sigma_t^{(N)}(t)\| + \|\Phi^{(N)}(t)\|_1 + \|\Sigma^{(N)}(t)\|_1 \le C, \ \ 0 \le t \le T.$$

3. Error Estimation

Now we turn to an error estimate for the solution of the spectral method. The system (1.1)–(1.2) can be written as

$$
\begin{aligned}
&(\varphi_{t\bar{t}}^{(N)}(t),\psi_j)+\tfrac{1}{2}(\nabla(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau)),\nabla\psi_j)\\
&\quad+\tfrac{\mu^2}{2}(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau),\psi_j)+\tfrac{\nu^2}{2}((\chi^{(N)}(t))^2(\varphi^{(N)}(t+\tau)\\
&\quad+\varphi^{(N)}(t-\tau)),\psi_j)+\tfrac{1}{2}(F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)(\varphi^{(N)}(t+\tau)\\
&\quad+\varphi^{(N)}(t-\tau)),\psi_j)=\sum_{i=1}^{4}(E_i(t),\psi_j)+(\nabla E_5(t),\nabla\psi_j),
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&(\chi_{t\bar{t}}^{(N)}(t),\psi_j)+\tfrac{1}{2}(\nabla(\chi^{(N)}(t+\tau)+\chi^{(N)}(t-\tau)),\nabla\psi_j)+\tfrac{\delta^2}{2}(\chi^{(N)}(t+\tau)\\
&\quad+\chi^{(N)}(t-\tau),\psi_j)+\tfrac{\nu^2}{2}(|\varphi^{(N)}(t)|^2(\chi^{(N)}(t+\tau)+\chi^{(N)}(t-\tau)),\psi_j)\\
&\quad+(H(\chi^{(N)}(t+\tau),\chi^{(N)}(t-\tau)),\psi_j)\\
&\quad=\sum_{i=6}^{9}(E_i(t),\psi_j)+(\nabla E_{10}(t),\nabla\psi_j),\quad |j|\le N,
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
E_1(t)&=\varphi_{t\bar{t}}^{(N)}(t)-\frac{\partial^2\varphi(t)}{\partial t^2},\\
E_2(t)&=\tfrac{\mu^2}{2}[\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau)-2\varphi(t)],\\
E_3(t)&=\tfrac{\nu^2}{2}[(\chi^{(N)}(t))^2(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau))-2(\chi(t))^2\varphi(t)],\\
E_4(t)&=\tfrac{1}{2}[F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)(\varphi^{(N)}(t+\tau)\\
&\qquad+\varphi^{(N)}(t-\tau))-2f(|\varphi(t)|^2)\varphi(t)],\\
E_5(t)&=\tfrac{1}{2}[\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau)-2\varphi(t)],\\
E_6(t)&=\chi_{t\bar{t}}^{(N)}(t)-\frac{\partial^2\chi(t)}{\partial t^2},\\
E_7(t)&=\tfrac{\delta^2}{2}[\chi^{(N)}(t+\tau)+\chi^{(N)}(t-\tau)-2\chi(t)],\\
E_8(t)&=\tfrac{\nu^2}{2}[|\varphi^{(N)}(t)|^2(\chi^{(N)}(t+\tau)+\chi^{(N)}(t-\tau))-2\chi(t)|\varphi(t)|^2],\\
E_9(t)&=H(\chi^{(N)}(t+\tau),\chi^{(N)}(t-\tau))-h(\chi(t)),\\
E_{10}(t)&=\tfrac{1}{2}[\chi^{(N)}(t+\tau)+\chi^{(N)}(t-\tau)-2\chi(t)].
\end{aligned}
$$

Note $\tilde{\varphi}^{(N)}(t)=\varphi^{(N)}(t)-\Phi^{(N)}(t)=\sum\limits_{|l|\le N}\alpha_l(t)\psi_l(x),\tilde{\chi}^{(N)}(t)=\chi^{(N)}(t)-\Sigma^{(N)}(t)$ $=\sum\limits_{|l|\le N}\beta_l(t)\psi_l$. From (3.1), (3.2) and (1.4), (1.6) we can obtain that $\tilde{\varphi}^{(N)}(x,t)$ and

$\tilde{\chi}^{(N)}(x,t)$ should satisfy respectively

$$(\tilde{\varphi}_{t\bar{t}}^{(N)}(t),\psi_j)+\tfrac{1}{2}(\nabla(\tilde{\varphi}^{(N)}(t+\tau)+\tilde{\varphi}^{(N)}(t-\tau)),\nabla\psi_j)+\tfrac{\mu^2}{2}(\tilde{\varphi}^{(N)}(t+\tau)+\tilde{\varphi}^{(N)}(t-\tau),\psi_j)$$
$$+\tfrac{\nu^2}{2}((\chi^{(N)}(t))^2(\tilde{\varphi}^{(N)}(t+\tau)+\tilde{\varphi}^{(N)}(t-\tau))+\tilde{\chi}^{(N)}(t)(\chi^{(N)}(t)+\Sigma^{(N)}(t))(\Phi^{(N)}(t+\tau)$$
$$+\Phi^{(N)}(t-\tau)),\psi_j)+\tfrac{1}{2}(F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)(\tilde{\varphi}^{(N)}(t+\tau)+\tilde{\varphi}^{(N)}(t-\tau)),\psi_j)$$
$$+\tfrac{1}{2}([F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)-F(|\Phi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)](\Phi^{(N)}(t+\tau)$$
$$+\Phi^{(N)}(t-\tau)),\psi_j)=\Sigma_{i=1}^4(E_i(t),\psi_j)+(\nabla E_5(t),\nabla\psi_j), \tag{3.3}$$

$$(\tilde{\chi}_{t\bar{t}}^{(N)}(t),\psi_j)+\tfrac{1}{2}(\nabla(\tilde{\chi}^{(N)}(t+\tau)+\tilde{\chi}^{(N)}(t-\tau)),\nabla\psi_j)+\tfrac{\delta^2}{2}(\tilde{\chi}^{(N)}(t+\tau)$$
$$+\tilde{\chi}^{(N)}(t-\tau),\psi_j)+\tfrac{\nu^2}{2}(|\varphi^{(N)}(t)|^2(\tilde{\chi}^{(N)}(t+\tau)+\tilde{\chi}^{(N)}(t-\tau)),\psi_j)$$
$$+\tfrac{\nu^2}{2}((\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau))(\varphi^{(N)}(t)\overline{\tilde{\varphi}^{(N)}(t)}+\overline{\Phi^{(N)}(t)}\tilde{\varphi}^{(N)}(t)),\psi_j)$$
$$+(H(\chi^{(N)}(t+\tau),\chi^{(N)}(t-\tau))-H(\Sigma^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau)),\psi_j)$$
$$=\sum_{i=6}^{9}(E_i(t),\psi_j)+(\nabla E_{10}(t),\nabla\psi_j),\quad |j|\le N. \tag{3.4}$$

Multiplying (2.3) by $\frac{1}{2\tau}(\alpha_j(t+\tau)-\alpha_j(t-\tau))$ and summing for j, we have

$$\tfrac{1}{2}(\tilde{\varphi}_{t\bar{t}}^{(N)}(t),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}^{(N)}(t-\tau))+\tfrac{1}{4\tau}(\nabla(\tilde{\varphi}^{(N)}(t+\tau)+\tilde{\varphi}^{(N)}(t-\tau)),$$
$$\nabla(\tilde{\varphi}^{(N)}(t+\tau)-\tilde{\varphi}^{(N)}(t-\tau)))+\tfrac{\mu^2}{4\tau}(\tilde{\varphi}^{(N)}(t+\tau)+\tilde{\varphi}^{(N)}(t-\tau),\tilde{\varphi}^{(N)}(t+\tau)$$
$$-\tilde{\varphi}^{(N)}(t-\tau))+\tfrac{\nu^2}{4\tau}((\chi^{(N)}(t))^2(\tilde{\varphi}^{(N)}(t+\tau)+\tilde{\varphi}^{(N)}(t-\tau)),\tilde{\varphi}^{(N)}(t+\tau)$$
$$-\tilde{\varphi}^{(N)}(t-\tau))+\tfrac{\nu^2}{4\tau}(\tilde{\chi}^{(N)}(t)(\chi^{(N)}(t)+\Sigma^{(N)}(t))(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),$$
$$\tilde{\varphi}^{(N)}(t+\tau)-\tilde{\varphi}^{(N)}(t-\tau))+\tfrac{1}{4\tau}(F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)(\tilde{\varphi}^{(N)}(t+\tau)$$
$$+\tilde{\varphi}^{(N)}(t-\tau)),\tilde{\varphi}^{(N)}(t+\tau)-\tilde{\varphi}^{(N)}(t+\tau))+\tfrac{1}{4\tau}([F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)$$
$$-F(|\Phi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)](\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\tilde{\varphi}^{(N)}(t+\tau)$$
$$-\tilde{\varphi}^{(N)}(t-\tau))=\sum_{i=1}^{4}\frac{1}{2\tau}(E_i(t),\tilde{\varphi}^{(N)}(t+\tau)-\tilde{\varphi}^{(N)}(t-\tau))$$
$$+\tfrac{1}{2\tau}(\nabla E_5(t),\nabla(\tilde{\varphi}^{(N)}(t+\tau)-\tilde{\varphi}^{(N)}(t-\tau))). \tag{3.4$'$}$$

Similarly , we have

$$\mathrm{Re}\tfrac{1}{2\tau}(\tilde{\varphi}_{t\bar{t}}^{(N)}(t),\tilde{\varphi}^{(N)}(t+\tau)-\tilde{\varphi}^{(N)}(t-\tau))=\tfrac{1}{2\tau}[\|\tilde{\varphi}_t^{(N)}(t)\|^2-\|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2],$$
$$\mathrm{Re}\tfrac{1}{4\tau}(\nabla(\tilde{\varphi}^{(N)}(t+\tau)+\tilde{\varphi}^{(N)}(t-\tau)),\nabla(\tilde{\varphi}^{(N)}(t+\tau)-\tilde{\varphi}^{(N)}(t-\tau)))$$
$$=\tfrac{1}{4\tau}[(\|\nabla\tilde{\varphi}^{(N)}(t+\tau)\|^2+\|\nabla\tilde{\varphi}^{(N)}(t)\|^2)-(\|\nabla\tilde{\varphi}^{(N)}(t)\|^2+\|\nabla\tilde{\varphi}^{(N)}(t-\tau)\|^2)],$$

$$
\begin{aligned}
&\mathrm{Re}\frac{\mu^2}{4\tau}(\tilde\varphi^{(N)}(t+\tau)+\tilde\varphi^{(N)}(t-\tau),\tilde\varphi^{(N)}(t+\tau)-\tilde\varphi^{(N)}(t-\tau))\\
&\quad=\frac{\mu^2}{4\tau}[(\|\tilde\varphi^{(N)}(t+\tau)\|^2+\|\tilde\varphi^{(N)}(t)\|^2)-(\|\tilde\varphi^{(N)}(t)\|^2+\|\tilde\varphi^{(N)}(t-\tau)\|^2)],\\
&\mathrm{Re}\frac{\nu^2}{4\tau}((\chi^{(N)}(t))^2(\tilde\varphi^{(N)}(t+\tau)+\tilde\varphi^{(N)}(t-\tau)),\tilde\varphi^{(N)}(t+\tau)-\tilde\varphi^{(N)}(t-\tau))\\
&\quad\le\frac{\nu^2}{4}\|\chi^{(N)}(t)\|^2_{L^6}(\|\tilde\varphi^{(N)}(t+\tau)\|_{L^6}+\|\tilde\varphi^{(N)}(t-\tau)\|_{L^6})(\|\tilde\varphi^{(N)}_t(t)\|+\|\tilde\varphi^{(N)}_t(t-\tau)\|)\\
&\quad\le C\|\chi^{(N)}(t)\|_1^2(\|\tilde\varphi^{(N)}(t+\tau)\|_1^2+\|\tilde\varphi^{(N)}(t-\tau)\|_1^2+\|\tilde\varphi^{(N)}_t(t)\|^2+\|\tilde\varphi^{(N)}_t(t-\tau)\|^2),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{\nu^2}{4\tau}(\tilde\chi^{(N)}(t)(\chi^{(N)}(t)+\Sigma^{(N)}(t))(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau)),\tilde\varphi^{(N)}(t+\tau)-\tilde\varphi^{(N)}(t-\tau))\right|\\
&\quad\le\frac{\nu^2}{4}\|\tilde\chi^{(N)}(t)\|_{L^6}(\|\chi^{(N)}(t)\|_{L^6}+\|\Sigma^{(N)}(t)\|_{L^6})\\
&\qquad\times(\|\Phi^{(N)}(t+\tau)\|_{L^6}+\|\Phi^{(N)}(t-\tau)\|_{L^6})(\|\tilde\varphi^{(N)}_t(t)\|+\|\tilde\varphi^{(N)}_t(t-\tau)\|)\\
&\quad\le C(\|\chi^{(N)}(t)\|_1+\|\Sigma^{(N)}(t)\|_1)(\|\Phi^{(N)}(t+\tau)\|_1+\|\Phi^{(N)}(t-\tau)\|_1)\\
&\qquad\times(\|\tilde\chi^{(N)}(t)\|_1^2+\|\tilde\varphi^{(N)}_t(t)\|^2+\|\tilde\varphi^{(N)}_t(t-\tau)\|^2).
\end{aligned}
$$

Using the property of difference quotient, we get

$$
\begin{aligned}
&F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)-F(|\Phi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)\\
&\quad=F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)-F(|\varphi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)\\
&\qquad+F(|\varphi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)-F(|\Phi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)\\
&\quad=\frac12F''(P^{(N)})(|\varphi^{(N)}(t-\tau)|^2-|\Phi^{(N)}(t-\tau)|^2)\\
&\qquad+\frac12F''(Q^{(N)})(|\varphi^{(N)}(t+\tau)|^2-|\Phi^{(N)}(t+\tau)|^2)\\
&\quad=\frac12f'(P^{(N)})[\varphi^{(N)}(t-\tau)\overline{\tilde\varphi^{(N)}(t-\tau)}+\overline{\Phi^{(N)}(t-\tau)}\tilde\varphi^{(N)}(t-\tau)]\\
&\qquad+\frac12f'(Q^{(N)})[\varphi^{(N)}(t+\tau)\overline{\tilde\varphi^{(N)}(t+\tau)}+\overline{\Phi^{(N)}(t+\tau)}\tilde\varphi^{(N)}(t+\tau)],
\end{aligned}
$$

where

$$
\begin{aligned}
P^{(N)}\in(&\min(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2,|\Phi^{(N)}(t-\tau)|^2),\\
&\max(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)),\\
Q^{(N)}\in(&\min(|\varphi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2,|\Phi^{(N)}(t+\tau)|^2),\\
&\max(|\varphi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2,|\Phi^{(N)}(t+\tau)|^2)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|\tfrac{1}{4\tau}[F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)-F(|\Phi^{(N)}(t+\tau)|^2,|\Phi^{(N)}(t-\tau)|^2)]\\
&\quad\cdot(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau),\tilde\varphi^{(N)}(t+\tau)-\tilde\varphi^{(N)}(t-\tau))|\\
&\quad\le \tfrac{1}{8}|(f'(P^{(N)})[\varphi^{(N)}(t-\tau)\overline{\tilde\varphi^{(N)}(t-\tau)}+\overline{\Phi^{(N)}(t-\tau)}\tilde\varphi^{(N)}(t-\tau)](\Phi^{(N)}(t+\tau)\\
&\quad+\Phi^{(N)}(t-\tau)),\tilde\varphi_t^{(N)}(t)+\tilde\varphi_t^{(N)}(t-\tau))|\\
&\quad+\tfrac{1}{8}|f'(Q^{(N)})[\varphi^{(N)}(t+\tau)\tilde\varphi^{(N)}(t+\tau)+\Phi^{(N)}(t+\tau)\tilde\varphi^{(N)}(t+\tau)]\\
&\quad(\Phi^{(N)}(t+\tau)+\Phi^{(N)}(t-\tau),\tilde\varphi_t^{(N)}(t)+\tilde\varphi_t^{(N)}(t-\tau))|\\
&\quad\le \tfrac{A_1}{8}(\|\Phi^{(N)}(t+\tau)\|_{L^6}+\|\Phi^{(N)}(t-\tau)\|_{L^6})(\|\tilde\varphi_t^{(N)}(t)\|+\|\tilde\varphi_t^{(N)}(t-\tau)\|)\\
&\quad[\|\tilde\varphi^{(N)}(t-\tau)\|_{L^6}(\|\varphi^{(N)}(t-\tau)\|_{L^6}+\|\Phi^{(N)}(t-\tau)\|_{L^6})\\
&\quad+\|\tilde\varphi^{(N)}(t+\tau)\|_{L^6}(\|\varphi^{(N)}(t+\tau)\|_{L^6}+\|\Phi^{(N)}(t+\tau)\|_{L^6})]\\
&\quad\le C(\|\Phi^{(N)}(t+\tau)\|_1+\|\Phi^{(N)}(t-\tau)\|_1(\|\varphi^{(N)}(t-\tau)\|_1+\|\varphi^{(N)}(t+\tau)\|_1\\
&\quad+\|\Phi^{(N)}(t-\tau)\|_1+\|\Phi^{(N)}(t+\tau)\|_1)\ (\|\tilde\varphi^{(N)}(t-\tau)\|_1^2+\|\tilde\varphi^{(N)}(t+\tau)\|_1^2\\
&\quad+\|\tilde\varphi_t^{(N)}(t)\|^2+\|\tilde\varphi_t^{(N)}(t-\tau)\|^2).
\end{aligned}
$$

Multiplying (2.4) by $\frac{1}{2\tau}(\beta_j(t+\tau)-\beta_j(t-\tau))$ and then summing for j, we have

$$
\begin{aligned}
&\tfrac{1}{2\tau}(\tilde\chi_{tt}^{(N)}(t),\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau))+\tfrac{1}{4\tau}(\nabla(\tilde\chi^{(N)}(t+\tau)+\tilde\chi^{(N)}(t-\tau)),\nabla(\tilde\chi^{(N)}(t+\tau)\\
&-\tilde\chi^{(N)}(t-\tau))+\tfrac{\delta^2}{4\tau}(\tilde\chi^{(N)}(t+\tau)+\tilde\chi^{(N)}(t-\tau),\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau))\\
&+\tfrac{\nu^2}{4\tau}(|\varphi^{(N)}(t)|^2(\tilde\chi^{(N)}(t+\tau)+\tilde\chi^{(N)}(t-\tau)),\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau))+\tfrac{\nu^2}{4\tau}((\Sigma^{(N)}(t+\tau)\\
&+\Sigma^{(N)}(t-\tau))(\varphi^{(N)}(t)\overline{\tilde\varphi^{(N)}(t)}+\overline{\Phi^{(N)}(t)}\tilde\varphi^{(N)}(t)),\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau))\\
&+\tfrac{1}{2\tau}(H(\chi^{(N)}(t+\tau),\chi^{(N)}(t-\tau))-H(\Sigma^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau),\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}t-\tau))\\
&=\frac{1}{2\tau}\sum_{i=6}^{9}(E_i(t),\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau))+\frac{1}{2\tau}(\nabla E_{10}(t),\nabla(\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau))).
\end{aligned}
\tag{3.5}
$$

Since

$$
\begin{aligned}
&\tfrac{1}{2\tau}(\tilde\chi_{tt}^{(N)}(t),\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau))=\tfrac{1}{2\tau}(\|\tilde\chi_t^{(N)}(t)\|^2-\|\tilde\chi_t^{(N)}(t-\tau)\|^2),\\
&\tfrac{1}{4\tau}(\nabla(\tilde\chi^{(N)}(t+\tau)+\tilde\chi^{(N)}(t-\tau)),\nabla(\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau)))\\
&\quad=\tfrac{1}{4\tau}[(\|\nabla\tilde\chi^{(N)}(t+\tau)\|^2+\|\nabla\tilde\chi^{(N)}(t)\|^2)-(\|\nabla\tilde\chi^{(N)}(t)\|^2+\|\nabla\tilde\chi^{(N)}(t-\tau)\|^2)],\\
&\tfrac{\delta^2}{4\tau}(\tilde\chi^{(N)}(t+\tau)+\tilde\chi^{(N)}(t-\tau),\tilde\chi^{(N)}(t+\tau)-\tilde\chi^{(N)}(t-\tau))\\
&\quad=\tfrac{\delta^2}{4\tau}[(\|\tilde\chi^{(N)}(t+\tau)\|^2+\|\tilde\chi^{(N)}(t)\|^2)-(\|\tilde\chi^{(N)}(t)\|^2+\|\tilde\chi^{(N)}(t-\tau)\|^2)],
\end{aligned}
$$

$$
\begin{aligned}
&|\tfrac{\nu^2}{4\tau}(|\varphi^{(N)}(t)|^2(\tilde{\chi}^{(N)}(t+\tau)+\tilde{\chi}^{(N)}(t-\tau),\tilde{\chi}^{(N)}(t+\tau)-\tilde{\chi}^{(N)}(t-\tau))| \\
&\le \tfrac{\nu^2}{4}\|\varphi^{(N)}(t)\|^2_{L^6}(\|\tilde{\chi}^{(N)}(t+\tau)\|_{L^6}+\|\tilde{\chi}^{(N)}(t-\tau)\|_{L^6})(\|\tilde{\chi}_t^{(N)}(t)\|+\|\tilde{\chi}^{(N)}(t-\tau)\|) \\
&\le C\|\varphi^{(N)}(t)\|_1^2(\|\tilde{\chi}^{(N)}(t+\tau)\|_1^2+\|\tilde{\chi}^{(N)}(t-\tau)\|_1^2+\|\tilde{\chi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2),
\end{aligned}
$$

$$
\begin{aligned}
&|\tfrac{\nu^2}{4\tau}((\Sigma^{(N)}(t+\tau)+\Sigma^{(N)}(t-\tau))(\varphi^{(N)}(t)\overline{\tilde{\varphi}^{(N)}(t)} \\
&\quad+\overline{\Phi^{(N)}(t)}\tilde{\varphi}^{(N)}(t)),\tilde{\chi}^{(N)}(t+\tau)-\tilde{\chi}^{(N)}(t-\tau))| \\
&\le \tfrac{\nu^2}{4}(\|\Sigma^{(N)}(t+\tau)\|_{L^6}+\|\Sigma^{(N)}(t-\tau)\|_{L^6})(\|\varphi^{(N)}(t)\|_{L^6} \\
&\quad+\|\Phi^{(N)}(t)\|_{L^6})\|\tilde{\varphi}^{(N)}(t)\|_{L^6}(\|\tilde{\chi}_t^{(N)}(t)\|+\|\tilde{\chi}_t^{(N)}(t-\tau)\|) \\
&\le C(\|\Sigma^{(N)}(t+\tau)\|_1+\|\Sigma^{(N)}(t-\tau)\|_1)(\|\varphi^{(N)}(t)\|_1 \\
&\quad+\|\Phi^{(N)}(t)\|_1)(\|\tilde{\varphi}^{(N)}(t)\|_1^2+\|\tilde{\chi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2)
\end{aligned}
$$

and similarly to the above estimate, we have

$$
\begin{aligned}
&H(\chi^{(N)}(t+\tau),\chi^{(N)}(t-\tau))-H(\Sigma^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau)) \\
&=\tfrac{1}{2}h'(R^{(N)})\tilde{\chi}^{(N)}(t+\tau)+\tfrac{1}{2}h'(s^{(N)})\tilde{\chi}^{(N)}(t+\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
R^{(N)} &\in (\min(\chi^{(N)}(t-\tau),\chi^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau)), \\
&\quad \max(\chi^{(N)}(t-\tau),\chi^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau))), \\
S^{(N)} &\in (\min(\chi^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau),\Sigma^{(N)}(t+\tau)), \\
&\quad \max(\chi^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau),\Sigma^{(N)}(t+\tau))).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|\tfrac{1}{2\tau}(H(\chi^{(N)}(t+\tau),\chi^{(N)}(t-\tau))-H(\Sigma^{(N)}(t+\tau),\Sigma^{(N)}(t-\tau)),\tilde{\chi}^{(N)}(t+\tau)-\tilde{\chi}^{(N)}(t-\tau))| \\
&=|\tfrac{1}{4\tau}(h'(R^{(N)})\tilde{\chi}^{(N)}(t-\tau)+h'(s^{(N)})\tilde{\chi}^{(N)}(t+\tau),\tilde{\chi}^{(N)}(t+\tau)-\tilde{\chi}^{(N)}(t-\tau))| \\
&\le \tfrac{B_1}{4\tau}|([(\chi^{(N)}(t-\tau))^2+(\chi^{(N)}(t+\tau))^2+(\Sigma^{(N)}(t-\tau))^2]|\tilde{\chi}^{(N)}(t-\tau)|+[(\chi^{(N)}(t+\tau))^2 \\
&\quad+(\Sigma^{(N)}(t-\tau))^2+(\Sigma^{(N)}(t+\tau))^2]|\tilde{\chi}^{(N)}(t+\tau)|,|\tilde{\chi}^{(N)}(t+\tau)-\tilde{\chi}^{(N)}(t-\tau)|)| \\
&\le \tfrac{B_1}{4}[(\|\chi^{(N)}(t-\tau)\|^2_{L^6}+\|\chi^{(N)}(t+\tau)\|^2_{L^6}+\|\Sigma^{(N)}(t-\tau)\|^2_{L^6})\|\tilde{\chi}^{(N)}(t-\tau)\|_{L^6} \\
&\quad+(\|\chi^{(N)}(t+\tau)\|^2_{L^6}+\|\Sigma^{(N)}(t-\tau)\|^2_{L^6}+\|\Sigma^{(N)}(t+\tau)\|^2_{L^6})\|\tilde{\chi}^{(N)}(t+\tau)\|_{L^6}](\|\tilde{\chi}_t^{(N)}(t)\| \\
&\quad+\|\tilde{\chi}_t^{(N)}(t-\tau)\|)\le C(\|\chi^{(N)}(t-\tau)\|_1^2+\|\chi^{(N)}(t+\tau)\|_1^2+\|\Sigma^{(N)}(t-\tau)\|_1^2 \\
&\quad+\|\Sigma^{(N)}(t+\tau)\|_1^2)(\|\tilde{\chi}^{(N)}(t-\tau)\|_1^2+\|\tilde{\chi}^{(N)}(t+\tau)\|_1^2+\|\tilde{\chi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2).
\end{aligned}
$$

Finally we estimate the terms with $E_i(t)$. In the first place,

$$
\begin{aligned}
E_1(t) &= \varphi_{t\bar{t}}^{(N)}(t) - \frac{\partial^2\varphi(t)}{\partial t^2} = \varphi_{t\bar{t}}^{(N)}(t) - \frac{\partial^2\varphi^{(N)}(t)}{\partial t^2} + \frac{\partial^2\varphi^{(N)}(t)}{\partial t^2} - \frac{\partial^2\varphi(t)}{\partial t^2} \\
&= 0(\tau^2) - R^{(N)}(\frac{\partial^2\varphi(t)}{\partial t^2}), \\
E_2(t) &= \frac{\mu^2}{2}[\varphi^{(N)}(t+\tau) + \varphi^{(N)}(t-\tau) - 2\varphi(t)] = \frac{\mu^2}{2}[0(\tau^2) - 2R^{(N)}(\varphi(t))], \\
E_3(t) &= \frac{\nu^2}{2}[(\chi^{(N)}(t))^2(\varphi^{(N)}(t+\tau) + \varphi^{(N)}(t-\tau)) - 2(\chi(t))^2\varphi(t)] \\
&= \frac{\nu^2}{2}\{(\chi^{(N)}(t))^2[\varphi^{(N)}(t+\tau) + \varphi^{(N)}(t-\tau) - 2\varphi(t)] \\
&\quad -2\varphi(t)(\chi^{(N)}(t) + \chi(t))R^{(N)}(\chi(t))\} \\
&= \frac{\nu^2}{2}\{(\chi^{(N)}(t))^2(0(\tau^2) - 2R^{(N)}(\varphi(t))) - 2\varphi(t)(\chi^{(N)}(t) + \chi(t))R^{(N)}(\chi(t))\}, \\
E_6(t) &= \chi_{t\bar{t}}^{(N)}(t) - \frac{\partial^2\chi(t)}{\partial t^2} = 0(\tau^2) - R^{(N)}(\frac{\partial^2\chi(t)}{\partial t^2}), \\
E_7(t) &= \frac{\delta^2}{2}[\chi^{(N)}(t+\tau) + \chi^{(N)}(t-\tau) - 2\chi(t)] = \frac{\delta^2}{2}[0(\tau^2) - 2R^{(N)}(\chi(t))], \\
E_8(t) &= \frac{\nu^2}{2}[|\varphi^{(N)}(t)|^2(\chi^{(N)}(t+\tau) + \chi^{(N)}(t-\tau) - 2\chi(t)|\varphi(t)|^2] \\
&= \frac{\nu^2}{2}\{|\varphi^{(N)}(t)|^2(0(\tau^2) - 2R^{(N)}(\chi(t))) + 2\chi(t)(\varphi^{(N)}\overline{R^{(N)}(\varphi(t))} \\
&\quad +\overline{\varphi(t)}R^{(N)}(\varphi(t)))\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|\frac{1}{2}(E_1(t), \tilde{\varphi}_t^{(N)}(t) + \tilde{\varphi}_t^{(N)}(t-\tau))| \le \frac{1}{2}(\|E_1(t)\|^2 + \|\tilde{\varphi}_t^{(N)}(t)\|^2 + \|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2) \\
&\quad \le C(\tau^4 + \|R^{(N)}(\frac{\partial^2\varphi(t)}{\partial t^2})\|^2 + \|\tilde{\varphi}_t^{(N)}(t)\|^2 + \|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2), \\
&|\frac{1}{2}(E_2(t), \tilde{\varphi}_t^{(N)}(t) + \tilde{\varphi}_t^{(N)}(t-\tau))| \\
&\quad \le C(\tau^4 + \|R^{(N)}(\varphi(t))\|^2 + \|\tilde{\varphi}_t^{(N)}(t)\|^2 + \|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2), \\
&|\frac{1}{2}(E_3(t), \tilde{\varphi}_t^{(N)}(t) + \tilde{\varphi}_t^{(N)}(t-\tau))| \\
&\quad \le \frac{\nu^2}{4}(\|\chi^{(N)}(t)\|_{L^6}^2\|\varphi^{(N)}(t+\tau) + \varphi^{(N)}(t-\tau) - 2\varphi(t)\|_{L^6} + 2\|\varphi(t)\|_{L^6} \\
&\qquad \times\|R^{(N)}(\chi(t))\|_{L^6}(\|\chi^{(N)}(t)\|_{L^6} + \|\chi(t)\|_{L^6}))(\|\tilde{\varphi}_t^{(N)}(t)\| + \|\tilde{\varphi}_t^{(N)}(t-\tau)\|) \\
&\quad \le C(\|\chi^{(N)}(t)\|_1^2\|\varphi^{(N)}(t+\tau) + \varphi^{(N)}(t-\tau) - 2\varphi(t)\|_1 + 2\|\varphi(t)\|_1\|R^{(N)}(\chi(t))\|_1 \\
&\quad (\|\chi^{(N)}(t)\|_1 + \|\chi(t)\|_1)) \cdot (\|\tilde{\varphi}_t^{(N)}(t)\| + \|\tilde{\varphi}_t^{(N)}(t-\tau)\|) \\
&\quad \le C(\tau^4 + \|R^{(N)}(\varphi(t))\|_1^2 + \|R^{(N)}(\chi(t))\|_1^2 + \|\tilde{\varphi}_t^{(N)}(t)\|^2 + \|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2), \\
&|\frac{1}{2}(E_6(t), \tilde{\chi}_t^{(N)}(t) + \tilde{\chi}_t^{(N)}(t-\tau))| \\
&\quad \le C(\tau^4 + \|R^{(N)}(\frac{\partial^2\chi(t)}{\partial t^2})\|^2 + \|\tilde{\chi}_t^{(N)}(t)\|^2 + \|\tilde{\chi}_t^{(N)}(t-\tau)\|^2),
\end{aligned}
$$

$$
\begin{aligned}
&|\tfrac{1}{2}(E_7(t),\tilde{\chi}_t^{(N)}(t)+\tilde{\chi}_t^{(N)}(t-\tau)|\\
&\quad\le C(\tau^4+\|R^{(N)}(\chi(t))\|^2+\|\tilde{\chi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2),\\
&\quad|\tfrac{1}{2}(E_8(t),\tilde{\chi}_t^{(N)}(t)+\tilde{\chi}_t^{(N)}(t-\tau))|\\
&\quad\le C(\tau^4+\|R^{(N)}(\varphi(t))\|_1^2+\|R^{(N)}(\chi(t))\|_1^2+\|\tilde{\chi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2).
\end{aligned}
$$

For $E_5(t)$ and $E_{10}(t)$, we have

$$
\begin{aligned}
&|\tfrac{1}{2}(\nabla E_5(t),\nabla(\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau)))|=|\tfrac{1}{2}(\Delta E_5(t),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))|\\
&\quad\le|\tfrac{1}{4}(\Delta(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau)-2\varphi^{(N)}(t)),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))|\\
&\qquad+\tfrac{1}{2}|(\Delta R^{(N)}(\varphi(t)),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))|\\
&\quad\le C(\tau^4+\|\tilde{\varphi}_t^{(N)}(t)\|^2+\|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2).
\end{aligned}
$$

Since $R^{(N)}(\varphi(t))$ is orthogonal with S_N, the second term is zero. Similarly,

$$
|\frac{1}{2}(\nabla E_{10}(t),\nabla(\tilde{\chi}_t^{(N)}(t)+\tilde{\chi}^{(N)}(t-\tau)))|\le C(\tau^4+\|\tilde{\chi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2).
$$

In the last place, we estimate $E_4(t)$ and $E_9(t)$.

$$
\begin{aligned}
E_4(t)&=\tfrac{1}{2}[F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau))\\
&\quad-2f(|\varphi(t)|^2)\varphi(t)]=\tfrac{1}{2}[F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)-F(|\varphi^{(N)}(t+\tau)|^2,\\
&\quad|\varphi(t-\tau)|^2)+F(|\varphi^{(N)}(t+\tau)|^2,|\varphi(t-\tau)|^2)-F(|\varphi(t+\tau)|^2,|\varphi(t-\tau)|^2)\\
&\quad+F(|\varphi(t+\tau)|^2,|\varphi(t-\tau)|^2)-f(|\varphi(t)|^2)](\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau))\\
&\quad+f(|\varphi(t)|^2)[\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau)-2\varphi(t)],
\end{aligned}
$$

$$
\begin{aligned}
&|\tfrac{1}{4}([F(|\varphi^{(N)}(t+\tau)|^2,|\varphi^{(N)}(t-\tau)|^2)-F(|\varphi^{(N)}(t+\tau)|^2,|\varphi(t-\tau)|^2)]\\
&\qquad\times(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau)),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))|\\
&\quad\le\frac{A_1}{8}\int_\Omega||\varphi^{(N)}(t-\tau)|^2-|\varphi(t-\tau)|^2||\varphi^{(N)}(t+\tau)\\
&\qquad+\varphi^{(N)}(t-\tau)||\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau)|dx\\
&\quad\le C(\|R^{(N)}(\varphi(t-\tau))\|_1^2+\|\tilde{\varphi}_t^{(N)}(t)\|^2+\|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2).
\end{aligned}
$$

In the same manner,

$$
\begin{aligned}
&|\tfrac{1}{4}([F(|\varphi^{(N)}(t+\tau)|^2,|\varphi(t-\tau)|^2)-F(|\varphi(t+\tau)|^2,|\varphi(t-\tau)|^2)]\\
&\qquad\times(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau)),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))|\\
&\quad\le C(\|R^{(N)}(\varphi(t+\tau))\|_1^2+\|\tilde{\varphi}_t^{(N)}(t)\|^2+\|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2)
\end{aligned}
$$

$$
\begin{aligned}
&\times|\tfrac{1}{4}([F(|\varphi(t+\tau)|^2,|\varphi(t-\tau)|^2)-f(|\varphi(t)|^2)]\\
&\times(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau)),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))|\\
&=|\tfrac{1}{4}\int_\Omega \frac{1}{|\varphi(t+\tau)|^2-|\varphi(t-\tau)|^2}\int_{|\varphi(t-\tau)|^2}^{|\varphi(t+\tau)|^2}[f(s)-f(|\varphi(t)|^2)]ds\\
&\times(\varphi^{(N)}(t+\tau)+\varphi^{(N)}(t-\tau))\overline{(\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))}dx|,
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{1}{|\varphi(t+\tau)|^2-|\varphi(t-\tau)|^2}\int_{|\varphi(t-\tau)|^2}^{|\varphi(t+\tau)|^2}[f(s)-f(|\varphi(t)|^2)]ds\\
&=\frac{1}{|\varphi(t+\tau)|^2-|\varphi(t-\tau)|^2}\int_{|\varphi(t-\tau)|^2}^{|\varphi(t+\tau)|^2}[f(s)-f(\frac{1}{2}(|\varphi(t+\tau)|^2+|\varphi(t-\tau)|^2))]ds\\
&\quad+[f(\tfrac{1}{2}(|\varphi(t+\tau)|^2+|\varphi(t-\tau)|^2))-f(|\varphi(t)|^2)]=I_1+I_2,
\end{aligned}
$$

$$
\begin{aligned}
|I_1|&\le\tfrac{1}{6}\|f''(s)\|_\infty(|\varphi(t+\tau)|^2-|\varphi(t-\tau)|^2)^2,\\
|I_2|&\le\tfrac{1}{2}A_1(|\varphi(t+\tau)|^2+|\varphi(t-\tau)|^2-2|\varphi(t)|^2)\\
&=\tfrac{1}{2}A_1[\overline{\varphi(t)}(\varphi(t+\tau)+\varphi(t-\tau)-2\varphi(t))\\
&\quad+\varphi(t+\tau)\overline{(\varphi(t+\tau)+\varphi(t-\tau)-2\varphi(t))}\\
&\quad+(\varphi(t-\tau)-\varphi(t+\tau))\overline{(\varphi(t-\tau)-\varphi(t))}].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|(\tfrac{1}{4}[F(|\varphi(t+\tau)|^2,|\varphi(t-\tau)|^2)-f(|\varphi(t)|^2)](\varphi^{(N)}(t+\tau)\\
&\quad+\varphi^{(N)}(t-\tau)),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))|\\
&\le C(\tau^4+\|\tilde{\varphi}_t^{(N)}(t)\|^2+\|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2).
\end{aligned}
$$

Summing them up , we get

$$
\begin{aligned}
&|\tfrac{1}{2}(E_4(t),\tilde{\varphi}_t^{(N)}(t)+\tilde{\varphi}_t^{(N)}(t-\tau))|\\
&\le C(\tau^4+\|R^{(N)}(\varphi(t-\tau))\|_1^2+\|R^{(N)}(\varphi(t))\|_1^2\\
&\quad+\|R^{(N)}(\varphi(t+\tau))\|_1^2+\|\tilde{\varphi}_t^{(N)}(t)\|^2+\|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|\tfrac{1}{2}(E_9(t),\tilde{\chi}_t^{(N)}(t)+\tilde{\chi}_t^{(N)}(t-\tau))|\le C(\tau^4+\|R^{(N)}(\chi(t-\tau))\|_1^2\\
&\quad+\|R^{(N)}(\chi(t))\|_1^2+\|R^{(N)}(\chi(t+\tau))\|_1^2+\|\tilde{\chi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2).
\end{aligned}
$$

Taking the real part on both sides of equality (3.4)' , substituting the above estimate into (3.4)' , (3.5) and then summing them up, we obtain

$$\|\tilde{\varphi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t)\|^2+\frac{1}{2}(\|\nabla\tilde{\varphi}^{(N)}(t+\tau)\|^2+\|\nabla\tilde{\varphi}^{(N)}(t)\|^2+\|\nabla\tilde{\chi}^{(N)}(t+\tau)\|^2$$
$$+\|\nabla\tilde{\chi}^{(N)}(t)\|^2)+\frac{\mu^2}{2}(\|\tilde{\varphi}^{(N)}(t+\tau)\|^2+\|\tilde{\varphi}^{(N)}(t)\|^2)+\frac{\delta^2}{2}(\|\tilde{\chi}^{(N)}(t+\tau)\|^2+\|\tilde{\chi}^{(N)}(t)\|^2)$$
$$\le\|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2+\frac{1}{2}(\|\nabla\tilde{\varphi}^{(N)}(t)\|^2+\|\nabla\tilde{\varphi}^{(N)}(t-\tau)\|^2+\|\nabla\tilde{\chi}^{(N)}(t)\|^2$$
$$+\|\nabla\tilde{\chi}^{(N)}(t-\tau)\|^2)+\frac{\mu^2}{2}(\|\tilde{\varphi}^{(N)}(t)\|^2+\|\tilde{\varphi}^{(N)}(t-\tau)\|^2)+\frac{\delta^2}{2}(\|\tilde{\chi}^{(N)}(t)\|^2+\|\tilde{\chi}^{(N)}(t-\tau)\|^2)$$
$$+C\tau(\|\tilde{\varphi}_t^{(N)}(t)\|^2+\|\tilde{\chi}_t^{(N)}(t)\|^2+\|\tilde{\varphi}^{(N)}(t+\tau)\|_1^2+\|\tilde{\varphi}^{(N)}(t)\|_1^2+\|\tilde{\chi}^{(N)}(t+\tau)\|_1^2+\|\tilde{\chi}^{(N)}(t)\|_1^2)$$
$$+C\tau(\|\tilde{\varphi}_t^{(N)}(t-\tau)\|^2+\|\tilde{\chi}_t^{(N)}(t-\tau)\|^2+\|\tilde{\varphi}^{(N)}(t)\|_1^2+\|\tilde{\varphi}^{(N)}(t-\tau)\|_1^2+\|\tilde{\chi}^{(N)}(t)\|_1^2$$
$$+\|\tilde{\chi}^{(N)}(t-\tau)\|_1^2)+C\tau(\tau^4+\|R^{(N)}(\frac{\partial^2\varphi(t)}{\partial t^2})\|^2+\|R^{(N)}(\frac{\partial^2\chi(t)}{\partial t^2})\|^2+\|R^{(N)}(\varphi(t+\tau))\|_1^2$$
$$+\|R^{(N)}(\varphi(t))\|_1^2+\|R^{(N)}(\varphi(t-\tau))\|_1^2+\|R^{(N)}(\chi(t+\tau))\|_1^2+\|R^{(N)}(\chi(t))\|_1^2$$
$$+\|R^{(N)}(\chi(t-\tau))\|_1^2). \tag{3.6}$$

Here we have used Lemma 2 and

$$\|\varphi^{(N)}(t)\|_1\le\|\varphi(t)\|_1+\|\varphi^{(N)}(t)-\varphi(t)\|_1\le C,\quad 0\le t\le T,$$
$$\|\chi^{(N)}(t)\|_1\le\|\chi(t)\|_1+\|\chi^{(N)}(t)-\chi(t)\|_1\le C,\quad 0\le t\le T.$$

According to the Gronwall inequality, from (3.6) we can obtain

$$\|\tilde{\varphi}_t^{(N)}(t)\|+\|\tilde{\chi}_t^{(N)}(t)\|+\|\tilde{\varphi}^{(N)}(t+\tau)\|_1+\|\tilde{\varphi}^{(N)}(t)\|_1+\|\tilde{\chi}^{(N)}(t+\tau)\|_1+\|\tilde{\chi}^{(N)}(t)\|_1$$
$$\le\|\tilde{\varphi}_t^{(N)}(0)\|+\|\tilde{\chi}_t^{(N)}(0)\|+\|\tilde{\varphi}^{(N)}(\tau)\|_1+\|\tilde{\varphi}^{(N)}(0)\|_1+\|\tilde{\chi}^{(N)}(\tau)\|_1+\|\tilde{\chi}^{(N)}(0)\|_1$$
$$+C(\tau^2+\max_{0\le t\le T}(\|R^{(N)}(\frac{\partial^2\varphi(t)}{\partial t^2})\|+\|R^{(N)}(\frac{\partial^2\chi(t)}{\partial t^2})\|$$
$$+\|R^{(N)}(\varphi(t))\|_1+\|R^{(N)}(\chi(t))\|_1)),\quad 0\le t\le T-\tau.$$

Noting

$$\varphi(t)-\Phi^{(N)}(t)=R^{(N)}(\varphi(t))+\tilde{\varphi}^{(N)}(t),\chi(t)-\Sigma^{(N)}(t)=R^{(N)}(\chi(t))+\tilde{\chi}^{(N)}(t),$$

finally we have

$$\|\varphi(t)-\Phi^{(N)}(t)\|_1+\|\chi(t)-\Sigma^{(N)}(t)\|_1$$
$$\le\|\tilde{\varphi}_t^{(N)}(0)\|+\|\tilde{\chi}_t^{(N)}(0)\|+\|\tilde{\varphi}^{(N)}(\tau)\|_1+\|\tilde{\varphi}^{(N)}(0)\|_1$$
$$+\|\tilde{\chi}^{(N)}(\tau)\|_1+\|\tilde{\chi}^{(N)}(0)\|_1+C[\tau^2+\max_{0\le t\le T}(\|R^{(N)}(\frac{\partial^2\varphi(t)}{\partial t^2})\|$$
$$+\|R^{(N)}(\frac{\partial^2\chi(t)}{\partial t^2})\|+\|R^{(N)}(\varphi(t))\|_1+\|R^{(N)}(\chi(t))\|_1],\quad 0\le t\le T. \tag{3.7}$$

According to the choice of $\Phi^{(N)}(0)$ and $\sum^{(N)}(0)$, we have $\|\tilde{\varphi}^{(N)}(0)\|_1=\|\tilde{\chi}^{(N)}(0)\|_1=0$. Besides, if we take $\Phi^{(N)}(\tau)$ and $\sum^{(N)}(\tau)$ properly, not only the condition of Lemma 2 can be satisfied, but

$$\|\tilde{\varphi}_t^{(N)}(0)\|+\|\tilde{\chi}_t^{(N)}(0)\|+\|\tilde{\varphi}^{(N)}(\tau)\|_1+\|\tilde{\chi}^{(N)}(\tau)\|_1\le C\tau^2$$

can also be obtained. For example , $\Phi^{(N)}(\tau)$ can be obtained as follows. First we expand $\varphi(\tau)$ at $\tau = 0$ and neglect higher order terms; then we project it on S_N. Similarly, $\sum^{(N)}(\tau)$ can be obtained.

Lemma 3. *[6] For any $0 \le \mu \le \sigma$, there is a constant C independent of N, such that*

$$\|u - u^{(N)}\|_\mu \le CN^{-(\sigma-\mu)}\|u\|_\sigma, \qquad \forall u \in H_p^\sigma(\Omega).$$

According to Lemma 3, we can obtain immediately

Theorem. *Suppose that the conditions (1)–(3) are satisfied and the solution of (1.1)–(1.3), $\varphi(x,t), \chi(x,t)$ have fourth continuous derivatives with respect to t, $\frac{\partial^4\varphi}{\partial t^4}, \frac{\partial^4\chi}{\partial t^4} \in L^\infty(0,T;H_p^0(\Omega)), \frac{\partial^2\varphi}{\partial t^2}, \frac{\partial^2\chi}{\partial t^2} \in L^\infty(0,T;H_p^{r-1}(\Omega)), \varphi,\chi \in L^\infty(0,T;H_p^r(\Omega)), r \ge 2; \varphi_0, \varphi_1, \chi_0, \chi_1 \in H_p^r(\Omega)$. Then for the solution of the spectral method , we have*

$$\|\varphi(t) - \Phi^{(N)}(t)\|_1 + \|\chi(t) - \Sigma^{(N)}(t)\|_1 \le C(N^{-(r-1)} + \tau^2),$$

where C is a constant independent of N and τ.

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