SOLUTION FOR A NON-STATIONARY RADIATIVE TRANSFER EQUATION

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Abstract

The operators radiative transfer equation constructed by Chandrasekhar has been extended to the non-stationary case by Bellman and Wang. The local existence of solution of such non-stationary equation is established based on the construction of scattering matrices from a co-propagation group with unbounded generator. In case the system is dissipative, the local existended to the global existence.

1. Introduction

Based on the "Principle of Invariance," Chandrasekhar [1] established differential-integral equations which govern radiative transfer of diffuse reflection and transmission by plane-parallel atmospheres of arbitrary optical thickness and stationary. Bellman [2] and Wang [3,4] have extended Chandrasekhar's result to the non-stationary case. The non-stationary scattering matrix is

$$S = S(x,y;\Omega,\Omega_0;t,t_0) = \begin{pmatrix} t & \rho \\ r & \tau \end{pmatrix}, \tag{1.1}$$

where x, y are the spacial point, Ω and Ω_0 are input and output direction cosines, t and t_0 are input and output times. The left-hand reflection operator ρ satisfies a differential-integro ono-linear equation of the form

$$-\frac{\partial\rho}{\partial x} + \beta(x)\frac{\partial\rho}{\partial t} - \delta(x)\frac{\partial\rho}{\partial t_0} = a(x) + d(x)\rho + \rho b(x) + \rho c(x)\rho, \tag{1.2}$$

where β and δ are propagation coefficients, and a, b, c, d are bounded compact integral operators. For more details and other operators differential-integro equations for t, τ, and r, see Wang [4]. It should be pointed out that once ρ is solved, other operators t, τ and r can be solved by a system of linear equations. Therefore equation (1.2) is the most important and interesting one. The purpose of this paper is to find local and global solutions for ρ in equation (1.2), and more generally for S. Operators t, τ, ρ and r are nonpredictive.

2. Propagation Operator $\overrightarrow{S}$

Using the propgation operator [5],

$$\overrightarrow{S} = \overrightarrow{S}(x,y) = \overrightarrow{S}(x,y;\Omega,\Omega_0;t,t_0) = \begin{pmatrix} \overrightarrow{t} & \overrightarrow{\rho} \\ \overrightarrow{r} & \overrightarrow{\tau} \end{pmatrix} \tag{2.1}$$

with stable generator

$$M(x) = \begin{pmatrix} B(x) & A(x) \\ -C(x) & -D(x) \end{pmatrix}, \tag{2.2}$$

satisfying: (i) There is a Banach space Y, continuously and densely embedded in H with $Y \subset$ Domain $B(x)$ and $Y \subset$ Domain $D(x)$. Each $B(x)$ and $D(x)$ generate C_0-groups of operators on H, and the families $\{B(x)\}$ and $\{-D(x)\}$ generate propagation operators on $H, G_1(x,y), G_2(x,y)$ respectively with $G_1(Y) \subset Y$ and $G_2(Y) \subset Y$. (2.3)

(ii) For each $x, M(x)$ is closed densely defined, with $Y \oplus Y \subset$ Domain $(M(x))$, and generates a C_0-group on $H \oplus H$, and the family $\{M(x)\}$ is stable and generates propagation operators $\{\overrightarrow{S}(x,y)\}$ on $H \oplus H$ such that,

(a) $\overrightarrow{S}(x,y)$ is strongly continuous in x and y jointly.

(b) $\overrightarrow{S}(x,y)(H \oplus H) \subset H \oplus H$.

(c) for $\begin{pmatrix} f \\ k \end{pmatrix} \in H \oplus H, x \leq y$,

$$\frac{d}{dy}\overrightarrow{S}(x,y)\begin{pmatrix} f \\ k \end{pmatrix} = M(y)\overrightarrow{S}(x,y)\begin{pmatrix} f \\ k \end{pmatrix}.$$

To show dependencies of S and $\overrightarrow{S}$ on (x,y) we have used $S = S(x,y)$ and $\overrightarrow{S} = \overrightarrow{S}(x,y)$. If $\overrightarrow{S}(x,y)$ is a propagation group, we denote

$$\overrightarrow{S}^{-1} = \overleftarrow{S} = \begin{pmatrix} \overleftarrow{t} & \overleftarrow{\rho} \\ \overleftarrow{r} & \overleftarrow{\tau} \end{pmatrix}.$$

The following Lemmas are stated. Details of proofs are not presented here.

Lemma 2.1. *If $M(x)$ satisfies (2.3) and A and C are compact valued continuous function of x, then the operators $\rho(x,y)$ and $r(x,y)$ are compact operators on H.*

Lemma 2.2. *Under the conditions of Lemma 2.1, $\overrightarrow{\tau}, \overleftarrow{\tau}, \overrightarrow{t}$ and $\overleftarrow{t}$ are Fredholm.*

Lemma 2.3. *Under the condition of Lemma 2.1, for each x there exist $\varepsilon > 0$, such that if $x \le y \le x+\varepsilon$, then $\overrightarrow{\tau}(x,y)$ has a bounded inverse.*

It is well known [4] that if $\overrightarrow{\tau}$ is invertible, then S is single valued. And S is related to $\overrightarrow{S}$ by the relation

$$t = \overrightarrow{t} - \overrightarrow{\rho}\,\overrightarrow{\tau}^{-1}\overrightarrow{r}, \quad \tau = \overrightarrow{\tau}^{-1}, \quad \rho = \overrightarrow{\rho}\,\overrightarrow{\tau}^{-1} \quad \text{and} \quad r = -\overrightarrow{\tau}^{-1}\overrightarrow{r}. \tag{2.5}$$

In addition,

$$\frac{d}{dy}S = \begin{pmatrix} 0 & A(y) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B(y) & 0 \\ 0 & 0 \end{pmatrix} S + S \begin{pmatrix} 0 & 0 \\ 0 & D(y) \end{pmatrix} + S \begin{pmatrix} 0 & 0 \\ C(y) & 0 \end{pmatrix} S, \tag{2.6}$$

with

$$S(x,x) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Summarize Lemma 1–3, and relations between $\overrightarrow{S}$ and S.

Theorem 2.1. *Under the condition of (2.3) with C and A continuous (in norm) and compact, for each x there exists $\varepsilon > 0$ such that $x \le y \le x+\varepsilon$ we can construct a bounded single valued transport matrix $S(x,y)$ and $S(x,x) = I$. In particular we have*

$$\frac{d}{dy}\rho = A(y) + B(y)\rho + \rho D(y) + \rho C(y)\rho, \quad \rho(x,x) = 0,$$

where $\rho = \rho(x,y)$.

In case the initial value of reflection operator is K instead of 0, then the solution is given by [7],

$$P(K; x,y) = \rho(x,y) + t(x,y)K(I - r(x,y)K)^{-1}\tau(x,y).$$

That the map P from $B(H)$ to $B(H)$ is weak operator topology continuous if $r(x,y)$ is compact was shown by Krein [8]. The converse is proved by Shew [5].

Assuming such stability for $x \ge 0$ we may partially extend the construction of S , in the non-dissipative case [see section 4] past the point

$$z = glb\{x | \overrightarrow{\tau}(0,x) \text{ is not invertible }\}.$$

Let $z < x_0 < x_1 < \cdots < x_n < \cdots$ be a sequence of points such that $\{M(x)\}$ satisfies the above conditions on $[0, x_n]$ for each n. Define H_n by

$$H_n^{\perp} = \bigoplus_{i=0}^{n}(\ \ker \overrightarrow{T}(0, x_n)).$$

Since $\overrightarrow{T}(0, x_n)$ is Fredholm $H_n^{\perp}$ is finite dimensional and restriction

$$\overrightarrow{T}(0, x_n) : H_n \to \overrightarrow{T}(0, x_n) H_n$$

has a bounded inverse. Thus, $S(0, x_n)$ can be constructed on all but a finite number of dimensions, and S is partially extended beyond the conjugate point, z.

3. Non–stationary Radiative Transfer

To investigate the non-stationary radiative transfer of diffuse reflection and transmission, we shall consider, Wang [4], the decomposition of

$$M(x) = E(x) + e(x), \tag{3.1}$$

where $E(x) = \begin{pmatrix} \beta(x)\frac{\partial}{\partial t} & 0 \\ 0 & -\delta(x)\frac{\partial}{\partial t} \end{pmatrix}$ and $e(x) = \begin{pmatrix} b(x) & a(x) \\ -c(x) & -d(x) \end{pmatrix}$.

The operators $a(x), b(x), c(x)$ and $d(x)$ are compact operators (on H) valued. And $\beta(x), \delta(x)$ are bounded and continuous. It is well-known [9] that each $E(x)$, and thus $M(x)$, is closable (identify E and M with their closure) densed defind, and generates a group on $H \oplus H$. The operator $e(x)$ is continuous in x in the uniform operator topology on $B(H \oplus H)$, the family $\{M(x)\}_{x \geq 0}$ is stable.

One takes the space Y to be

$$Y = H'((-\infty, \infty), R^n) = \{f \in H : \frac{\partial}{\partial t} f \in H \text{ and } \|f\|_Y^2 < \infty\},$$

where

$$\|f\|_Y = \int_R (1 + |z|^2)[\frac{1}{\sqrt{2\pi}} \int_R e^{izs} f(s) ds]^2 dz.$$

Noting that Y is continuously embedded in Domain $(\frac{\partial}{\partial t})$ [cf. 10] and that Y is dense in H and that B, D and M are identified with their closures we may take Y to be Domain (B) and Domain (D) and $Y \oplus Y$ to be Domain (M). Since $\{M(x)\}_{x \geq 0}$ is stable, conditions (2.3 (ii) a and b) of Section 2 are satisfied [cf. 11]. It remains to show that the forward and backward propagation operators $\overrightarrow{S}(x, y)$ and $\overleftarrow{S}(x, y)$ so generated by $M(x)$ satisfy the regularity condition of (2.3 (ii) c) of Section 2, i.e., invariance of Y under $\overrightarrow{S}$ and $\overleftarrow{S}$ and the differentiability with respect to the second argument.

To this end, note that $k = \left(I - \frac{\partial^2}{\partial t^2}\right)^{\frac{1}{2}}$ is an isometry from Y to H, [10]. Let

$$K = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix},$$

then K is an isometry from $Y \oplus Y$ to $H \oplus H$. Let $\mathcal{L}$ denote the Schwartz class of rapidly decreasing functions at infinity. Then for $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{L} \oplus \mathcal{L}$, and each x, it can be shown that

$$KM(x)K^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = M(x)\begin{pmatrix} u \\ v \end{pmatrix} + (KE(x) - E(x)K)K^{-1}\begin{pmatrix} u \\ v \end{pmatrix} + (Ke(x) - e(x)K)K^{-1}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.2}$$

Since $\beta(x), \delta(x), a(x), b(x), c(x)$, and $d(x)$ are independent of t, and since K commutes with $\frac{\partial}{\partial t}$, the operator actions on $\mathcal{L} \oplus \mathcal{L}$ determined by the commutators $(KE - EK)$ and $(Ke - eK)$ can be extended to bounded operators from $H \oplus H$ to $H \oplus H$, [12]. Since $\left\| \frac{\partial}{\partial t}\left(I - \frac{\partial^2}{\partial t^2}\right)^{\frac{1}{2}} \right\|_H \leq 1$, and β and δ are bounded continuous in x, and a, b, c, d bounded continuous in x in the uniform operator topology of $B(H)$ we have by (3.2), for $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{L} \oplus \mathcal{L}$ that

$$KM(x)K^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = M(x)\begin{pmatrix} u \\ v \end{pmatrix} + P(x)\begin{pmatrix} u \\ v \end{pmatrix},$$

where $P(x) \in B(H \oplus H)$ is continuous in the uniform operator topology. We can extend (3.3) to $Y \oplus Y$. Details are not presented here. And

$$\text{Domain}(M(x) + P(x)) = \text{Domain} KM(x)K^{-1}.$$

Hence

$$KM(x)K^{-1} = M(x) + P(x).$$

where $P(\cdot) \in B(H \oplus H)$ is strongly continuous (indeed, norm continuous).

Under this condition, the work of Kato [13, 14] and (see theorem 6.3.7 [11]), assures that the propagation group $\overrightarrow{S}(x, y)$ associated with (3.1) satisfies the regularity condition (2.3 (ii) c), and the construction of the scatering operator $S(x, y)$ follows the development in Section 2.

4. A Dissipative Case

If the output energy is always less than or equal to the input energy, the system is called dissipative. It is equivalent to the condition $\|S(x,y)\|^2 = S^*(x,y)S(x,y) \leq 1$. S is called locally dissipative (at x) if there exists a $\Delta(x) > 0$ such that $S(x,y)$ is dissipative for $x \leq y \leq x + \Delta(x)$.

Theorem 4.1. *Under the assumption of Theorem 2.1 and S locally dissipative for all $x \leq y$ the result of Theorem (2.1) can be extended to all finite $y \geq x$, i.e. the solution for (2.6) exists for all finite $y \geq x$.*

Proof. The finite interval (x,y) is partitioned into $x = x_0, x_1, x_2, \cdots, x_n = y$ such that for all i,

$$x_{i+1} - x_i \leq \text{Min}[\Delta(x_i), \varepsilon(x_i)]$$

where $\varepsilon(x_i)$ is as given in Lemma 2.3 and $\Delta(x_i)$ is as in the above definition of locally dissipative. For convenience, let $\overrightarrow{\tau_i}$, $\overrightarrow{S_i}$ and S_i and etc., denote $\overrightarrow{\tau}(x_i, x_i + \Delta x_i)$, $\overrightarrow{S}_i(x_i, x_i + \Delta x_i)$ and $S(x_i, x_i + \Delta x_i)$ and etc. It follows from Lemma 2.3, $\overrightarrow{\tau_i} \in B(Y,Y)$ is nonsingular for all i and $\tau_i = \overrightarrow{\tau}_i^{-1} \in B(Y,Y)$ is also nonsingular, and,

$$\|\tau_i\|^2 = \tau_i^* \tau_i > 0. \tag{4.1}$$

Since each S_i exists and is locally dissipative, it follows that

$$\|r_i\| \leq 1 \quad \text{and} \quad \|\rho_i\| \leq 1, \quad \text{for all } i.$$

Furthermore,

$$1 \geq S_i^* S_i \quad \text{and} \quad 1 \geq r_i^* r_i + \tau_i^* \tau_i. \tag{4.2}$$

Therefore, $\|r_i\| < 1$ and $(E - r_{i+1}\rho_i)^{-1} \in B[Y,Y]$. Hence, by the star product, [cf. 7],

$$r(x_i, x_{i+2}) = \tau_i(E - r_{i+1}\rho_i)^{-1}\tau_{i+1} \tag{4.3}$$

is well defined as an element of $B(H,H)$ and has bounded inverse $\overrightarrow{\tau}_i(x_i, x_{i+2})$. Therefore $\overrightarrow{S}(x_i, x_{i+2}) \in B[H \oplus H, H \oplus H]$. $S(x_i, x_{i+2}) \in B[H \oplus H, H \oplus H]$ and is a solution of (2.6) related to (3.1) on the interval (x_i, x_{i+2}). By repeating, we have $\overrightarrow{\tau}(x,y)$ has a bounded inverse and result followed.

The remaining section will apply the above theorems to the time-dependent radiative transfer equation. We shall establish the existence of a scattering solution based on a local dissipative condition obtained by the following analysis.

Let us consider the generator for S, $\hat{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M$, with M given in (3.1). Then local regularity for S, i.e.,

$$\lim_{n \to 0} \frac{1}{h}(S(x+h, x) - E) = \hat{M}(x). \tag{4.4}$$

If $S = S(z, z + \Delta z)$ is locally dissipative, then for $\Phi \in Y \oplus Y$, then

$$0 \geq \Phi^*[\hat{M}^*(z) + \hat{M}(z)]\Phi, \tag{4.5}$$

where the notation $\psi^* A\psi$, means inner product $< A\psi, \psi >$ in H. We assume strict locally dissipative, i.e. $\geq$ in (4.5) is replaced by $>$. Then (4.5) reduces to

$$0 > \Phi^*[e^*(z) + e(z)]\Phi \tag{4.6}$$

for β and δ are independent of t.

Lemma 4.1. *If the condition of theroem* (2.1) *and* (4.6) *are satisfied then equation* (1.2) *has a solution for all finite* $y \geq x$, *i.e. the strict locally dissipative condition implies the global existence of solution.*

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