A SPECTRAL-DIFFERENCE METHOD FOR SOLVING TWO-DIMENSIONAL VORTICITY EQUATIONS*

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Abstract

We construct a spectral-difference scheme for solving two-dimensional vorticity equation with a single periodical boundary condition. The conservation, the generalized stability and the convergence are proved. Both steady and unsteady problems are considered.

§ 1. Introduction

Let $\xi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ be the vorticity function and the stream function respectively. ν is a positive constant. $f_j(x_1, x_2, t)$ and $\xi_0(x_1, x_2)$ are given. Let

$$I=\{x_2/0<x_2<2\pi\}, \quad Q=\{(x_1, x_2)/0<x_1<1,\ x_2\in I\}.$$

We consider the following two-dimensional vorticity equation

$$\begin{cases} \dfrac{\partial\xi}{\partial t}+\dfrac{\partial\psi}{\partial x_2}\dfrac{\partial\xi}{\partial x_1}-\dfrac{\partial\psi}{\partial x_1}\dfrac{\partial\xi}{\partial x_2}-\nu\Big(\dfrac{\partial^2\xi}{\partial x_1^2}+\dfrac{\partial^2\xi}{\partial x_2^2}\Big)=f_1, & \text{in } Q\times(0, T], \\ -\dfrac{\partial^2\psi}{\partial x_1^2}-\dfrac{\partial^2\psi}{\partial x_2^2}=\xi+f_2, & \text{in } Q\times[0, T], \\ \xi(x_1, x_2, 0)=\xi_0(x_1, x_2), & \text{in } \bar{Q}. \end{cases} \tag{1.1}$$

There is a lot of literature concerning the finite element methods and difference methods for solving (1.1); see, e.g., Raviart[1] and Guo Ben-yu[2,3]. But for any fixed scheme, the accuracy of the approximate solution is limited even if the solution of (1.1) is infinitely smooth.

In the past ten years, the spectral method for P. D. E. has developed rapidly, see Gottlieb, Orszag[4], Pasciak[5], Kreiss, Oliger[6] and Guo Ben-yu[7]. In particular, Guo Ben-yu[8] and Ma He-ping, Guo Ben-yu[9] proposed some spectral and pseudospectral schemes to solve (1.1). But all their works are for periodical problems.

In this paper, we assume that all functions are periodical only for the variable x_2 and thus we cannot use the full Fourier-spectral method. Such problems take place in the study of fluid flow in a tub. Following [10], we construct a class of spectral-difference scheme by using the Fourier-spectral method for the variable x_2 and the difference method for the variable x_1. If we choose the parameters in the scheme suitably, then the semi-discrete energy is kept unchanged. We strictly prove

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the generalized stability (see Guo Ben-yu[2, 11], and Griffiths[12]), from which the convergence follows with some assumption.

§ 2. The Scheme and the Conservations of the Approximate Solution

Suppose that all functions in (1.1) have the period 2π for the variable x_2.

Let h be the mesh spacing of x_1, $Mh=1$ and $I_h=\{x_1=jh/1\leqslant j\leqslant M-1\}$, $Q_h=I_h\times I$. Let τ be the mesh spacing of t, $S_\tau=\{t=k\tau/k=0, 1, 2, \cdots\}$. We define

$$u_{x_1}(x_1, x_2, t)=\frac{1}{h}(u(x_1+h, x_2, t)-u(x_1, x_2, t)),$$

$$u_{\bar{x}_1}(x_1, x_2, t)=u_{x_1}(x_1-h, x_2, t),$$

$$u_{\hat{x}_1}(x_1, x_2, t)=\frac{1}{2}(u_{x_1}(x_1, x_2, t)+u_{\bar{x}_1}(x_1, x_2, t)),$$

$$\Delta u(x_1, x_2, t)=u_{x_1,\bar{x}_1}(x_1, x_2, t)+\frac{\partial^2 u}{\partial x_2^2}(x_1, x_2, t),$$

$$u_t(x_1, x_2, t)=\frac{1}{\tau}(u(x_1, x_2, t+\tau)-u(x_1, x_2, t)).$$

The key problem for constructing a reasonable scheme is to simulate as many as possible the properties of the solution of (1.1). Indeed we have the following conservations

$$\begin{aligned}&\iint_Q \xi(x_1, x_2, t)dx_1dx_2\\&+\int_0^t\left[\int_I\left\{\frac{\partial\psi}{\partial x_2}(1, x_2, y)\xi(1, x_2, y)-\frac{\partial\psi}{\partial x_2}(0, x_2, y)\xi(0, x_2, y)\right\}dx_2\right]dy\\&-\nu\int_0^t\left[\int_I\left\{\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\Big|_{x_1=1}-\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\Big|_{x_1=0}\right\}dx_2\right]dy\\&=\iint_Q\xi_0(x_1, x_2)dx_1dx_2+\int_0^t\left[\iint_Q f_1(x_1, x_2, y)dx_1dx_2\right]dy\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\iint_Q \xi^2(x_1, x_2, t)dx_1dx_2\\&+\int_0^t\left[\int_I\left\{\frac{\partial\psi}{\partial x_2}(1, x_2, y)\xi^2(1, x_2, y)-\frac{\partial\psi}{\partial x_2}(0, x_2, y)\xi^2(0, x_2, y)\right\}dx_2\right]dy\\&+2\nu\int_0^t\left[\iint_Q\left\{\left(\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\right)^2+\left(\frac{\partial\xi}{\partial x_2}(x_1, x_2, y)\right)^2\right\}dx_1dx_2\right]dy\\&-2\nu\int_0^t\left[\int_I\left\{\xi(x_1, x_2, y)\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\Big|_{x_1=1}\right.\right.\\&\left.\left.-\xi(x_1, x_2, y)\frac{\partial\xi}{\partial x_1}(x_1, x_2, y)\Big|_{x_1=0}\right\}dx_2\right]dy\\&=\iint_Q\xi_0^2(x_1, x_2)dx_1dx_2+2\int_0^t\left[\iint_Q\xi(x_1, x_2, y)f_1(x_1, x_2, y)dx_1dx_2\right]dy.\end{aligned}\tag{2.2}$$

We shall construct a scheme, the solution of which satisfies the conservations similar to (2.1) and (2.2). Noticing that

$$\frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_1}-\frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_2}=\frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_2}u\right)-\frac{\partial}{\partial x_2}\left(\frac{\partial w}{\partial x_1}u\right)$$
$$=\frac{\partial}{\partial x_2}\left(w\frac{\partial u}{\partial x_1}\right)-\frac{\partial}{\partial x_1}\left(w\frac{\partial u}{\partial x_2}\right),$$

we define

$$J_1(u,\ w)=\frac{\partial w}{\partial x_2}u_{\hat{x}_1}-w_{\hat{x}_1}\frac{\partial u}{\partial x_2},$$
$$J_2(u,\ w)=\left(\frac{\partial w}{\partial x_2}u\right)_{\hat{x}_1}-\frac{\partial}{\partial x_2}(w_{\hat{x}_1},\ u),$$
$$J_3(u,\ w)=\frac{\partial}{\partial x_2}(wu_{\hat{x}_1})-\left(w\frac{\partial u}{\partial x_2}\right)_{\hat{x}_1}$$

and

$$J^{(\alpha)}(u,\ w)=\sum_{l=1}^{3}\alpha_l J_l(u,\ w),$$

where $\alpha=(\alpha_1,\ \alpha_2,\ \alpha_3)$, $\alpha_l\geqslant 0$ and $\alpha_1+\alpha_2+\alpha_3=1$.

Let $V_N=\text{span}\{e^{inx_2}/|n|\leqslant N\}$ and P_N be the orthogonal projection operator, i.e.,

$$\int_I P_N u\cdot\bar{v}\,dx_2=\int_I u\cdot\bar{v}\,dx_2,\quad \forall v\in V_N.$$

Let $\eta^{(N)}$ and $\varphi^{(N)}$ be the approximations to ξ and ψ respectively, where

$$z^{(N)}(x_1,\ x_2,\ t)=\sum_{|n|<N} z_n^{(N)}(x,\ t)e^{inx_2},\quad z=\eta \text{ or } \varphi.$$

The spectral-difference scheme for (1.1) is the following

$$\begin{cases}\eta_t^{(N)}+P_N J^{(\alpha)}(\eta^{(N)}+\delta\tau\eta_t^{(N)},\ \varphi^{(N)})-\nu\Delta(\eta^{(N)}+\sigma\tau\eta_t^{(N)})=P_N f_1, & \text{in } Q_h\times S_\tau,\\ -\Delta\varphi^{(N)}=\eta^{(N)}+P_N f_2, & \text{in } Q_h\times S_\tau,\\ \eta^{(N)}(x_1,\ x_2,\ 0)=\eta_0^{(N)}(x_1,\ x_2)=P_N\xi_0(x_1,\ x_2), & \text{in } \bar{Q}_h,\end{cases}\tag{2.3}$$

where δ and σ are parameters, $0\leqslant\delta,\ \sigma\leqslant 1$. If $\delta=\sigma=0$, then (2.3) is an explicit scheme. Otherwise we need the iteration to solve $\eta^{(N)}(x_1,\ x_2,\ t)$ for each $t\in S_\tau$. But only the one-dimensional iteration is needed, because we apply the spectral method to the variable x_2. This is one of the advantages of (2.3).

We are going to check the conservations. We first introduce some notations as follows:

$$(u(x_1),\ v(x_1))_I=\frac{1}{2\pi}\int_I u(x_1,\ x_2)\bar{v}(x_1,\ x_2)dx_2,$$
$$(u(x_2),\ v(x_2))_{I_h}=h\sum_{x_1\in I_h}u(x_1,\ x_2)\bar{v}(x_1,\ x_2),$$
$$(u,\ v)=h\sum_{x_1\in I_h}(u(x_1),\ v(x_1))_I,$$
$$\|u(x_1)\|_I^2=(u(x_1),\ u(x_1))_I,\quad \|u(x_1)\|_{I_h}^2=(u(x_2),\ u(x_2))_{I_h},\quad \|u\|^2=(u,\ u),$$
$$|u|_1^2=\frac{1}{2}\|u_{x_1}\|^2+\frac{1}{2}\|u_{\bar{x}_1}\|^2+\left\|\frac{\partial u}{\partial x_2}\right\|^2,$$

$$|u|_2^2=\frac{1}{2}\|u_{x_1\bar{x}_1}\|^2+\frac{h}{4}\sum_{\substack{x_1\in I_h\\x_1<1-2h}}\|u_{x_1x_1}(x_1)\|_I^2+\frac{h}{4}\sum_{\substack{x_1\in I_h\\x_1>2h}}\|u_{\bar{x}_1\bar{x}_1}(x_1)\|_I^2$$
$$+\left\|\frac{\partial}{\partial x_2}u_{x_1}\right\|^2+\left\|\frac{\partial}{\partial x_2}u_{\bar{x}_1}\right\|^2+\left\|\frac{\partial^2 u}{\partial x_2^2}\right\|^2,$$
$$S(u)=\frac{1}{2h}(\|u(h)\|_I^2+\|u(1-h)\|_I^2).$$

From the Abel formula, we have

$$(u_{\hat{x}_1}, v)+(v_{\hat{x}_1}, u)=\frac{1}{2}[(u(1), v(1-h))_I+(u(1-h), v(1))_I$$
$$-(u(h), v(0))_I-(u(0), v(h))_I], \tag{2.4}$$
$$\left(\frac{\partial u}{\partial x_2}, v\right)+\left(\frac{\partial v}{\partial x_2}, u\right)=0, \tag{2.5}$$

which leads to

$$(J_1, (u, w), 1)=\left(\frac{\partial w}{\partial x_2}, u_{\hat{x}_1}\right)+\left(\left(\frac{\partial w}{\partial x_2}\right)_{\hat{x}_1}, u\right)=A_1(u, w), \tag{2.6}$$

where

$$A_1(u, w)=\frac{1}{2}\left[\left(u(1), \frac{\partial w}{\partial x_2}(1-h)\right)_I+\left(u(1-h), \frac{\partial w}{\partial x_2}(1)\right)_I\right.$$
$$\left.-\left(u(h), \frac{\partial w}{\partial x_2}(0)\right)_I-\left(u(0), \frac{\partial w}{\partial x_2}(h)\right)_I\right]. \tag{2.7}$$

Similarly,

$$(J_2(u, w), 1)=\left(\left(\frac{\partial w}{\partial x_2}u\right)_{\hat{x}_1}, 1\right)=A_2(u, w)$$

and

$$(J_3(u, w), 1)=-\left(\left(w\frac{\partial u}{\partial x_2}\right)_{\hat{x}_1}, 1\right)=-A_2(w, u)=A_2(u, w), \tag{2.8}$$

where

$$A_2(u, w)=\frac{1}{2}\left[\left(u(1), \frac{\partial w}{\partial x_2}(1)\right)_I+\left(u(1-h), \frac{\partial w}{\partial x_2}(1-h)\right)_I\right.$$
$$\left.-\left(u(h), \frac{\partial w}{\partial x_2}(h)\right)_I-\left(u(0), \frac{\partial w}{\partial x_2}(0)\right)_I\right].$$

We have from (2.4) and (2.5) that

$$\left(\frac{\partial w}{\partial x_2}u_{\hat{x}_1}, v\right)+\left(\left(\frac{\partial w}{\partial x_2}v\right)_{\hat{x}_1}, u\right)=A_3(u, v, w),$$
$$\left(w_{\hat{x}_1}\frac{\partial u}{\partial x_2}, v\right)+\left(\frac{\partial}{\partial x_2}(w_{\hat{x}_1}, v), u\right)=0,$$

where

$$A_3(u, v, w)=\frac{1}{2}\left[\left(u(1), v(1-h)\frac{\partial w}{\partial x_2}(1-h)\right)_I+\left(u(1-h), v(1)\frac{\partial w}{\partial x_2}(1)\right)_I\right.$$
$$\left.-\left(u(h), v(0)\frac{\partial w}{\partial x_2}(0)\right)_I-\left(u(0), v(h)\frac{\partial w}{\partial x_2}(h)\right)_I\right].$$

Thus

$$(J_1(u, w), v) + (J_2(v, w), u) = A_3(u, v, w). \tag{2.9}$$

Similarly, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial x_2}(wu_{\hat{x}_1}), v\right) + \left(\frac{\partial v}{\partial_2}, wu_{\hat{x}_1}\right) = 0,\\ &-\left(\left(w\frac{\partial u}{\partial x_2}\right)_{\hat{x}_1}, v\right) - \left(\frac{\partial u}{\partial x_2}, wv_{\hat{x}_1}\right) = A_4(u, v, w),\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}A_4(u, v, w) = -\frac{1}{2}\Big[&\left(\frac{\partial u}{\partial x_2}(1)w(1), v(1-h)\right)_I + \left(\frac{\partial u}{\partial x_2}(1-h)w(1-h), v(1)\right)_I\\ &-\left(\frac{\partial u}{\partial x_2}(h)w(h), v(0)\right)_I - \left(\frac{\partial u}{\partial x_2}(0)w(0), v(h)\right)_I\Big]\end{aligned}$$

and thus

$$\left(J_3(u, w), v\right) + \left(\frac{\partial v}{\partial x_2}, wu_{\hat{x}_1}\right) - \left(\frac{\partial u}{\partial x_2}, wv_{\hat{x}_1}\right) = A_4(u, v, w),$$

from which and (2.6)—(2.10), we obtain

$$(J^{(\alpha)}(u, w), 1) = \alpha_1 A_1(u, w) + (\alpha_2 + \alpha_3) A_2(u, w) \tag{2.11}$$

and for $\alpha_1 = \alpha_2$,

$$\begin{aligned}(J^{(\alpha)}(u, w), v) + (J^{(\alpha)}(v, w), u) = \alpha_1 A_3(u, v, w) + \alpha_1 A_3(v, u, w)&\\ + \alpha_3 A_4(u, v, w) + \alpha_3 A_4(v, u, w).&\end{aligned} \tag{2.12}$$

In particular, if $\alpha_1 = \alpha_2$, then

$$(J^{(\alpha)}(u, w), u) = \alpha_1 A_3(u, u, w) + \alpha_3 A_4(u, u, w).$$

It is easy to show that

$$(u, \Delta v) + \frac{1}{2}(u_{x_1}, v_{x_1}) + \frac{1}{2}(u_{\bar{x}_1}, v_{\bar{x}_1}) + \left(\frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2}\right) = B(u, v), \tag{2.13}$$

where

$$B(u, v) = \frac{1}{2}(u(1) + u(1-h), v_{\bar{x}_1}(1))_I - \frac{1}{2}(u(h) + u(0), v_{x_1}(0))_I.$$

In particular,

$$(\Delta u, u) + |u|_1^2 = B(u, u). \tag{2.14}$$

We next check the conservations of the solution of (2.3). Firstly, we sum up the first formula of (2.2) for all $(x_1, x_2) \in Q_h$ and get from (2.11) and (2.13) that

$$\begin{aligned}&(\eta^{(N)}(t), 1)_t + \alpha_1 A_1(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \varphi^{(N)}(t))\\ &+ (\alpha_2 + \alpha_3) A_2(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \varphi^{(N)}(t))\\ &- \nu B(1, \eta^{(N)}(t) + \sigma\tau\eta_t^{(N)}(t)) = (f_1(t), 1).\end{aligned}$$

Hence

$$\begin{aligned}&(\eta^{(N)}(t), 1)_t + \tau \sum_{\substack{y\in B_\tau\\ y\le t-\tau}} [\alpha_1 A_1(\eta^{(N)}(y) + \delta\tau\eta_t^{(N)}(y), \varphi^{(N)}(y))\\ &\quad + (\alpha_2 + \alpha_3) A_2(\eta^{(N)}(y) + \delta\tau\eta_t^{(N)}(y), \varphi^{(N)}(y)) - \nu B(1, \eta^{(N)}(y) + \sigma\tau\eta_t^{(N)}(y))]\\ &= (\eta^{(N)}(0), 1) + \tau \sum_{\substack{y\in B_\tau\\ y\le t-\tau}} (f_1(y), 1)\end{aligned}$$

which is a reasonable analogy to (2.1).

Secondly, we put $\delta=\sigma=\frac{1}{2}$, $\alpha_1=\alpha_2$, and (2.15)

$$\hat{\eta}^{(N)}(x_1, x_2, t)=\frac{1}{2}(\eta^{(N)}(x_1, x_2, t)+\eta^{(N)}(x_1, x_2, t+\tau)).$$

By multiplying the first formula of (2.3) by $2\hat{\eta}^{(N)}(x_1, x_2, t)$ and summing it up for all $(x_1, x_2)\in Q_h$, we have from (2.12)—(2.14) that

$$\begin{aligned}&\|\eta^{(N)}(t)\|_t^2+2\nu|\hat{\eta}^{(N)}(t)|_1^2+2\alpha_1A_3(\hat{\eta}^{(N)}(t), \hat{\eta}^{(N)}(t), \varphi^{(N)}(t))\\&\quad+2\alpha_3A_4(\hat{\eta}^{(N)}(t), \hat{\eta}^{(N)}(t), \varphi^{(N)}(t))-2\nu B(\hat{\eta}^{(N)}(t), \hat{\eta}^{(N)}(t))\\&=2(f_1(t), \hat{\eta}^{(N)}(t))\end{aligned}$$

and thus

$$\begin{aligned}&\|\eta^{(N)}(t)\|^2+2\tau\sum_{\substack{y\in S_\tau\\y\le t-\tau}}[\nu|\hat{\eta}^{(N)}(t)|_1^2+\alpha_1A_3(\hat{\eta}^{(N)}(y), \hat{\eta}^{(N)}(y), \varphi^{(N)}(y))\\&\quad+\alpha_3A_4(\hat{\eta}^{(N)}(y), \hat{\eta}^{(N)}(y), \varphi^{(N)}(y))-\nu B(\hat{\eta}^{(N)}(y), \hat{\eta}^{(N)}(y))]\\&=\|\eta^{(N)}(0)\|^2+2\tau\sum_{\substack{y\in S_\tau\\y\le t-\tau}}(f_1(y), \hat{\eta}^{(N)}(y))\end{aligned}\tag{2.16}$$

which is a reasonable analogy to (2.2). Therefore the scheme (2.3) can give better numerical results if $\alpha_1=\alpha_2$.

§ 3. Some Lemmas

In order to estimate the error, we need some lemmas.

Lemma 1. *For all $u(x_1, x_2, t)$, we have*

$$2(u(t), u_t(t))_I=(\|u(t)\|_I^2)_t-\tau\|u_t(t)\|_I^2,$$
$$2(u(t), u_t(t))=\|u(t)\|_t^2-\tau\|u_t(t)\|^2.$$

Lemma 2.

$$2(u_t(t), \Delta u(t))+(|u(t)|_1^2)_t-\tau|u_t(t)|_1^2=2B(u_t(t), u(t)),$$
$$2(u(t), \Delta u_t(t))+(|u(t)|_1^2)_t-\tau|u_t(t)|_1^2=2B(u(t), u_t(t)).$$

Lemma 3. *If $u(x_1, x_2)\in V_N$ for all $x_1\in I_h$, then* $\left\|\frac{\partial u}{\partial x_2}\right\|^2\le N^2\|u\|^2$.

Lemma 4. *For all $u(x_1, x_2)$, we have*

$$\|u_{\bar{x}_1}\|^2\le\frac{4}{h^2}\|u\|^2+h\|u_{x_1}(0)\|_I^2,\quad \|u_{x_1}\|^2\le\frac{4}{h^2}\|u\|^2+h\|u_{\bar{x}_1}(1)\|_I^2$$

and

$$\|u_{\bar{x}_1}\|^2\le\frac{4}{h^2}\|u\|^2+\frac{2}{h}\|u(0)\|_I^2,\quad \|u_{x_1}\|^2\le\frac{4}{h^2}\|u\|^2+\frac{2}{h}\|u(1)\|_I^2.$$

Lemma 5. *If $u(x_1, x_2)\in V_N$ for all $x_1\in I_h$ and $u(0, x_2)=u(1, x_2)=0$, then*

$$\|u\|^2\le c_1[|u|_1^2+S(u)],$$

where c_1 is a positive constant depending only on the domain Q_h.

Proof. Consider the eigenvalue problem

$$\begin{cases} -\Delta u(x_1, x_2) - \lambda u(x_1, x_2) = 0, & \text{in } Q_h, \\ u(x_1, x_2) = u(x_1, x_2 + 2\pi), & \text{in } Q_h, \\ u(0, x_2) = u(1, x_2) = 0, & x_2 \in I. \end{cases}$$

By taking the scalar product of the above equality with $u(x_1, x_2)$, we have

$$|u|_1^2 - B(u, u) = \lambda \|u\|^2.$$

Because $B(u, u) = -S(u)$, we get the conclusion.

Lemma 6. *If $u(x_1, x_2) \in V_N$ for all $x_1 \in I_h$ and $u(0, x_2) = u(1, x_2) = 0$, then*

$$\|\Delta u\|^2 = |u|_2^2 + \frac{h}{4}(\|u_{x_1x_1}(0)\|_I^2 + \|u_{\bar{x}_1\bar{x}_1}(1)\|_I^2) + \frac{1}{h}\left(\left\|\frac{\partial u}{\partial x_2}(h)\right\|_I^2 + \left\|\frac{\partial u}{\partial x_2}(1-h)\right\|_I^2\right).$$

Proof. We have

$$\|\Delta u\|^2 = \|u_{x_1\bar{x}_1}\|^2 + \left\|\frac{\partial^2 u}{\partial x_2^2}\right\|^2 + 2\left(u_{x_1\bar{x}_1}, \frac{\partial^2 u}{\partial x_2^2}\right).$$

Clearly

$$\|u_{x_1\bar{x}_1}\|^2 = \frac{1}{2}\|u_{x_1\bar{x}_1}\|^2 + \frac{h}{4}\Big(\sum_{\substack{x_1 \in I_h \\ x_1 < 1-2h}} \|u_{x_1x_1}(x_1)\|_I^2 + \sum_{\substack{x_1 \in I_h \\ x_1 > 2h}} \|u_{\bar{x}_1\bar{x}_1}(x_1)\|_I^2$$
$$+ \|u_{x_1x_1}(0)\|_I^2 + \|u_{\bar{x}_1\bar{x}_1}(1)\|_I^2\Big).$$

On the other hand,

$$2\left(u_{x_1\bar{x}_1}, \frac{\partial^2 u}{\partial x_2^2}\right) = \left\|\frac{\partial}{\partial x_2} u_{x_1}\right\|^2 + \left\|\frac{\partial}{\partial x_2} u_{\bar{x}_1}\right\|^2 + Q(u),$$

where

$$Q(u) = -\left(\frac{\partial u}{\partial x_2}(1) + \frac{\partial u}{\partial x_2}(1-h), \frac{\partial}{\partial x_2} u_{\bar{x}_1}(1)\right)_I$$
$$+ \left(\frac{\partial u}{\partial x_2}(h) + \frac{\partial u}{\partial x_2}(0), \frac{\partial}{\partial x_2} u_{x_1}(0)\right)_I$$
$$= \frac{1}{h}\left(\left\|\frac{\partial u}{\partial x_2}(h)\right\|_I^2 + \left\|\frac{\partial u}{\partial x_2}(1-h)\right\|_I^2\right).$$

Lemma 7. *If $h < 2\varepsilon$, then for all $x_1 \in I_h$,*

$$\|u(x_1)\|_I^2 \leq \varepsilon(\|u_{x_1}\|^2 + \|u_{\bar{x}_1}\|^2) + c_0(\varepsilon)\|u\|^2,$$

where $c_0(\varepsilon)$ is a positive constant depending only on ε and the domain Q_h.

Lemma 8. *If $u(x_1, x_2), v(x_1, x_2) \in V_N$ for all $x_1 \in I_h$, then*

$$\|u(x_1)v(x_1)\|_I^2 \leq (2N+1)\|u(x_1)\|_I^2 \|v(x_1)\|_I^2,$$

$$\|u(x_2)v(x_2)\|_{I_h}^2 \leq \frac{1}{h}\|u(x_2)\|_{I_h}^2 \|v(x_2)\|_{I_h}^2,$$

$$\|uv\|^2 \leq \frac{2N+1}{h}\|u\|^2\|v\|^2.$$

Lemma 9. *If the following conditions are fulfilled:*

(i) $u(x_1, x_2), v(x_1, x_2) \in V_N$ *for all* $x_1 \in I_h$,

(ii) $u(0, x_2) = 0$ *or* $u(1, x_2) = 0$,

(iii) *for all* $x_1 \in I_h$, $\int_I v(x_1, x_2)\,dx_2 = 0$, *then*

$$\|uv\|^2 \le \frac{8}{h}\sqrt{c_2 h(2N+1)}\,\|u\|\,\|v\|^{\frac{3}{2}}\,|u|_1|v|_1^{\frac{1}{2}},$$

where

$$c_2=\int_0^\infty \frac{dz}{(1+z^2)^2}.$$

Proof. Assume $u(0, x_2)=0$. Then

$$\max_{x_1\in I_h} u^2(x_1, x_2)=\max_{x_1\in I_h}\Big(h\sum_{\substack{y\in I_h\\ y<x_1}}[u^2(y, x_2)]_y\Big)\le 2\|u(x_2)\|_{I_h}\|u_{\bar{x}_1}(x_2)\|_{I_h},$$

from which and Lemma 8, we obtain

$$\begin{aligned}\|u^2\|^2 &\le \frac{2}{\pi}\int_0^{2\pi}\|u(x_2)\|_{I_h}^2\|u_{\bar{x}_1}(x_2)\|_{I_h}^2 dx_2\\ &\le \frac{2}{\pi}\Big(\int_0^{2\pi}\|u(x_2)\|_{I_h}^4 dx_2\Big)^{\frac{1}{2}}\Big(\int_0^{2\pi}\|u_{\bar{x}_1}(x_2)\|_{I_h}^4 dx_2\Big)^{\frac{1}{2}}\\ &\le 4(2N+1)\|u\|^2\|u_{\bar{x}_1}\|^2.\end{aligned}$$

Next, suppose

$$v(x_1, x_2)=\sum_{n=-\infty}^{\infty} v_n(x_1)e^{inx_2}.$$

From the Yong-Hausedoff inequality (see Hardy, Littlewood and Polya[13], we have

$$\|v^2(x_1)\|_I^2=\frac{1}{2\pi}\int_0^{2\pi}|v(x_1, x_2)|^4 dx_2 \le \Big(\sum_{n=-\infty}^{\infty}|v_n(x_1)|^{\frac{4}{3}}\Big)^3.$$

On the other hand,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}|v_n(x_1)|^{\frac{4}{3}} &= \sum_{n=-\infty}^{\infty}\left\{|v_n(x_1)|^2\left[1+\frac{n^2\|v(x_1)\|_I^2}{\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I^2}\right]\right\}^{\frac{2}{3}}\left\{\frac{1}{1+\frac{n^2\|v(x_1)\|_I^2}{\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I^2}}\right\}^{\frac{2}{3}}\\ &\le \left\{\sum_{n=-\infty}^{\infty}|v_n(x_1)|^2\left[1+\frac{n^2\|v(x_1)\|_I^2}{\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I^2}\right]\right\}^{\frac{2}{3}}\left\{\sum_{n=-\infty}^{\infty}\frac{1}{\left[1+\frac{n^2\|v(x_1)\|_I^2}{\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I^2}\right]^2}\right\}^{\frac{1}{3}}.\end{aligned}$$

Moreover,

$$\sum_{n=-\infty}^{\infty}|v_n(x_1)|^2\left[1+\frac{n^2\|v(x_1)\|^2}{\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I^2}\right]=2\|v(x_1)\|_I^2$$

and

$$\sum_{n=1}^{\infty}\frac{1}{\left[1+\frac{n^2\|v(x_1)\|_I^2}{\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I^2}\right]^2}\le\int_0^\infty\frac{dz}{\left(1+\frac{z^2\|v(x_1)\|_I^2}{\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I^2}\right)^2}=\frac{c_2\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I}{\|v(x_1)\|_I}.$$

Therefore,

$$\|v^2(x_1)\|_I^2 \leqslant 4c_2\|v(x_1)\|_I^3\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I,$$

from which and Lemma 8,

$$\|v^2\|^2 = 4c_2 h\sum_{x_1\in I_h}\|v(x_1)\|_I^3\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I$$

$$\leqslant 4c_2\Big(h\sum_{x_1\in I_h}\|v(x_1)\|_I^6\Big)^{\frac{1}{2}}\Big(h\sum_{x_1\in I_h}\left\|\frac{\partial v}{\partial x_2}(x_1)\right\|_I^2\Big)^{\frac{1}{2}} \leqslant \frac{4c_2}{h}\|v\|^3\left\|\frac{\partial v}{\partial x_2}\right\|.$$

The conclusion follows from the above statements and $\|uv\|^2 \leqslant \|u^2\|\,\|v^2\|$.

Lemma 10. *If the following conditions are fulfilled:*

(i) *$z(t)$ is non-negative function defined on S_τ,*

(ii) *ρ, a, b, M_i are non-negative constants,*

(iii) *$H(z)$ is such a function that if $z \leqslant M_3$, then $H(z) \leqslant 0$,*

(iv) *for all $t\in S_\tau$,*

$$z(t) \leqslant \rho + \sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}[M_1 z(y) + M_2 N^a h^{-b} z^2(y) + H(z(y))],$$

(v) *$\rho e^{(M_1+M_2)T} \leqslant \min\left(M_3, \frac{h^b}{N^a}\right)$ and $z(0) \leqslant \rho$,*

then for all $t\in S_\tau$ and $t\leqslant T$, we have

$$z(t) \leqslant \rho e^{(M_1+M_2)t}.$$

In particular, if $M_2=0$ and $H(z)\leqslant 0$ for all z, then for all ρ and t, we have $z(t) \leqslant \rho e^{M_1 t}$.

§ 4. The Error Estimation for the Problem with the First Boundary Condition

In this section, we suppose $\alpha_1=\alpha_2$, $\tau=O(h^2)$, $\tau=O\left(\frac{1}{N^2}\right)$ and

$$\eta^{(N)}(0, x_2, t) = P_N\xi(0, x_2, t) = P_N g_0(x_2, t),$$
$$\eta^{(N)}(1, x_2, t) = P_N\xi(1, x_2, t) = P_N g_1(x_2, t).$$

Let $\tilde f_1$, $\tilde f_2$, $\tilde\xi_0$, $\tilde g_0$ and $\tilde g_1$ be the errors of f_1, f_2, ξ_0, g_0 and g_1 respectively which induce the errors of $\eta^{(N)}$ and $\varphi^{(N)}$, denoted by $\tilde\eta^{(N)}$ and $\tilde\varphi^{(N)}$. For simplicity, we assume $\tilde\varphi^{(N)}(0, x_2, t) = \tilde\varphi^{(N)}(1, x_2, t) = 0$. The errors satisfy the following equation

$$\begin{cases} \tilde\eta_t^{(N)} + P_N J^{(\alpha)}(\tilde\eta^{(N)} + \delta\tau\tilde\eta_t^{(N)}, \varphi^{(N)} + \tilde\varphi^{(N)}) + P_N J^{(\alpha)}(\eta^{(N)} + \delta\tau\eta_t^{(N)}, \tilde\varphi^{(N)}) \\ \quad -\nu\Delta(\tilde\eta^{(N)} + \sigma\tau\tilde\eta_t^{(N)}) = P_N\tilde f_1, \quad \text{in } Q_h\times S_\tau, \\ -\Delta\tilde\varphi^{(N)} = \tilde\eta^{(N)} + P_N\tilde f_2, \quad \text{in } Q_h\times S_\tau, \\ \tilde\eta^{(N)}(x_1, x_2, 0) = P_N\tilde\xi_0(x_1, x_2), \quad \text{in } \bar Q_h. \end{cases} \tag{4.1}$$

By taking the scalar product of the first formula of (4.1) with $2\tilde\eta^{(N)}$, we obtain from (2.12), (2.13) and Lemmas 1 and 2 that

$$\|\tilde\eta^{(N)}(t)\|_t^2 - \tau\|\tilde\eta_t^{(N)}(t)\|^2 - 2\delta\tau(\tilde\eta_t^{(N)}(t), J^{(\alpha)}(\tilde\eta^{(N)}(t), \tilde\varphi^{(N)}(t)))$$
$$+2(\tilde\eta^{(N)}(t), J^{(\alpha)}(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \tilde\varphi^{(N)}(t)) + J^{(\alpha)}(\tilde\eta^{(N)}(t)$$
$$+\delta\tau\tilde\eta_t^{(N)}(t), \varphi^{(N)}(t))) + 2\nu|\tilde\eta^{(N)}(t)|_1^2 + \nu\sigma\tau(|\tilde\eta^{(N)}(t)|_1^2)_t$$

$$-\nu\sigma\tau^2|\tilde{\eta}_t^{(N)}(t)|_1^2+\sum_{l=1}^{6}D_l(t)+B_1(t)+B_2(t)=2(\tilde{\eta}^{(N)}(t),\tilde{f}_1(t)),\tag{4.2}$$

where

$$\begin{aligned}
D_1(t)&=2\alpha_1A_3(\tilde{\eta}^{(N)}(t),\tilde{\eta}^{(N)}(t),\tilde{\varphi}^{(N)}(t)),\\
D_2(t)&=2\alpha_3A_4(\tilde{\eta}^{(N)}(t),\tilde{\eta}^{(N)}(t),\tilde{\varphi}^{(N)}(t)),\\
D_3(t)&=-2\alpha_1\delta\tau A_3(\tilde{\eta}_t^{(N)}(t),\tilde{\eta}^{(N)}(t),\tilde{\varphi}^{(N)}(t)),\\
D_4(t)&=-2\alpha_1\delta\tau A_3(\tilde{\eta}^{(N)}(t),\tilde{\eta}_t^{(N)}(t),\tilde{\varphi}^{(N)}(t)),\\
D_5(t)&=-2\alpha_3\delta\tau A_4(\tilde{\eta}_t^{(N)}(t),\tilde{\eta}^{(N)}(t),\tilde{\varphi}^{(N)}(t)),\\
D_6(t)&=-2\alpha_3\delta\tau A_4(\tilde{\eta}^{(N)}(t),\tilde{\eta}_t^{(N)}(t),\tilde{\varphi}^{(N)}(t)),\\
B_1(t)&=-2\nu B(\tilde{\eta}^{(N)}(t),\tilde{\eta}^{(N)}(t)),\\
B_2(t)&=-2\nu\sigma\tau B(\tilde{\eta}^{(N)}(t),\tilde{\eta}_t^{(N)}(t)).
\end{aligned}$$

Let m be an undetermined positive constant. By taking the scalar product of the first formula of (4.1) with $m\tau\tilde{\eta}_t^{(N)}(t)$, we get

$$\begin{aligned}
&m\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+m\tau(\tilde{\eta}_t^{(N)}(t),J^{(\alpha)}(\tilde{\eta}^{(N)}(t),\tilde{\varphi}^{(N)}(t)))+m\tau(\tilde{\eta}_t^{(N)}(t),J^{(\alpha)}(\tilde{\eta}^{(N)}(t)\\
&\quad+\delta\tau\tilde{\eta}_t^{(N)}(t),\varphi^{(N)}(t))+J^{(\alpha)}(\eta^{(N)}(t)+\delta\tau\eta_t^{(N)}(t),\tilde{\varphi}^{(N)}(t)))\\
&\quad+\frac{m\nu\tau}{2}(|\tilde{\eta}^{(N)}(t)|_1^2)_t-\frac{m\nu\tau^2}{2}|\tilde{\eta}_t^{(N)}(t)|_1^2+m\nu\sigma\tau^2|\tilde{\eta}_t^{(N)}(t)|_1^2+D_7(t)\\
&\quad+D_8(t)+B_3(t)+B_4(t)=m\tau(\tilde{\eta}_t^{(N)}(t),\tilde{f}_1(t)),
\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}
D_7(t)&=m\delta\alpha_1\tau^2A_3(\tilde{\eta}_t^{(N)}(t),\tilde{\eta}_t^{(N)}(t),\tilde{\varphi}^{(N)}(t)),\\
D_8(t)&=m\delta\alpha_3\tau^2A_4(\tilde{\eta}_t^{(N)}(t),\tilde{\eta}_t^{(N)}(t),\tilde{\varphi}^{(N)}(t)),\\
B_3(t)&=-m\nu\tau B(\tilde{\eta}_t^{(N)}(t),\tilde{\eta}^{(N)}(t)),\\
B_4(t)&=-m\nu\sigma\tau^2B(\tilde{\eta}_t^{(N)}(t),\tilde{\eta}_t^{(N)}(t)).
\end{aligned}$$

Let $\varepsilon>0$ and c denote a positive constant which may be different in different formulas. Putting (4.2) and (4.3) together, we obtain

$$\begin{aligned}
&\|\tilde{\eta}^{(N)}(t)\|_t^2+\tau(m-1-\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2+2\nu|\tilde{\eta}^{(N)}(t)|_1^2+\nu\tau\Big(\sigma+\frac{m}{2}\Big)(|\tilde{\eta}^{(N)}(t)|_1^2)_t\\
&\quad+\nu\tau^2\Big(m\sigma-\sigma-\frac{m}{2}\Big)|\tilde{\eta}_t^{(N)}(t)|_1^2+\sum_{l=1}^{3}G_l(t)+\sum_{l=1}^{8}D_l(t)+\sum_{l=1}^{4}B_l(t)\\
&\leqslant\|\tilde{\eta}^{(N)}(t)\|^2+\Big(1+\frac{m^2\tau}{4\varepsilon}\Big)\|\tilde{f}_1(t)\|^2,
\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}
G_1(t)&=(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t),J^{(\alpha)}(\eta^{(N)}(t)+\delta\tau\eta_t^{(N)}(t),\tilde{\varphi}^{(N)}(t))),\\
G_2(t)&=(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t),J^{(\alpha)}(\tilde{\eta}^{(N)}(t)+\delta\tau\tilde{\eta}_t^{(N)}(t),\varphi^{(N)}(t))),\\
G_3(t)&=\tau(m-2\delta)(\tilde{\eta}_t^{(N)}(t),J^{(\alpha)}(\tilde{\eta}^{(N)}(t),\tilde{\varphi}^{(N)}(t))).
\end{aligned}$$

By taking the scalar product of the second formula of (4.1) with $\tilde{\varphi}^{(N)}(t)$, we have from (2.13)

$$\begin{aligned}
|\tilde{\varphi}^{(N)}(t)|_1^2+S(\tilde{\eta}^{(N)}(t))&=(\tilde{\varphi}^{(N)}(t),\tilde{\eta}^{(N)}(t)+\tilde{f}_2(t))\\
&\leqslant\frac{1}{2c_1}\|\tilde{\varphi}^{(N)}(t)\|^2+\frac{c_1}{2}(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2)
\end{aligned}$$

from which and Lemma 5,

$$|\tilde{\varphi}^{(N)}(t)|_1^2+S(\tilde{\eta}^{(N)}(t))\leqslant c_1(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2). \tag{4.5}$$

Now we are going to estimate the terms in (4.4). For simplicity, we shall use the following notations

$$\|\tilde{g}(t)\|_I^2=\|\tilde{g}_0(t)\|_I^2+\|\tilde{g}_1(t)\|_I^2,\quad \left\|\frac{\partial\tilde{g}}{\partial x_2}(t)\right\|^2=\left\|\frac{\partial\tilde{g}_0}{\partial x_2}(t)\right\|^2+\left\|\frac{\partial\tilde{g}_1}{\partial x_2}(t)\right\|^2,$$

$$\|\tilde{g}_t(t)\|_I^2=\|\tilde{g}_{0t}(t)\|_I^2+\|\tilde{g}_{1t}(t)\|_I^2,\qquad \|u\|_{a,\infty}=\max_{\substack{t\in S_\tau\\ t<T}}\|u(t)\|_{W^{a,\infty}(Q)}.$$

We first have from (4.5)

$$|G_1(t)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\frac{c}{\varepsilon}\|\eta^{(N)}\|_{1,\infty}^2(\|\tilde{\eta}^{(N)}(t)\|^2+|\tilde{\varphi}^{(N)}(t)|_1^2)$$

$$\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\frac{c}{\varepsilon}\|\eta^{(N)}\|_{1,\infty}^2(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2). \tag{4.6}$$

The computation gives

$$|G_2(t)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2$$

$$+\frac{c}{\varepsilon}\|\varphi^{(N)}(t)\|_{2,\infty}^2\left(\|\tilde{\eta}^{(N)}(t)\|^2+\frac{\tau}{h}\|\tilde{g}(t)\|_I^2+\tau h\|\tilde{g}_t(t)\|_I^2\right). \tag{4.7}$$

Lemma 8 leads to

$$|G_3(t)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\frac{c\tau N(m-2\delta)^2}{\varepsilon h}|\tilde{\varphi}^{(N)}(t)|_1^2|\tilde{\eta}^{(N)}(t)|_1^2$$

$$\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\frac{c\tau N(m-2\delta)^2}{\varepsilon h}(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2)|\tilde{\eta}^{(N)}(t)|_1^2. \tag{4.8}$$

Next, we estimate $|D_i(t)|$. From Lemma 8, we have

$$|D_1(t)|=2\alpha_1|A_3(\tilde{\eta}^{(N)}(t),\ \tilde{\eta}^{(N)}(t),\ \tilde{\varphi}^{(N)}(t))|$$

$$\leqslant\varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{chN}{\varepsilon}\|\tilde{g}(t)\|_I^2\left(\left\|\frac{\partial\tilde{\varphi}}{\partial x_2}(h,t)\right\|_I^2+\left\|\frac{\partial\tilde{\varphi}}{\partial x_2}(1-h,t)\right\|_I^2\right)$$

$$\leqslant\varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{cN}{\varepsilon}\|\tilde{g}(t)\|_I^2|\tilde{\varphi}^{(N)}(t)|_1^2$$

$$\leqslant\varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{cN}{\varepsilon}\|\tilde{g}(t)\|_I^2(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2). \tag{4.9}$$

It is clear that

$$\left(\frac{\partial u}{\partial x_2}(h),v(0)w(h)\right)_I=-\left(\frac{\partial w}{\partial x_2}(h),\ u(h)v(0)\right)_I-\left(\frac{\partial v}{\partial x_2}(0),\ u(h)w(h)\right)_I,\quad \text{etc.}$$

from which and Lemma 8,

$$|D_2(t)|\leqslant\varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{chN}{\varepsilon}\left(\|\tilde{g}(t)\|_I^2+\left\|\frac{\partial\tilde{g}}{\partial x_2}(t)\right\|_I^2\right)$$

$$\times\left(\|\tilde{\varphi}^{(N)}(h,t)\|_I^2+\|\tilde{\varphi}^{(N)}(1-h,t)\|_I^2+\left\|\frac{\partial\tilde{\varphi}^{(N)}}{\partial x_2}(h,t)\right\|_I^2+\left\|\frac{\partial\tilde{\varphi}^{(N)}}{\partial x_2}(1-h,t)\right\|_I^2\right)$$

$$\leqslant\varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{cN}{\varepsilon}\left(\|\tilde{g}(t)\|_I^2+\left\|\frac{\partial\tilde{g}}{\partial x_2}(t)\right\|_I^2\right)(\|\tilde{\varphi}^{(N)}(t)\|^2+|\tilde{\varphi}^{(N)}(t)|_1^2)$$

$$\leqslant\varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\frac{cN}{\varepsilon}\left(\|\tilde{g}(t)\|_I^2+\left\|\frac{\partial\tilde{g}}{\partial x_2}(t)\right\|_I^2\right)(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2). \tag{4.10}$$

Similarly,

$$|D_3(t)|+|D_4(t)|\leqslant\varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\varepsilon\nu\tau^2S(\tilde{\eta}_t^{(N)}(t))$$
$$+\frac{cN}{\varepsilon}(\|\tilde{g}(t)\|_I^2+\tau^2\|\tilde{g}_t(t)\|_I^2)(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2),\quad(4.11)$$

$$|D_5(t)|+|D_6(t)|\leqslant\varepsilon\nu S(\tilde{\eta}^{(N)}(t))+\varepsilon\nu\tau^2S(\tilde{\eta}_t^{(N)}(t))$$
$$+\frac{cN}{\varepsilon}\left(\|\tilde{g}(t)\|_I^2+\left\|\frac{\partial\tilde{g}}{\partial x_2}(t)\right\|_I^2+\tau^2\|\tilde{g}_t(t)\|_I^2+\tau^2\left\|\frac{\partial\tilde{g}}{\partial x_2}(t)\right\|_I^2\right)$$
$$\times(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2),\quad(4.12)$$

$$|D_7(t)|+|D_8(t)|\leqslant\varepsilon\nu\tau^2S(\tilde{\eta}_t^{(N)}(t))+\frac{cN\tau^2}{\varepsilon}\left(\|\tilde{g}_t(t)\|_I^2+\left\|\frac{\partial\tilde{g}_t}{\partial x_2}(t)\right\|_I^2\right)$$
$$\times(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2).\quad(4.13)$$

Finally, we estimate $B_l(t)$. From the boundary condition, we have

$$B_1(t)\geqslant2\nu S(\tilde{\eta}^{(N)}(t))+\frac{c}{\varepsilon h}\|\tilde{g}(t)\|_I^2.\quad(4.14)$$

By Lemma 1,

$$B_2(t)+B_3(t)\geqslant\nu\tau\left(\sigma+\frac{m}{2}\right)[S(\tilde{\eta}^{(N)}(t))]_t-\nu\tau^2\left(\sigma+\frac{m}{2}\right)S(\tilde{\eta}_t^{(N)}(t))$$
$$-\varepsilon\nu\tau^2S(\tilde{\eta}_t^{(N)}(t))-\varepsilon\nu S(\tilde{\eta}^{(N)}(t))$$
$$-\frac{c}{\varepsilon}[\tau h\|\tilde{g}_t(t)\|_I^2+h^{-1}\|\tilde{g}(t)\|_I^2].\quad(4.15)$$

Similarly,

$$B_4(t)\geqslant m\nu\sigma\tau^2S(\tilde{\eta}_t^{(N)}(t))-\frac{c\tau h}{\varepsilon}\|\tilde{g}_t(t)\|_I^2.\quad(4.16)$$

Substituting (4.6)—(4.16) into (4.4), we obtain

$$\|\tilde{\eta}^{(N)}(t)\|_t^2+\tau\left(m-1-4\varepsilon-\frac{\tau c}{\varepsilon}\|\varphi^{(N)}(t)\|_{2,\infty}\right)\|\tilde{\eta}_t^{(N)}(t)\|^2+\nu|\tilde{\eta}^{(N)}(t)|_1^2$$
$$+\nu\tau\left(\sigma+\frac{m}{2}\right)(|\tilde{\eta}^{(N)}(t)|_1^2)_t+\nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}\right)|\tilde{\eta}_t^{(N)}(t)|_1^2$$
$$+(2\nu-5\varepsilon)S(\tilde{\eta}^{(N)}(t))+\nu\tau\left(\sigma+\frac{m}{2}\right)[S(\tilde{\eta}^{(N)}(t))]_t$$
$$+\nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}-4\varepsilon\right)S(\tilde{\eta}_t^{(N)}(t))$$
$$\leqslant H_0(t)\|\tilde{\eta}^{(N)}(t)\|^2+H_1(t)|\tilde{\eta}^{(N)}(t)|_1^2+R(t),\quad(4.17)$$

where

$$H_0(t)=\frac{c}{\varepsilon}\left(\|\eta^{(N)}\|_{1,\infty}^2+\|\varphi^{(N)}\|_{1,\infty}^2+N\left(\|\tilde{g}(t)\|_I^2+\left\|\frac{\partial\tilde{g}}{\partial x_2}(t)\right\|_I^2\right.\right.$$
$$\left.\left.+\tau^2\|\tilde{g}_t(t)\|_I^2+\tau^2\left\|\frac{\partial\tilde{g}_t}{\partial x_2}(t)\right\|_I^2\right)\right),$$

$$H_1(t)=-\nu+\varepsilon+\frac{c\tau N(m-2\delta)^2}{\varepsilon h}(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2),$$

$$R(t)=c\left(1+\frac{1}{\varepsilon}\right)\|\tilde{f}_1(t)\|^2+\frac{c}{\varepsilon}\|\tilde{f}_2(t)\|^2$$

$$\times\Big[\ \|\tilde{\eta}^{(N)}\|_{1,\infty}^2+\Big(N\|\tilde{g}(t)\|_I^2+\Big\|\frac{\partial\tilde{g}}{\partial x_2}(t)\Big\|_I^2+\tau^2\|\tilde{g}_t(t)\|^2+\tau^2\Big\|\frac{\partial\tilde{g}_t}{\partial x_2}(t)\Big\|^2\Big)\Big]$$

$$+\frac{c}{\varepsilon h}(1+\|\varphi^{(N)}\|_{1,\infty}^2)(\|\tilde{g}(t)\|_I^2+\tau h^2\|\tilde{g}_t(t)\|_I^2).$$

Let ε be suitably small. We choose the value of m in the following way.

Case 1. $\sigma>\frac{1}{2}$. In this case we take

$$m>m_1=\max\Big(\frac{2\sigma+8\varepsilon}{2\sigma-1},\ 1+p_0+4\varepsilon\Big),\quad p_0\geqslant 0.$$

Then (4.17) leads to

$$\begin{aligned}&\|\tilde{\eta}^{(N)}(t)\|_t^2+p_0\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\nu\,|\tilde{\eta}^{(N)}(t)|_1^2+\nu S(\tilde{\eta}^{(N)}(t))\\&\quad+\nu\tau\Big(\sigma+\frac{m}{2}\Big)[\,|\tilde{\eta}^{(N)}(t)|_1^2+S(\tilde{\eta}^{(N)}(t))]_t\\&\quad\leqslant H_0(t)\|\tilde{\eta}^{(N)}(t)\|^2+H_1(t)\,|\tilde{\eta}^{(N)}(t)|_1^2+R(t).\end{aligned}\tag{4.18}$$

Case 2. $\sigma=\frac{1}{2}$. We take

$$m>m_2=1+p_0+\frac{1}{2}\ \nu\tau N^2+\frac{9\nu\tau}{4h^2}+\frac{2\varepsilon\nu\tau}{h^2}+4\varepsilon.$$

We have from Lemmas 3 and 4 that

$$|\tilde{\eta}_t^{(N)}(t)|_1^2\leqslant\Big(N^2+\frac{4}{h^2}\Big)\|\tilde{\eta}_t^{(N)}(t)\|^2+\frac{2}{h}\|\tilde{g}_t(t)\|^2,\tag{4.19}$$

$$S(\tilde{\eta}_t^{(N)}(t))\leqslant\frac{1}{2h^2}\|\tilde{\eta}_t^{(N)}(t)\|^2.\tag{4.20}$$

Thus

$$\begin{aligned}&\tau(m-1-4\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2+\nu\tau^2\Big(m\sigma-\sigma-\frac{m}{2}\Big)|\tilde{\eta}_t^{(N)}(t)|_1^2\\&\quad+\nu\tau^2\Big(m\sigma-\sigma-\frac{m}{2}-4\varepsilon\Big)S(\tilde{\eta}_t^{(N)}(t))\\&\quad\geqslant p_0\tau\|\tilde{\eta}_t^{(N)}(t)\|^2-\frac{c\tau}{h}\|\tilde{g}_t(t)\|^2.\end{aligned}\tag{4.21}$$

Hence (4.18) holds still.

Case 3. $\sigma<\frac{1}{2}$ and $\tau<\dfrac{4h^2}{\nu(1-2\sigma)(9+2N^2h^2)}$. We take

$$\begin{aligned}m>m_3=&\Big(1+p_0+\nu\sigma\tau N^2+\frac{9\nu\sigma\tau}{2h^2}+\frac{2\varepsilon\nu\tau}{h^2}+4\varepsilon\Big)\\&\times\Big(1+\nu\tau N^2\Big(\sigma-\frac{1}{2}\Big)+\frac{9\nu\tau}{2h^2}\Big(\sigma-\frac{1}{2}\Big)\Big)^{-1}.\end{aligned}$$

From (4.19) and (4.20), we get (4.21) also and then (4.18) follows.

Now put

$$\begin{aligned}E_1^{(N)}(t)=&\|\tilde{\eta}^{(N)}(t)\|^2+\nu\tau(|\tilde{\eta}^{(N)}(t)|_1^2+S(\tilde{\eta}^{(N)}(t)))\\&+\tau\sum_{\substack{y\in R_\tau\\ y\leqslant t-\tau}}(p_0\tau\|\tilde{\eta}_t^{(N)}(y)\|^2+\nu|\tilde{\eta}^{(N)}(y)|_1^2+\nu S(\tilde{\eta}^{(N)}(y))),\end{aligned}$$

$$\rho_1^{(N)}(t)=\|\tilde{\eta}^{(N)}(0)\|^2+\tau\sum_{\substack{y\in S_\tau\\ y<t-\tau}}\|R(y)\|^2.$$

By summing up (4.18), we get

$$E^{(N)}(t)\leqslant\rho_1^{(N)}(t)+\tau\sum_{\substack{y\in S_\tau\\ y<t-\tau}}(H_0(y)E^{(N)}(y)+H_1(y)|\tilde{\eta}^{(N)}(y)|_1^2). \tag{4.22}$$

In particular, if

$$2\delta>\begin{cases} m_1, & \text{for } \sigma>\frac{1}{2},\\ m_2, & \text{for } \sigma=\frac{1}{2},\\ m_3, & \text{for } \sigma<\frac{1}{2},\end{cases} \tag{4.23}$$

then we can take $m=2\delta$ and thus $H_1(t)=-\nu+s<0$. Applying Lemma 10 to (4.22) we get the following result.

Theorem 1. *If the following conditions are satisfied:*

(i) $\alpha_1=\alpha_2$, $\tau=O(h^2)$, $\tau=O\left(\frac{1}{N^2}\right)$,

(ii) $\sigma\geqslant\frac{1}{2}$ *or* $\tau<\frac{4h^2}{\nu(1-2\sigma)(9+2N^2h^2)}$,

(iii) *for all* $t\leqslant T$,

$$\|\tilde{f}_2(t)\|^2\leqslant b_1,\quad \|\tilde{g}(t)\|_l^2+\left\|\frac{\partial\tilde{g}}{\partial x_2}(t)\right\|_l^2\leqslant\min\left(\frac{b_2}{N},\,b_3h\right),\quad \rho_1^{(N)}(t)\leqslant b_4,$$

where b_i are positive constants depending only on $\|\varphi^{(N)}\|_{2,\infty}$, $\|\eta^{(N)}\|_{1,\infty}$ and ν, then for all $t\leqslant T$,

$$E_1^{(N)}(t)\leqslant b_5e^{b_6t}\rho_1^{(N)}(t). \tag{4.24}$$

In particular, if (4.23) holds, then for all $\rho_1^{(N)}(t)$ and t, (4.24) holds.

For the convergence, we put $\xi^{(N)}=P_N\xi$, $\psi^{(N)}=P_N\psi$, $\tilde{\xi}^{(N)}=\eta^{(N)}-\xi^{(N)}$ and $\tilde{\psi}^{(N)}=\varphi^{(N)}-\psi^{(N)}$. Then

$$\begin{cases}\xi_t^{(N)}+P_NJ^{(\alpha)}(\xi^{(N)}+\delta\tau\xi_t^{(N)},\,\psi^{(N)})-\nu\Delta(\eta^{(N)}+\sigma\tau\xi_t^{(N)})=P_Nf_1+\sum\limits_{i=1}^{5}M_i^{(N)}, & \text{in } Q_h\times S_\tau,\\ -\Delta\psi^{(N)}=\xi^{(N)}+P_Nf_2+M_6^{(N)}, & \text{in } Q_h\times S_\tau,\\ \xi^{(N)}(x_1,x_2,0)=P_N\xi_0(x_1,x_2), & \text{in } \bar{Q}_h,\end{cases}$$

where

$$M_1^{(N)}=\xi_t^{(N)}-\frac{\partial\xi^{(N)}}{\partial t},$$

$$M_2^{(N)}=P_NJ^{(\alpha)}(\xi^{(N)},\,\psi^{(N)})-P_N\left(\frac{\partial\psi}{\partial x_2}\frac{\partial\xi}{\partial x_1}-\frac{\partial\psi}{\partial x_1}\frac{\partial\xi}{\partial x_2}\right),$$

$$M_3^{(N)}=\delta\tau P_NJ^{(\alpha)}(\xi_t^{(N)},\,\psi^{(N)}),$$

$$M_4^{(N)}=\nu\frac{\partial^2\xi^{(N)}}{\partial x_1^2}-\nu\xi_{x_1\bar{x}_1}^{(N)},$$

$$M_5^{(N)}=\nu\sigma\tau\Delta\xi_t^{(N)},$$

$$M_6^{(N)}=\frac{\partial^2\psi^{(N)}}{\partial x_1^2}-\psi_{x_1\bar{x}_1}^{(N)}.$$

Furthermore,

$$
\begin{cases}
\tilde{\xi}_t^{(N)}+P_N J^{(\alpha)}(\tilde{\xi}^{(N)}+\delta\tau\tilde{\xi}_t^{(N)},\ \psi^{(N)}+\tilde{\psi}^{(N)})+P_N J^{(\alpha)}(\xi^{(N)}+\delta\tau\xi_t^{(N)},\ \tilde{\psi}^{(N)}) & \\
\quad -\nu\Delta(\tilde{\xi}^{(N)}+\sigma\tau\tilde{\xi}_t^{(N)})=-\sum\limits_{l=1}^{5}M_l^{(N)}, & \text{in } Q_h\times S_\tau, \\
-\Delta\tilde{\psi}^{(N)}=\tilde{\xi}^{(N)}-M_6^{(N)}, & \text{in } Q_h\times S_\tau, \\
\tilde{\xi}^{(N)}(x_1,\ x_2,\ t)=\tilde{\psi}^{(N)}(x_1,\ x_2,\ t)=0, & \text{on } \partial Q_h\times S_\tau, \\
\tilde{\xi}^{(N)}(x_1,\ x_2,\ 0)=0, & \text{in } \bar{Q}_h.
\end{cases}
\tag{4.25}
$$

Let $B_1(I)$, $B_2(0,1)$ and $B(Q)$ be Banach spaces and $B_2(B_1)=B_2(0,1;\ B_1(I))$. Let $\|u\|_B=\max\limits_{0<t<T}\|u(t)\|B(Q)$ and $\beta>0$, $\gamma>0$. Then

$$\|M_1^{(N)}(t)\|\leqslant c\tau\left\|\frac{\partial^2\xi}{\partial t^2}\right\|_{C(L^2)},$$

$$\|M_2^{(N)}(t)\|\leqslant c(h^2+N^{-\beta})[\|\psi\|_{H^{2+r}}(\|\xi\|_{H^{\frac{1}{2}+r}(H^{\beta+1})}+\|\xi\|_{H^{\frac{3}{2}+r}(H^\beta)}+\|\xi\|_{H^{\frac{7}{2}+r}(L^2)})$$
$$+\|\xi\|_{H^{2+r}}(\|\psi\|_{H^{\frac{1}{2}+r}(H^{\beta+1})}+\|\psi\|_{H^{\frac{3}{2}+r}(H^\beta)}),$$

$$\|M_3^{(N)}(t)\|\leqslant c\tau\|\psi\|_{H^{2+r}}\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{\frac{3}{2}+r}(L^2)}+\left\|\frac{\partial\xi}{\partial t}\right\|_{H^{\frac{1}{2}+r}(H^1)}\right),$$

$$\|M_4^{(N)}(t)\|\leqslant ch^2\|\xi\|_{C^4(L^2)},$$

$$\|M_5^{(N)}(t)\|\leqslant c\tau\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{C^2(L^2)}+\left\|\frac{\partial\xi}{\partial t}\right\|_{C(H^2)}\right),$$

$$\|M_6^{(N)}(t)\|\leqslant ch^2\|\psi\|_{C^4(L^2)}.$$

Thus, by an argument as in Theorem 1, we have the following result.

Theorem 2. *If conditions* (i), (ii) *of Theorem* 1 *hold,* $\beta>0$, $\gamma>0$ and $\xi\in C(0,T;\ H^{2+r}\cap C^4(L^2)\cap H^{\frac{1}{2}+r}(H^{\beta+1})\cap H^{\frac{3}{2}+r}(H^\beta)\cap H^{\frac{7}{2}+r}(L^2))$, $\frac{\partial\xi}{\partial t}\in C(0,T;\ C(H^2)\cap C^2(L^2)\cap H^{\frac{1}{2}+r}(H^1)\cap H^{\frac{3}{2}+r}(L^2))$, $\frac{\partial^2\xi}{\partial t^2}\in C(0,\ T;\ C(L^2))$, $\psi\in C(0,\ T;\ H^{2+r}\cap C^2(L^2)\cap H^{\frac{1}{2}+r}(H^{\beta+1})\cap H^{\frac{3}{2}+r}(H^\beta))$, then for all $t\leqslant T$,

$$\|\xi(t)-\eta^{(N)}(t)\|^2\leqslant cb^*(\tau^2+h^4+N^{-2\beta}),$$

where b^* is a positive constant depending on ν and the norms appearing in the estimations of $\|M_i^{(N)}(t)\|$.

§ 5. The Error Estimation for Problems with Other Boundary Conditions

In this section we suppose $b\geqslant0$ and

$$
\begin{cases}
-\eta_{x_1}^{(N)}(0,\ x_2,\ t)+\dfrac{b}{2}(\eta^{(N)}(0,\ x_2,\ t)+\eta^{(N)}(h,\ x_2,\ t)) \\
\quad =P_N\left(-\dfrac{\partial\xi}{\partial x_1}(0,\ x_2,\ t)+b\xi^{(N)}(0,\ x_2,\ t)\right)=P_N g_0(x_2,\ t), \\
\eta_{\bar{x}_1}^{(N)}(1,\ x_2,\ t)+\dfrac{b}{2}(\eta^{(N)}(1,\ x_2,\ t)+\eta^{(N)}(1-h,\ x_2,\ t)) \\
\quad =P_N\left(\dfrac{\partial\xi}{\partial x_1}(1,\ x_2,\ t)+b\xi^{(N)}(1,\ x_2,\ t)\right)=P_N g_1(x_2,\ t).
\end{cases}
\tag{5.1}
$$

For simplicity we assume $\delta=0$ and $\tilde{\varphi}^{(N)}(0, x_2, t)=\tilde{\varphi}^{(N)}(1, x_2, t)=0$. By an argument as in Section 4, we have

$$\begin{aligned}&\|\tilde{\eta}^{(N)}(t)\|_t^2+\tau(m-1-\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2+2\nu|\tilde{\eta}^{(N)}(t)|_1^2+\nu\tau\left(\sigma+\frac{m}{2}\right)(|\tilde{\eta}^{(N)}(t)|_1^2)_t\\&\quad+\nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}\right)|\tilde{\eta}_t^{(N)}(t)|_1^2+G_4(t)+G_5(t)+\sum_{l=1}^{4}B_l(t)\\&\leqslant\|\tilde{\eta}^{(N)}(t)\|^2+\left(1+\frac{m^2\tau}{4\varepsilon}\right)\|\tilde{f}_1(t)\|^2,\end{aligned}\tag{5.2}$$

where $B_l(t)$ are the same as those in the previous section and

$$G_4(t)=(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t),\ J^{(\alpha)}(\eta^{(N)}(t),\ \tilde{\varphi}^{(N)}(t))+J^{(\alpha)}(\tilde{\eta}^{(N)}(t),\ \varphi^{(N)}(t))),$$

$$G_5(t)=(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t),\ J^{(\alpha)}(\tilde{\eta}^{(N)}(t),\ \tilde{\varphi}^{(N)}(t))).$$

Theorem 3. *If $b=0$, $\sigma>\frac{1}{2}$ or $\tau<\frac{2h^2}{\nu(1-2\sigma)(4+N^2h^2)}$, and for all $t\leqslant T$,*

$$\|\tilde{f}_2(t)\|^2\leqslant b_8,\quad \tau h\|\tilde{g}(t)\|_I^2\leqslant b_9,\quad \rho_2^{(N)}(t)\leqslant\frac{b_{10}h}{N},$$

then for all $t\leqslant T$,

$$E_2^{(N)}(t)\leqslant b_{11}e^{b_{11}t}\rho_2^{(N)}(t),$$

where

$$E_2^{(N)}(t)=\|\tilde{\eta}^{(N)}(t)\|^2+\nu\tau|\tilde{\eta}^{(N)}(t)|_1^2+\tau\sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}(p_0\tau\|\tilde{\eta}_t^{(N)}(y)\|^2+\nu|\tilde{\eta}^{(N)}(y)|_1^2),$$

$$\rho_2^{(N)}(t)=\|\tilde{\eta}\eta^{(N)}(0)\|^2+\sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}(\|\tilde{f}_1(y)\|^2+\|\tilde{g}(y)\|_I^2).$$

Proof. We have

$$\begin{aligned}|G_4(t)|&\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2\\&\quad+\frac{c}{\varepsilon}(\|\eta^{(N)}\|_{1,\infty}^2+\|\varphi^{(N)}\|_{1,\infty}^2)(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2+\tau h\|\tilde{g}(t)\|_I^2),\\|G_5(t)|&\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2+\frac{cN}{\varepsilon h}\|\eta^{(N)}(t)\|^2(\|\tilde{\eta}^{(N)}(t))\|^2\\&\quad+\|\tilde{f}_2(t)\|^2+\tau h\|\tilde{g}(t)\|_I^2).\end{aligned}\tag{5.3}$$

We have from (5.1)

$$\begin{aligned}B(\tilde{\eta}^{(N)}(t),\ \tilde{\eta}^{(N)}(t))&=(\tilde{\eta}^{(N)}(0,t),\ \tilde{g}_0(t))_I+(\tilde{\eta}^{(N)}(1,t),\ \tilde{g}_1(t))_I\\&\quad-\frac{h}{2}(\|\tilde{g}_0(t)\|_I^2+\|\tilde{g}_1(t)\|_I^2)\end{aligned}$$

and thus Lemma 7 leads to

$$|B_1(t)|\leqslant\varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2+\frac{c}{\varepsilon}(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{g}(t)\|_I^2).$$

Similarly,

$$\begin{aligned}|B_2(t)|+|B_3(t)|&\leqslant\tau^2\|\tilde{\eta}_t^{(N)}(t)\|^2+\varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2+\varepsilon\nu\tau^2|\tilde{\eta}_t^{(N)}(t)|_1^2\\&\quad+\frac{c}{\varepsilon}(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{g}(t)\|_I^2+\tau^2\|\tilde{g}_t(t)\|_I^2),\end{aligned}$$

$$|B_4(t)|\leqslant\varepsilon\nu\tau^2|\tilde{\eta}_t^{(N)}(t)|_1^2+\frac{c\tau^2}{\varepsilon}(\|\tilde{\eta}_t^{(N)}(t)\|^2+\|\tilde{g}_t(t)\|_I^2).$$

We can complete the proof as in Theorem 1.

Theorem 4. *If* $b>0$, $\sigma>\frac{1}{2}$ *or* $\tau<\frac{2h^2}{\nu(1-2\sigma)(4+N^2h^2)}$ *and for all* $t<T$,

$$\|\tilde{f}_2(t)\|^2\leqslant b_{13},\quad \tau h\|\tilde{g}(t)\|_I^2\leqslant b_{14},\quad \rho_3^{(N)}(t)\leqslant\frac{b_{15}h}{N},$$

then for all $t<T$,

$$E_3^{(N)}(t)\leqslant b_{16}e^{b_{17}t}\rho_3^{(N)}(t),$$

where

$$E_3^{(N)}(t)=\|\tilde{\eta}^{(N)}(t)\|^2+\nu\tau(|\tilde{\eta}^{(N)}(t)|_1^2+S^*(\tilde{\eta}^{(N)}(t)))$$
$$+\tau\sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}(p_0\tau\|\tilde{\eta}_t^{(N)}(y)\|^2+\nu|\tilde{\eta}^{(N)}(y)|_1^2+\nu S^*(\tilde{\eta}^{(N)}(y))),$$
$$\rho_3^{(N)}(t)=\|\tilde{\eta}^{(N)}(0)\|^2+\sum_{\substack{y\in S_\tau\\ y\leqslant t-\tau}}(\|\tilde{f}_1(y)\|^2+\|\tilde{g}(y)\|_I^2).$$

Proof. We also have (5.2). Let

$$S^*(\tilde{\eta}^{(N)}(t))=\frac{1}{2}(\|\tilde{\eta}^{(N)}(0,t)+\tilde{\eta}^{(N)}(h,t)\|_I^2+\|\tilde{\eta}^{(N)}(1-h,t)+\tilde{\eta}^{(N)}(1,t)\|_I^2).$$

From (5.1), we get

$$\|\eta_{x_1}^{(N)}(0,t)\|_I^2+\|\eta_{\bar{x}_1}^{(N)}(1,t)\|_I^2\leqslant\varepsilon S^*(\tilde{\eta}^{(N)}(t))+\frac{c}{\varepsilon}\|\tilde{g}(t)\|_I^2$$

and so

$$|G_4(t)|+|G_5(t)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2$$
$$+\frac{c}{\varepsilon}\Big(\|\eta^{(N)}\|_{1,\infty}^2+\|\varphi^{(N)}\|_{1,\infty}^2+\frac{N}{h}\|\tilde{\eta}^{(N)}(t)\|^2\Big)$$
$$\times(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2+\tau hS^*(\tilde{\eta}^{(N)}(t))+\tau h\|\tilde{g}(t)\|_I^2).$$

On the other hand,

$$B_1(t)\geqslant b\nu(1-\varepsilon)S^*(\tilde{\eta}^{(N)}(t))-\frac{c}{\varepsilon}\|\tilde{g}(t)\|_I^2,$$

$$B_2(t)+B_3(t)\geqslant\frac{b(m+2\sigma)\nu\tau}{4}S_t^*(\tilde{\eta}^{(N)}(t))-\frac{b(m+2\sigma)\nu\tau^2}{4}(1+\varepsilon)S^*(\tilde{\eta}_t^{(N)}(t))$$
$$-\frac{c}{\varepsilon}(\|\tilde{g}(t)\|_I^2+\tau^2\|\tilde{g}_t(t)\|_I^2),$$

and

$$B_4(t)\geqslant\frac{b\nu\sigma m\tau^2}{2}(1-\varepsilon)S^*(\tilde{\eta}_t^{(N)}(t))-\frac{c}{\varepsilon}\tau^2\|\tilde{g}_t(t)\|_I^2.$$

The rest of the proof is obvious.

We can prove the convergence for $b=0$ and $b>0$ as in Theorem 2.

§ 6. The Steady Problem

We consider the steady problem

$$\begin{cases}\dfrac{\partial\psi}{\partial x_2}\dfrac{\partial\xi}{\partial x_1}-\dfrac{\partial\psi}{\partial x_1}\dfrac{\partial\xi}{\partial x_2}-\nu\Big(\dfrac{\partial^2\xi}{\partial x_1^2}+\dfrac{\partial^2\xi}{\partial x_2^2}\Big)=f_1, & \text{in } Q,\\ -\dfrac{\partial^2\psi}{\partial x_1^2}-\dfrac{\partial^2\psi}{\partial x_2^2}=\xi+f_2, & \text{in } Q.\end{cases}\tag{6.1}$$

Let

$$H=\{u/u(0,\ x_2)=u(1,\ x_2)=0,\ u(x_1,\ x_2)=u(x_1,\ x_2+2\pi),\ P_N u\in V_N\}$$

be a Hilbert space equipped with the scalar product and the norm as follows:

$$\langle u,\ v\rangle=\frac{1}{2}(u_{x_1},\ v_{x_1})+\frac{1}{2}(u_{\bar{x}_1},\ v_{\bar{x}_1})+\left(\frac{\partial u}{\partial x_2},\ \frac{\partial v}{\partial x_2}\right)+\frac{1}{2h}(u(h),\ v(h))_I$$
$$+\frac{1}{2h}(u(1-h),\ v(1-h))_I,$$
$$\|u\|_H^2=|u|_1^2+S(u).$$

For simplicity we suppose $\xi(0,\ x_2)=\xi(1,\ x_2)=\psi(0,\ x_2)=\psi(1,\ x_2)=0$. The spectral-difference scheme for (6.1) is

$$P_N J^{(\alpha)}(\eta^{(N)},\ \varphi^{(N)})-\nu\Delta\eta^{(N)}=P_N f_1. \tag{6.2}$$

Clearly for any fixed $w\in H$, $(w,\ f_1)$ is linear functional in H and so there exists $F\in H$ such that $\langle F,\ w\rangle=(w,\ f_1)$ and $|(w,\ f_1)|\leqslant\|F\|_H\|w\|_H$.

Theorem 5. *If $\|F\|_H$ is bounded uniformly for N and h, and $\alpha_1=\alpha_2$, then* (6.2) *has at least one solution which is bounded uniformly for N and h.*

Proof. From (6.2), we have

$$\nu\langle\eta^{(N)},\ w\rangle+(w,\ J^{(\alpha)}(\eta^{(N)},\ \varphi^{(N)}))=(f_1,\ w).$$

For any fixed $\eta^{(N)}$ and $\varphi^{(N)}$, $(w,\ J^{(\alpha)}(\eta^{(N)},\ \varphi^{(N)}))$ is linear functional in H and thus there exists $A\eta^{(N)}\in H$ such that $\langle A\eta^{(N)},\ w\rangle=(w,\ J^{(\alpha)}(\eta^{(N)},\ \varphi^{(N)}))$. Hence (6.2) is equivalent to the following operator equation

$$\eta^{(N)}-\frac{1}{\nu}(A\eta^{(N)}+F)=0. \tag{6.3}$$

Assume that the sequence $\{\eta_n^{(N)}\}$ satisfies $-\Delta\varphi_n^{(N)}=\eta_n^{(N)}$ and $\|\eta_n^{(N)}-\eta_0^{(N)}\|_H\to0$ as $n\to\infty$. If n is large enough, then $\|\eta_n^{(N)}\|_H$ and $\|\varphi_n^{(N)}\|_H$ are uniformly bounded. Let $z_{m,n}=\langle A\eta_m^{(N)}-A\eta_n^{(N)},\ w\rangle$. Then

$$|z_{m,n}|\leqslant|(\eta_m^{(N)}-\eta_n^{(N)},\ J^{(\alpha)}(w,\ \varphi_n^{(N)}))|+|(\eta_n^{(N)},\ J^{(\alpha)}(w,\ \varphi_m^{(N)}-\varphi_n^{(N)}))|$$
$$\leqslant c^*\|\eta_m^{(N)}-\eta_n^{(N)}\|_H\|w\|_H.$$

Put $w=A\eta_m^{(N)}-A\eta_n^{(N)}$. Then $\|A\eta_m^{(N)}-A\eta_n^{(N)}\|_H\leqslant c^*\|\eta_m^{(N)}-\eta_n^{(N)}\|_H$ and so A is a continuous operator.

On the other hand, (6.3) leads to that for $\lambda\in\left[0,\frac{1}{\nu}\right]$, the possible solution satisfies

$$\|\eta_\lambda^{(N)}\|_H\leqslant\lambda\|\eta_\lambda^{(N)}\|_H\|F\|_H\leqslant\frac{1}{\nu}\|\eta_\lambda^{(N)}\|_H\|F\|_H$$

and thus $\|\eta_\lambda^{(N)}\|_H$ is bounded.

The conclusion follows from the above statements and the Browder theorem.

Theorem 6. *If $\nu^2>c_1\sqrt{c^{**}c_1(1+c_1)}\|f_1\|$ and $\alpha_1=\alpha_2$, then* (6.2) *has only one solution where c^{**} is a positive constant depending only on the domain.*

Proof. Let $\eta^{(N)}$, $\varphi^{(N)}$ and $\eta_1^{(N)}$, $\varphi_1^{(N)}$ be the solutions of (6.2) and $\tilde{\eta}^{(N)}=\eta_1^{(N)}-\eta^{(N)}$, $\tilde{\varphi}^{(N)}=\varphi_1^{(N)}-\varphi^{(N)}$. Then

$$\begin{cases} P_N J^{(\alpha)}(\tilde{\eta}^{(N)}, \varphi^{(N)}+\tilde{\varphi}^{(N)}) - \nu\Delta\tilde{\eta}^{(N)}=0, & \text{in } Q, \\ -\Delta\tilde{\varphi}^{(N)}=\tilde{\eta}^{(N)}, & \text{on } \partial Q. \end{cases} \tag{6.4}$$

By taking the scalar product of the first formula of (6.4) and noticing

$$(\tilde{\eta}^{(N)}, J(\eta^{(N)}, \tilde{\varphi}^{(N)})) = -(\eta^{(N)}, J(\tilde{\eta}^{(N)}, \tilde{\varphi}^{(N)})),$$

we have

$$\nu\|\tilde{\eta}\|_H^2 \leqslant \|\eta^{(N)}\|_{L^\infty}|\tilde{\eta}^{(N)}|_1|\tilde{\varphi}^{(N)}|_1. \tag{6.5}$$

From the second formula of (6.4) and Lemma 5,

$$\|\tilde{\varphi}^{(N)}\|_H^2 \leqslant \|\tilde{\eta}^{(N)}\|\,\|\tilde{\varphi}^{(N)}\| \leqslant c_1\|\tilde{\eta}^{(N)}\|_H\|\tilde{\varphi}^{(N)}\|_H,$$

from which and (6.5)

$$(\nu - c_1\|\tilde{\eta}^{(N)}\|_{L^\infty})\|\tilde{\eta}^{(N)}\|_H^2 \leqslant 0. \tag{6.6}$$

On the other hand, (6.2) leads to

$$\nu\|\eta^{(N)}\|_H^2 \leqslant \|\eta^{(N)}\|\,\|f_1\| \leqslant \sqrt{c_1}\|\eta^{(N)}\|_H\|f_1\|$$

and so

$$\|\eta^{(N)}\|_{L^\infty}^2 \leqslant c^{**}(|\eta^{(N)}|_1^2 + \|\eta^{(N)}\|^2) \leqslant \frac{c^{**}}{\nu^2}c_1(1+c_1)\|f_1\|^2,$$

from which and (6.6) the conclusion follows.

By the technique in [3], we have the following results.

Theorem 7. *Let $\tilde{f}_l$ be the error of f_l. If the conditions of Theorem 6 hold, then*

$$\|\tilde{\eta}^{(N)}\|^2 \leqslant c(\|\tilde{f}_1\|^2 + \|\tilde{f}_2\|^2).$$

Theorem 8. *If the conditions of Theorem 6 hold, then the iteration*

$$\begin{cases} \eta_{n+1}^{(N)} = \eta_n^{(N)} + \tau[\nu\Delta\eta_{n+1}^{(N)} + P_N J^{(\alpha)}(\eta_{n+1}^{(N)}, \varphi_n^{(N)}) + f_1, & \tau>0,\ n\geqslant 0, \\ \Delta\varphi_n^{(N)} = \eta_n^{(N)} + f_2, & n\geqslant 0 \end{cases}$$

is convergent and

$$\|\eta_{n+1}^{(N)} - \eta^{(N)}\|_H^2 \leqslant \theta^n\|\eta_0^{(N)} - \eta^{(N)}\|_H^2, \quad 0<\theta<1.$$

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