A CHARACTERISTIC ITERATION FOR SOLVING COEFFICIENT INVERSE PROBLEMS OF A WAVE EQUATION*

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Abstract

In this paper, a characteristic iteration for solving coefficient inverse problems has been presented. This method is stable and fast converging, and may be extended to the 2-D case. Excellent numerical results have been obtained by this method.

§ 1. 1-D Wave Equation and Its Inverse Problem

$$\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial u}{\partial x}\right)-\sigma(x)\frac{\partial^2 u}{\partial t^2}=0, \quad x>0,\ t>0, \tag{1.1}$$

$$u(x,0)=\frac{\partial u}{\partial t}(x,0)=0, \quad x>0, \tag{1.2}$$

$$\frac{\partial u}{\partial x}(0,t)=\delta(t), \quad t>0. \tag{1.3}$$

Measured data:

$$u(0,t)=f(t). \tag{1.4}$$

Inverse problem:

To recover $\sigma(x)$ from $f(t)$ and (1.1)—(1.3), where $\sigma(x)$ is the coefficient of (1.1), which belongs to

$$\Sigma=\{\sigma(x)\,|\,\sigma(x)\in C^1(0,\infty),\ 0<\underline{\sigma}\leqslant\sigma(x)\leqslant\bar{\sigma}\}$$

and $\delta(t)$ is the generalized Delta function.

Assume that $f(t)$ satisfies a certain compatible condition, for example, $f(0)=-1$, such that the solution of the inverse problem exists.

§ 2. Singularity of the Solution of (1.1)—(1.3)

Lemma 1. *Suppose that* $\sigma(x)$ *belongs to* Σ *and* $u(x,t)$ *is the generalized solution of* (1.1)—(1.3). *Then,*

$$u(x,t)=a(x)H(t-x)+v(x,t), \tag{2.1}$$

where $a(x)$ *is the jump quantity,* $H(\cdot)$ *is a Heaviside function, and* $v(x,t)$ *is the*

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regular part of $u(x, t)$. Moreover, $a(x)$ and $v(x, t)$ satisfy respectively

$$a(x)=-\frac{\sqrt{\sigma(0)}}{\sqrt{\sigma(x)}}, \tag{2.2}$$

$$\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial v}{\partial x}\right)-\sigma(x)\frac{\partial^2 v}{\partial t^2}=-\left[\sigma(x)\frac{d^2 a}{dx^2}+\frac{d\sigma}{dx}\frac{da}{dx}\right]H(t-x), \tag{2.3}$$

$$v(x, 0)=v_t(x, 0)=0, \tag{2.4}$$

$$v_x(0, t)=-\frac{\sigma'(0)}{\sigma(0)}H(t). \tag{2.5}$$

Proof. Since $\sigma(x)\in\Sigma$, (1.1) is a wave equation. By [3], the singularities of the solution of (1.1)—(1.3) propagate along the characteristic line $t=x$. By (1.2) and (1.3), we have (2.1).

Since $u(x, t)$ is the generalized solution of (1.1)—(1.3), it should satisfy the weak equation of (1.1)—(1.3). By the theory of generalized function[3], we can make generalized calculus on both sides of (2.1). Thus we have

$$\frac{\partial u}{\partial x}=-a(x)\delta(t-x)+a_x H(t-x)+\frac{\partial v}{\partial x}, \tag{2.6}$$

$$\frac{\partial^2 u}{\partial x^2}=a(x)\delta'(t-x)-2a_x\delta(t-x)+a_{xx}H(t-x)+\frac{\partial^2 v}{\partial x^2}, \tag{2.7}$$

$$\frac{\partial u}{\partial t}=a(x)\delta(t-x)+\frac{\partial v}{\partial t}, \tag{2.8}$$

$$\frac{\partial^2 u}{\partial t^2}=a(x)\delta'(t-x)+\frac{\partial^2 v}{\partial t^2}. \tag{2.9}$$

Substituting (2.6)—(2.9) into (1.1) and making proper arrangement, we have

$$\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial v}{\partial x}\right)-\sigma(x)\frac{\partial^2 v}{\partial t^2}=(2a_x\sigma(x)+a(x)\sigma_x)\delta(t-x)$$
$$-(\sigma(x)a_{xx}+\sigma_x a_x)H(t-x). \tag{2.10}$$

Let

$$R(x, t)=\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial v}{\partial x}\right)-\sigma(x)\frac{\partial^2 v}{\partial t^2}, \tag{2.11}$$

where $R(x, t)$ does not include the δ function. This is because by (2.1) we have

$$v(x, t)=b(x)C(t-x)+w(x, t), \tag{2.12}$$

in which $w(x, t)$ belongs to C^1 and $C(t-x)=(t-x)^+$. Clearly, we have

$$R(x, t)=(-2b_x\sigma(x)-b(x)\sigma_x)H(t-x)+(\sigma(x)b_{xx}+\sigma_x b_x)C(t-x)$$
$$+\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial w}{\partial x}\right)-\sigma(x)\frac{\partial^2 w}{\partial t^2}. \tag{2.13}$$

In (2.13), since $w(x, t)$ belongs to C^1, it is obvious that there is no δ function in $\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial w}{\partial x}\right)-\sigma(x)\frac{\partial^2 w}{\partial t^2}$. Thus the coefficient of $\delta(t-x)$ in (2.10) should vanish, i.e.

$$2a_x\sigma(x)+a(x)\sigma_x=0, \tag{2.14}$$

and we have

$$\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial v}{\partial x}\right)-\sigma(x)\frac{\partial^2 v}{\partial t^2}=-(\sigma(x)a_{xx}+\sigma_x a_x)H(t-x),$$

which is (2.3). From (2.14), we have

$$a(x)=\frac{C_0}{\sqrt{\sigma(x)}}.$$

After using (1.3), we have

$$a(x)=-\frac{\sqrt{\sigma(0)}}{\sqrt{\sigma(x)}}.$$

Hence (2.2) has been proved.

From (2.6) and (2.2), (2.5) will be obtained. It is evident that (2.4) holds. Q.E.D.

Corollary. Under conditions of Lemma 1, we have

$$u(x,t)=-\frac{\sqrt{\sigma(0)}}{\sqrt{\sigma(x)}}H(t-x)+v(x,t), \tag{2.15}$$

$$u(x,x)=-\frac{\sqrt{\sigma(0)}}{\sqrt{\sigma(x)}}, \tag{2.16}$$

$$\sigma(x)=\frac{\sigma(0)}{u^2(x,x)}. \tag{2.17}$$

Proof. Obviously, from (2.1) and (2.2), we have (2.15). And by (2.3)—(2.5), we have

$$v(x,t)=0, \quad t<x.$$

Finally we get (2.16) and (2.17). Q.E.D.

§ 3. Iteration

(i) Choose $\sigma_0(x)$.

(ii) Suppose that $\sigma_n(x)$ has been obtained in the $(n-1)$th step, replace $\sigma(x)$ by $\sigma_n(x)$, in the direction of increase of x solve (1.1), (1.3), (1.4) and obtain

(iii) $\sigma_{n+1}(x)=\dfrac{\sigma(0)}{u_n^2(x,x)}$.

§ 4. The Difference Scheme of Characteristic Iteration

Subdivide the time-space domain Ω into a computing net in Fig. 1:

$$0=x_0<x_1<x_2<\cdots<x_M=a, \tag{4.1}$$

$$x_{i+1}-x_i=h=\frac{a}{M}, \tag{4.2}$$

$$0=t_0<t_1<t_2<\cdots<t_N=2a, \tag{4.3}$$

$$t_{i+1}-t_i=\tau. \tag{4.4}$$

Let $r=\frac{\tau}{h}$ be the net radio. Here we take $r=1$.

The characteristic iterative method described in Section 3 requires calculating $u(x,x)$ in each step of iteration. In order to compute $u(x,x)$ we establish the following special explicit difference scheme:

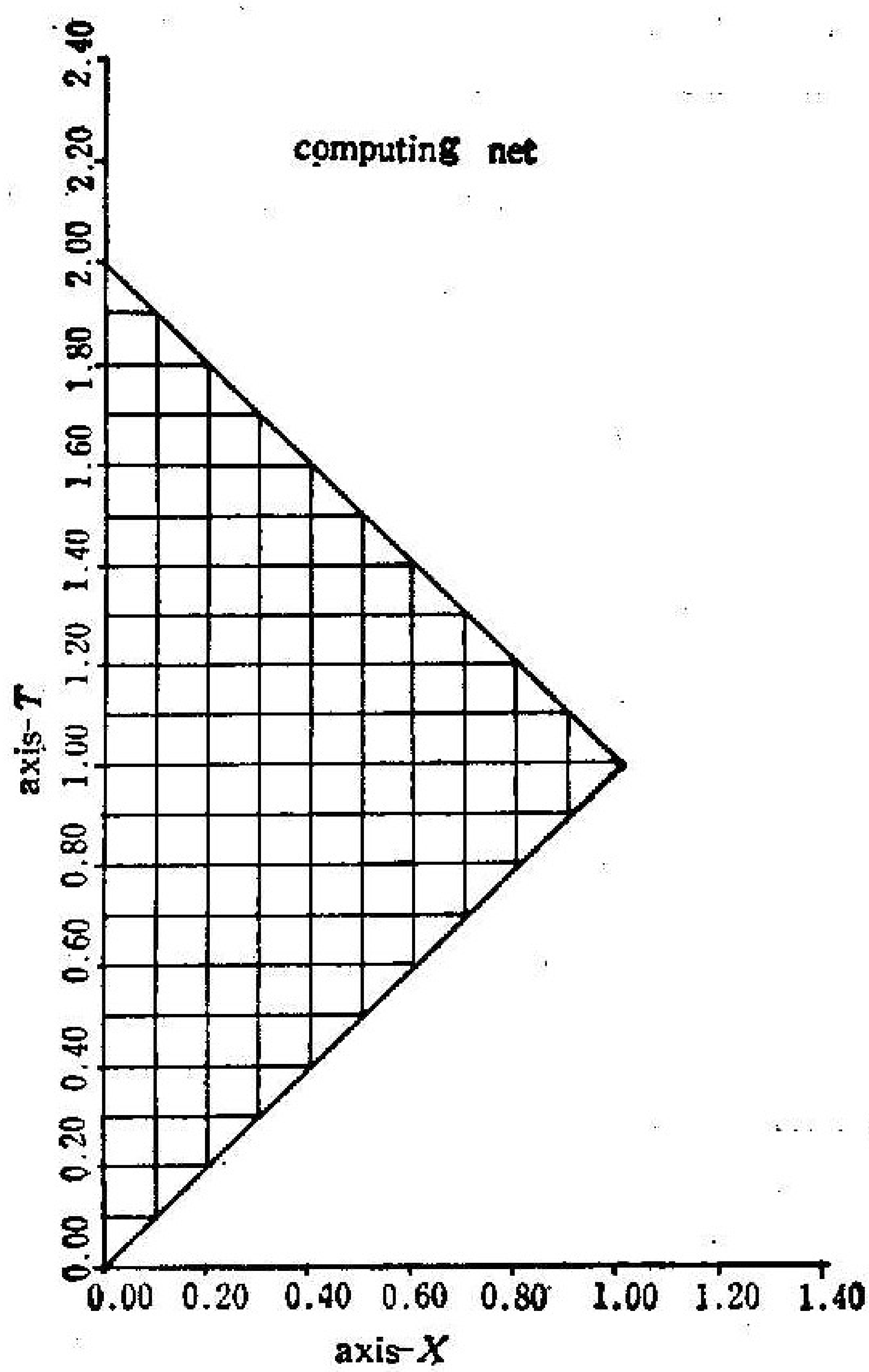

Fig. 1

$$u_{j+1}^n=\frac{\sigma_{j-\frac{1}{2}}+\sigma_{j+\frac{1}{2}}}{2\sigma_{j+\frac{1}{2}}}(u_j^{n+1}+u_j^{n-1})-\frac{\sigma_{j-\frac{1}{2}}}{\sigma_{j+\frac{1}{2}}}u_{j-1}^n, \tag{4.5}$$

$$j=1,\ 2,\ \cdots,\ M-1, \tag{4.6}$$

$$n=j+1,\ j+2,\ \cdots,\ N-j-1, \tag{4.7}$$

$$u_0^n=f(t_n),\quad n=0,\ 1,\ 2,\ \cdots,\ N, \tag{4.8}$$

$$u_1^n=u_0^n,\quad n=1,\ 2,\ \cdots,\ N-1. \tag{4.9}$$

By (4.5)—(4.9), we can calculate in the kth step of iteration (i)—(iii):

$$u_k(x_j,\ x_j)=u_{j,k}^j\quad j=1,\ 2,\ \cdots,\ M. \tag{4.10}$$

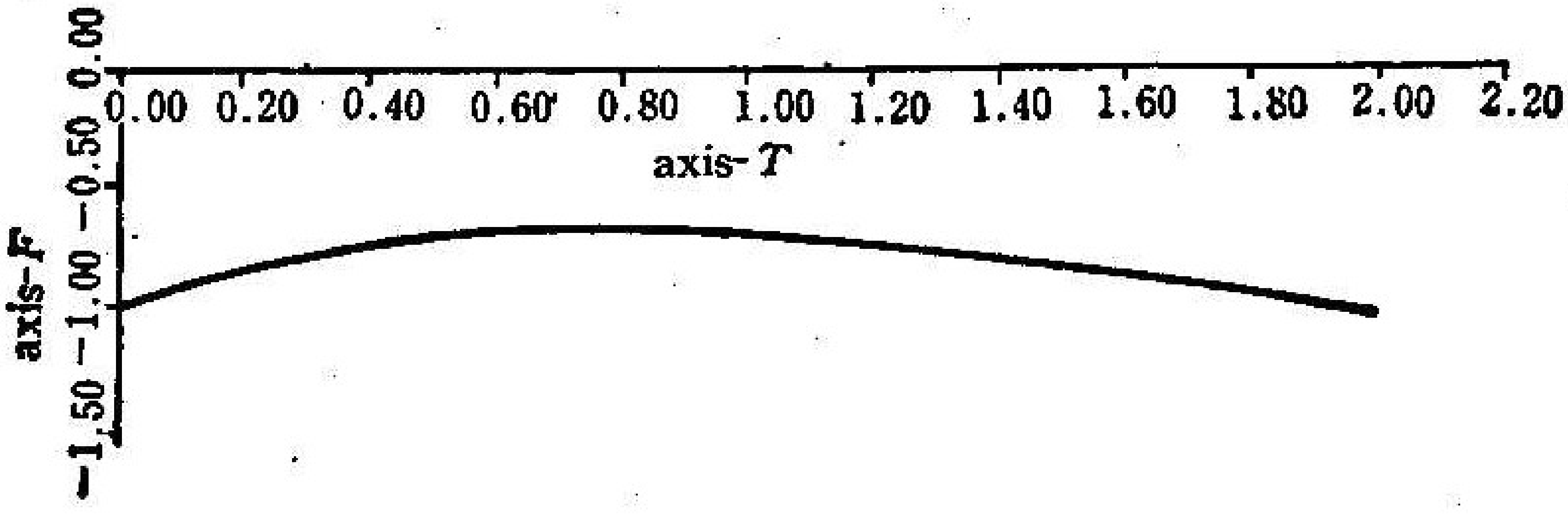

Fig. 2

Then, by using (4.10) and (iii) in Section 3, new iterative values $\sigma^j_{k+1}=\sigma_{k+1}(x_j)$ $(j=1, 2, \cdots, M)$ will be obtained.

§ 5. Numerical Simulation and Discussion

5.1. Numerical simulation

The procedure has been tested for many examples. We select five characteristic ones, which show very well the numerical effect of our method.

Example 1. We choose

$$a=1, \tag{5.1}$$

$$\sigma(x)=-2x^2+2x+1, \quad x\in[0, 1], \tag{5.2}$$

$$h=x_{j+1}-x_j=0.1, \quad j=0, 1, 2, \cdots, M-1.$$

In the case that the impulse response is calculated by solving numerically (1.1)—(1.3) and is given in Fig. 2, the initial guess σ_0 is taken to be

$$\sigma_0(x)=1, \quad x\in[0, 1]. \tag{5.3}$$

After five iterations we obtain excellent results listed in Table 1, where

σ^*_j denotes the exact $\sigma(x_j)$,

$\sigma^{(0)}_j$ denotes initial guess $\sigma_0(x_j)$,

....

$\sigma^{(5)}_j$ denotes the results of the 5th iteration $\sigma_5(x_j)$.

Table 1

x_j	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
σ^*_j	1.0	1.18	1.32	1.42	1.48	1.5	1.48	1.42	1.32	1.18	1.0
$\sigma^{(0)}_j$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\sigma^{(1)}_j$	1.0000	1.1800	1.3174	1.4103	1.4601	1.4697	1.4421	1.3807	1.2881	1.1671	1.0196
$\sigma^{(3)}_j$	1.0000	1.1800	1.3200	1.4200	1.4800	1.4800	1.4799	1.4196	1.3193	1.1791	0.9992
$\sigma^{(5)}_j$	1.0000	1.1800	1.3200	1.4200	1.4800	1.5000	1.4801	1.4202	1.3202	1.1802	1.0002

Figs. 3—4 show that the convergence of the iteration is very fast.

Example 2. In this example, we take

$$a=2.0, \tag{5.4}$$

$$\sigma(x)=\frac{x}{2}+1, \quad x\in[0, 2], \tag{5.5}$$

and initial guess $\sigma_0(x)=1.0$, $x\in[0, 2]$.

The results listed in the following Table 2 are also satisfactory:

In the following examples, $\sigma(x)\in L_\infty$, i.e. $\sigma(x)$ is a staircase function, and the datum are contaminated by noise in example 5, but we also obtain excellent results using the method described above. The three examples illustrate the effectiveness of the algorithm of characteristic iteration (CI) in reconstructing a layered medium of the earth.

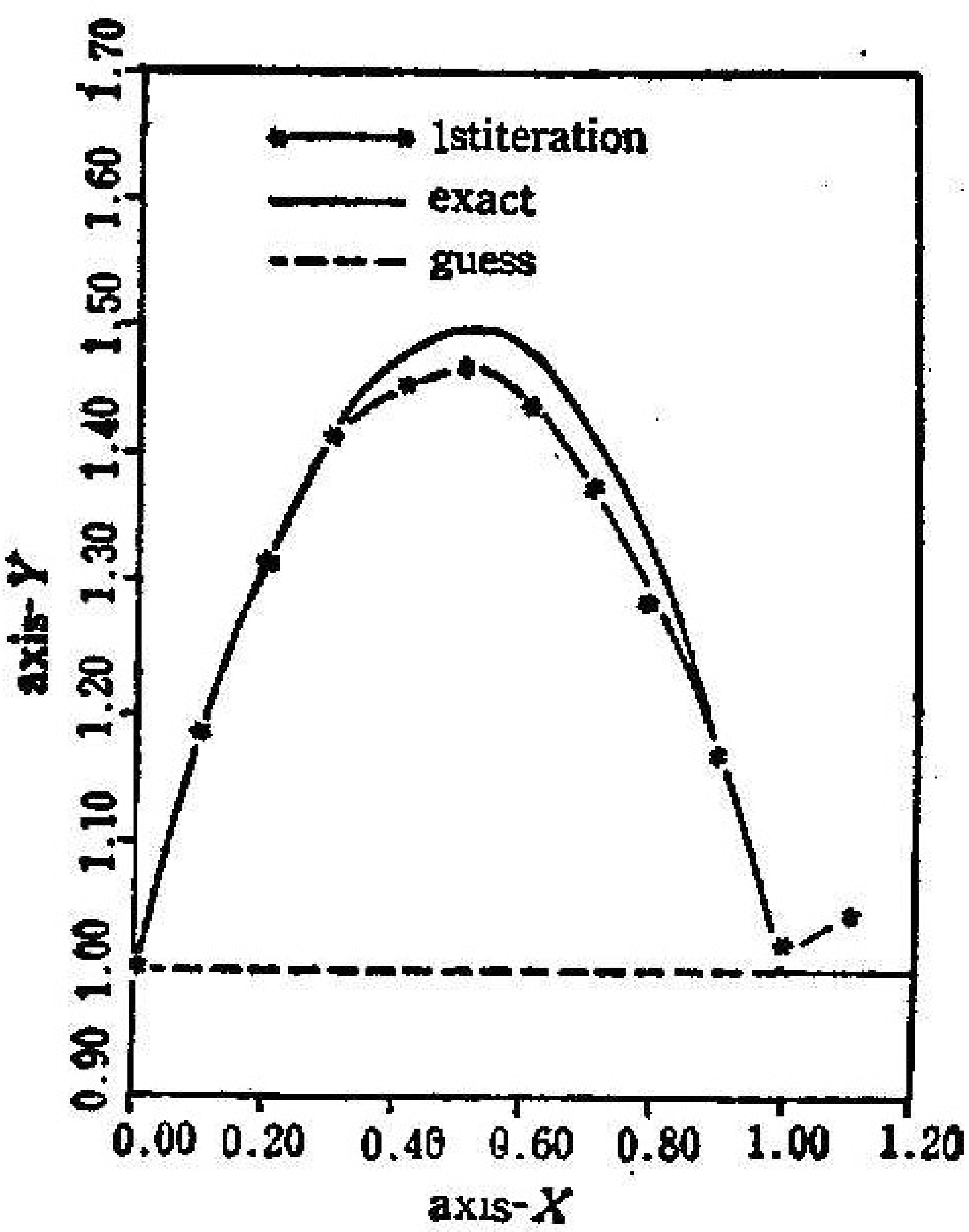

Fig. 3

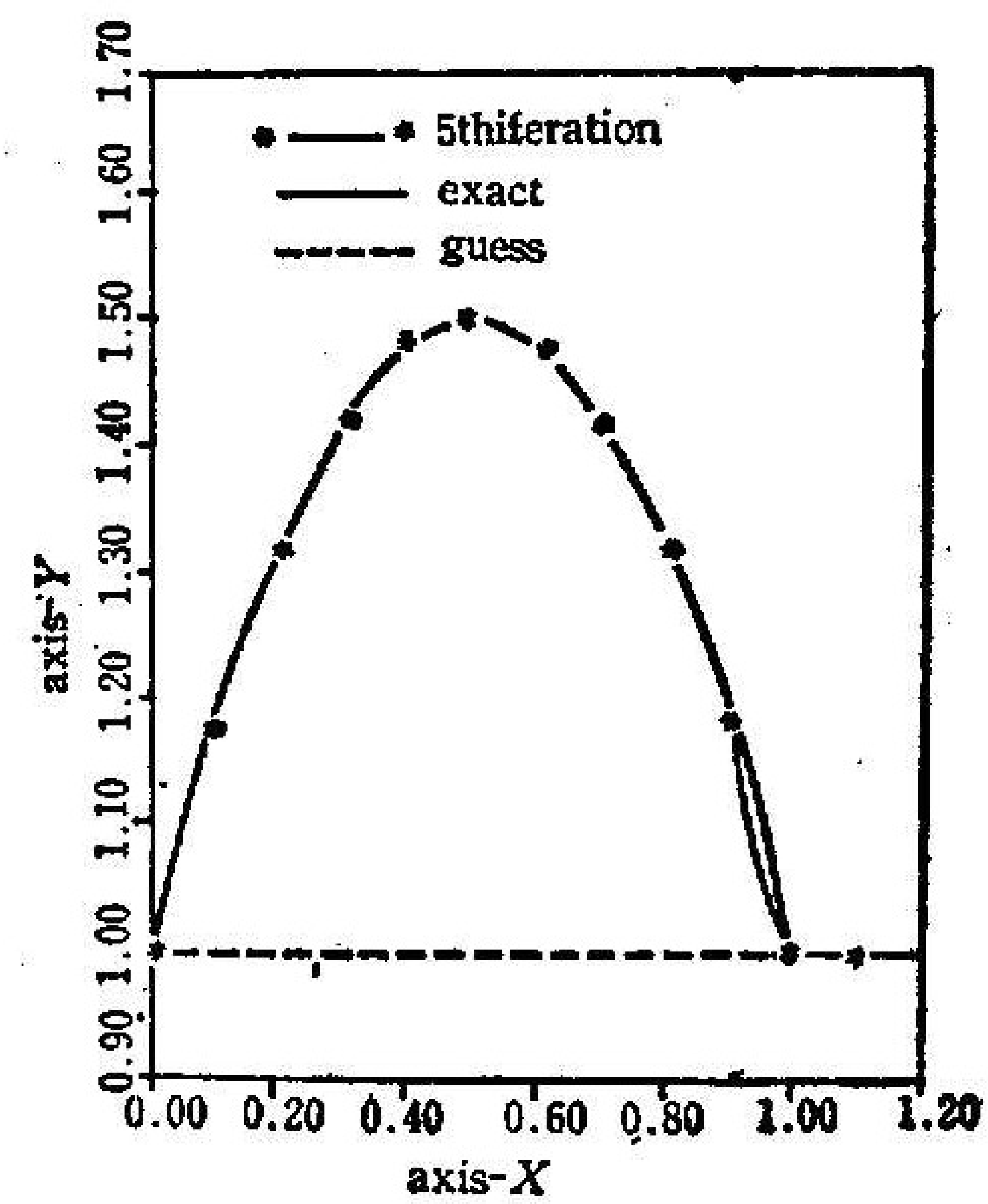

Fig. 4

Table 2

x_j	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
σ_j^*	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\sigma_j^{(0)}$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\sigma_j^{(1)}$	1.00000	1.09996	1.19948	1.29778	1.39449	1.48912	1.58131	1.67077	1.75731	1.84075	1.92105
$\sigma_j^{(3)}$	1.00000	1.10001	1.20003	1.30009	1.40019	1.50033	1.60053	1.70076	1.80107	1.90139	2.00176
$\sigma_j^{(5)}$	1.00000	1.10000	1.20004	1.30000	1.40020	1.50036	1.60057	1.70082	1.80117	1.90157	2.00206

Example 3. We take

$$a=1.0, \tag{5.6}$$

$$\sigma(x)=1+H(x-0.2), \quad x\in[0,1], \tag{5.7}$$

and initial guess $\sigma_0(x)=1.0$, $x\in[0,1]$.

The impulse response is shown in Fig. 5. Figs. 6 and 7 show the process for iterative convergence of example 3.

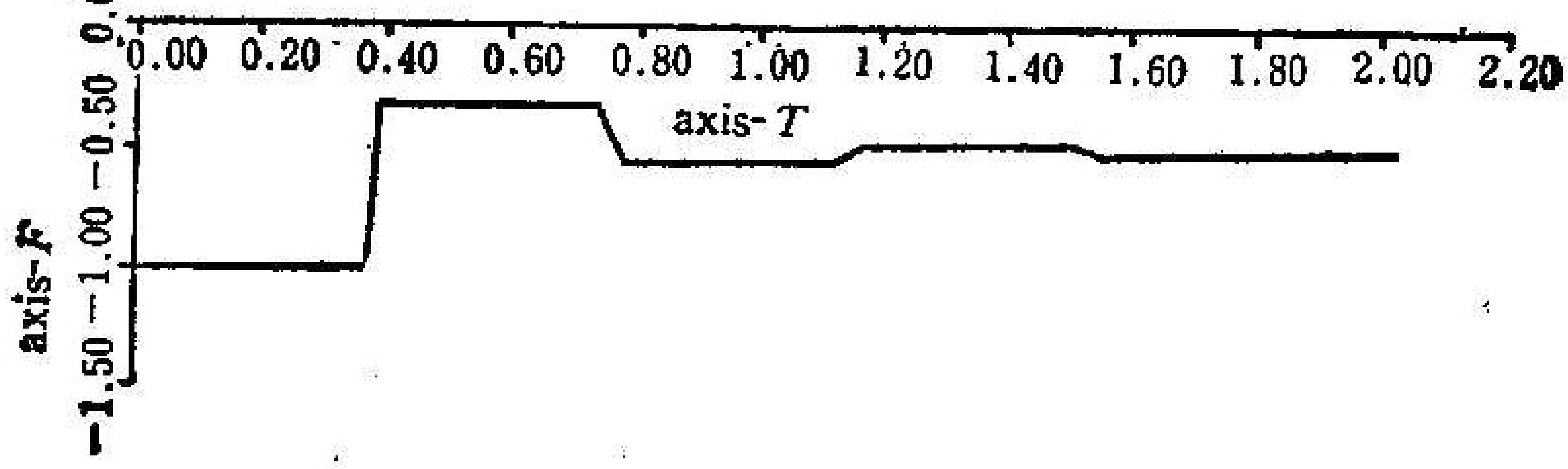

Fig. 5

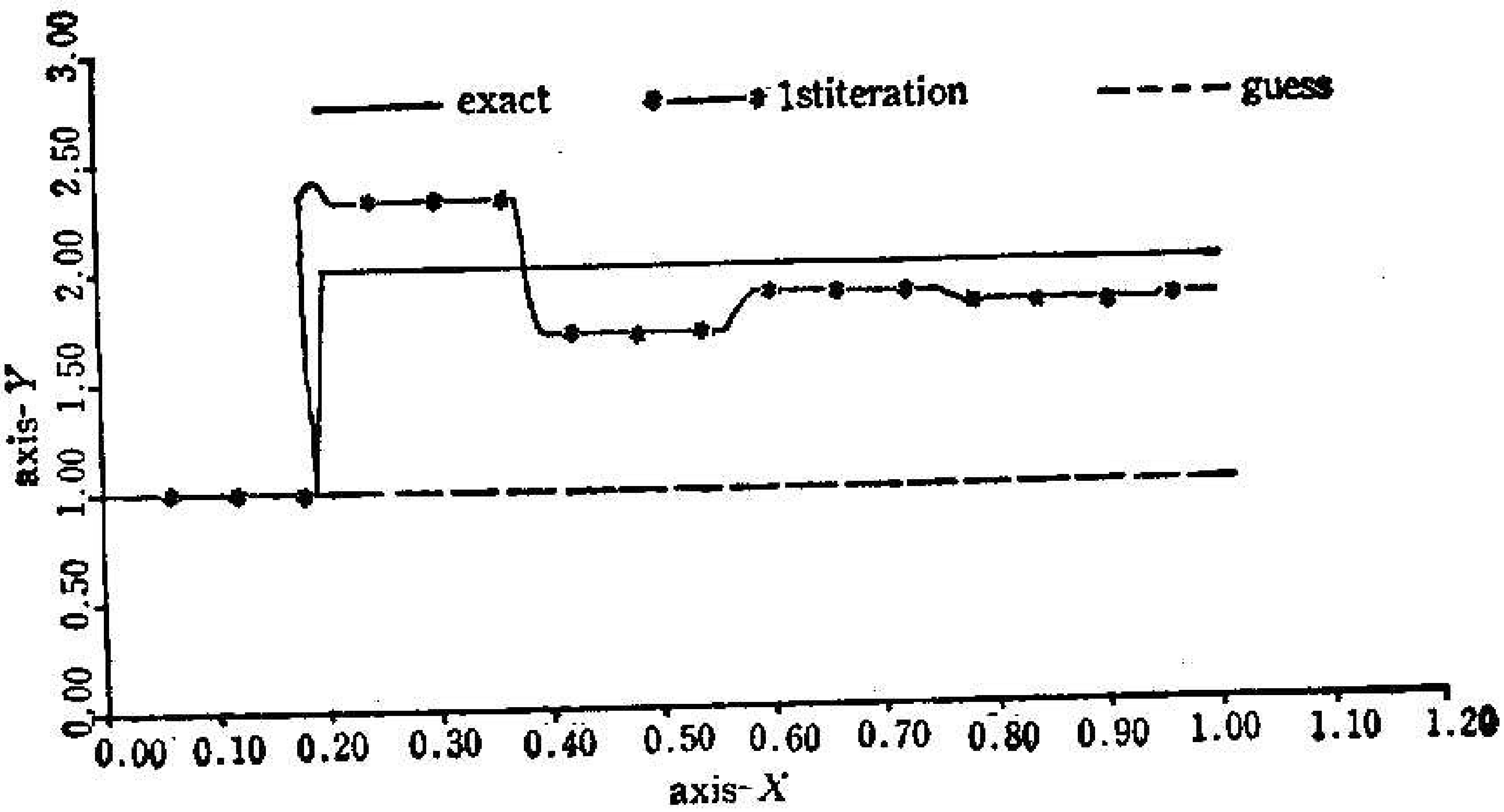

Fig. 6

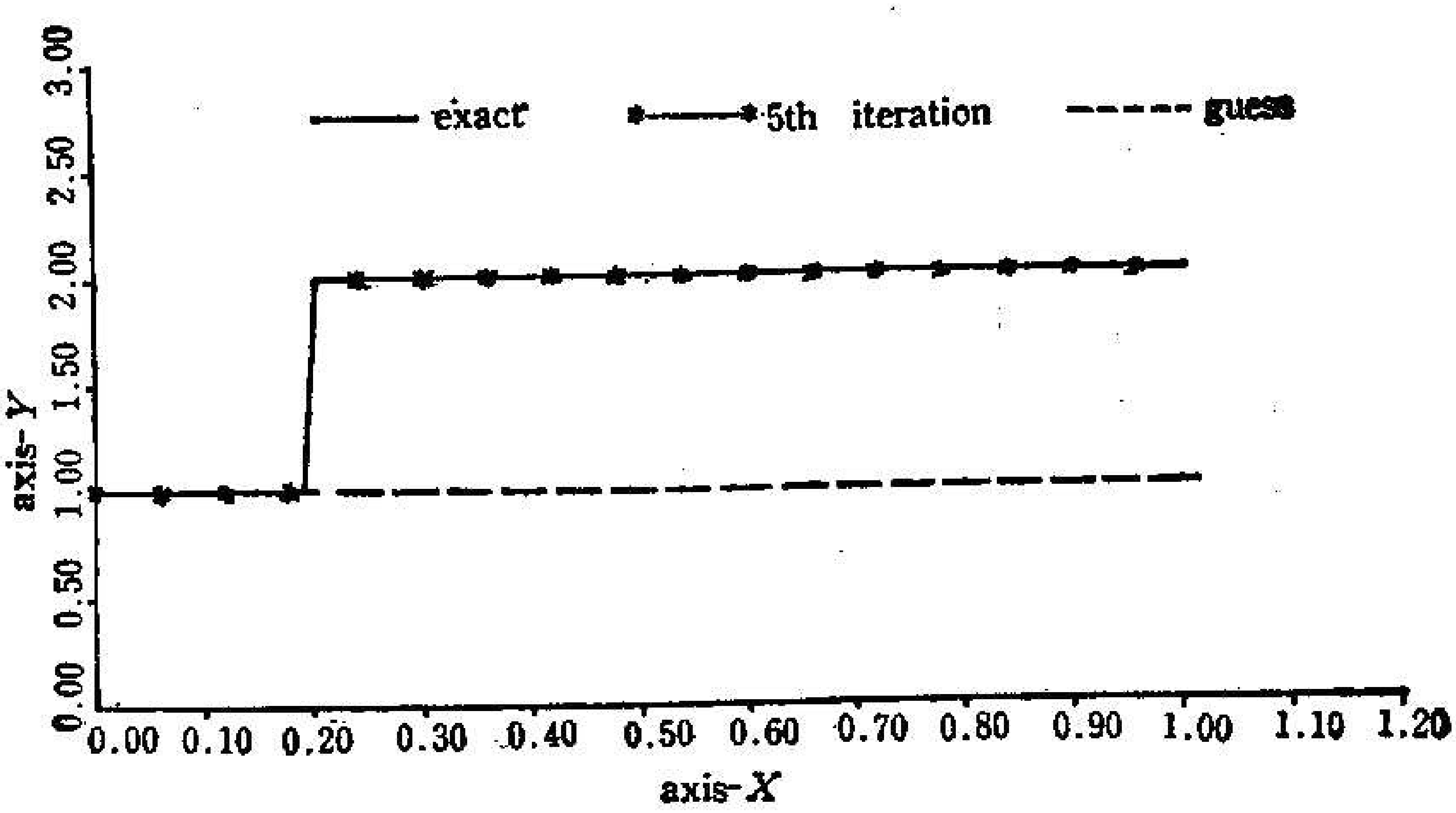

Fig. 7

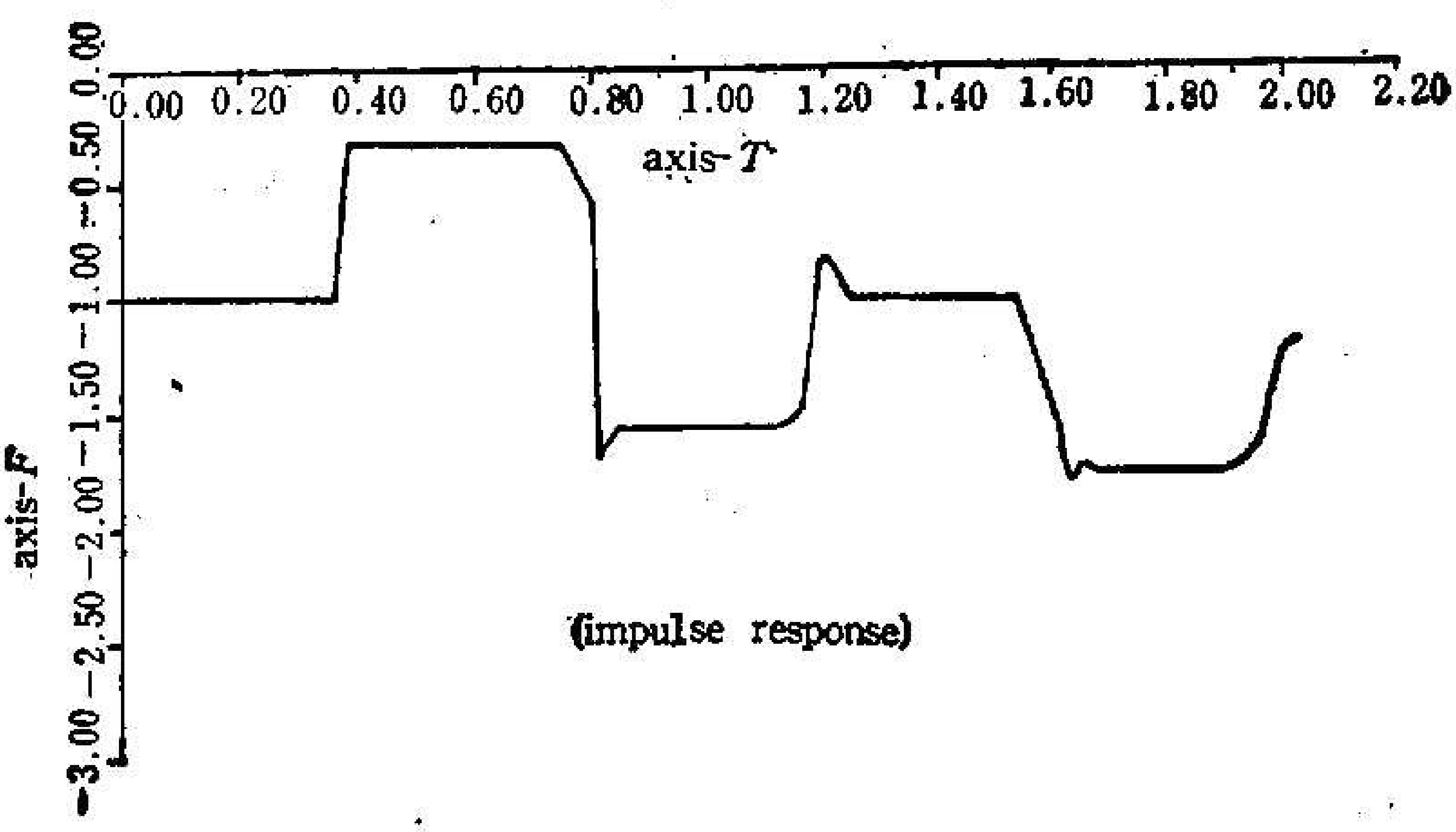

Fig. 8

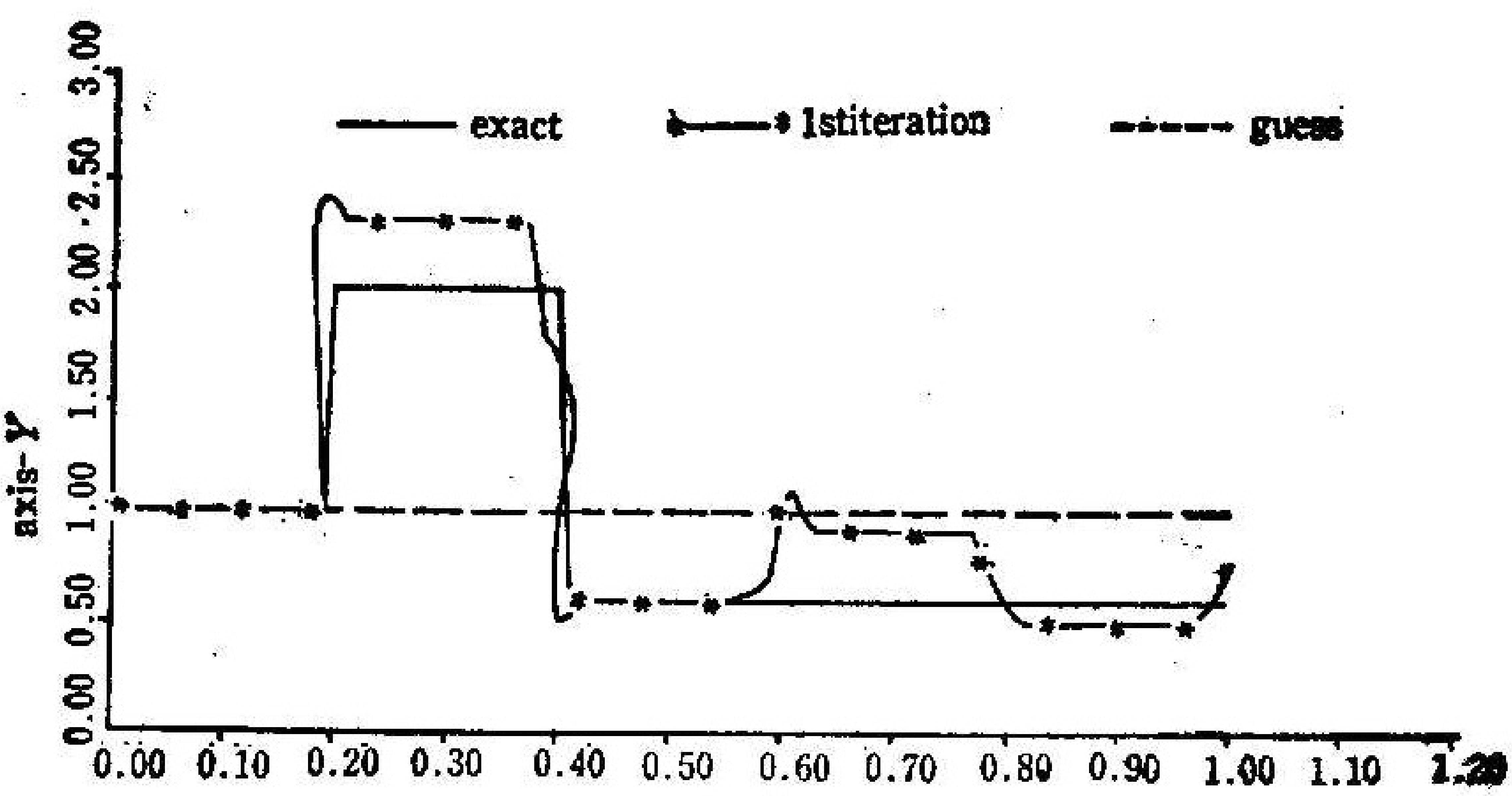

Fig. 9

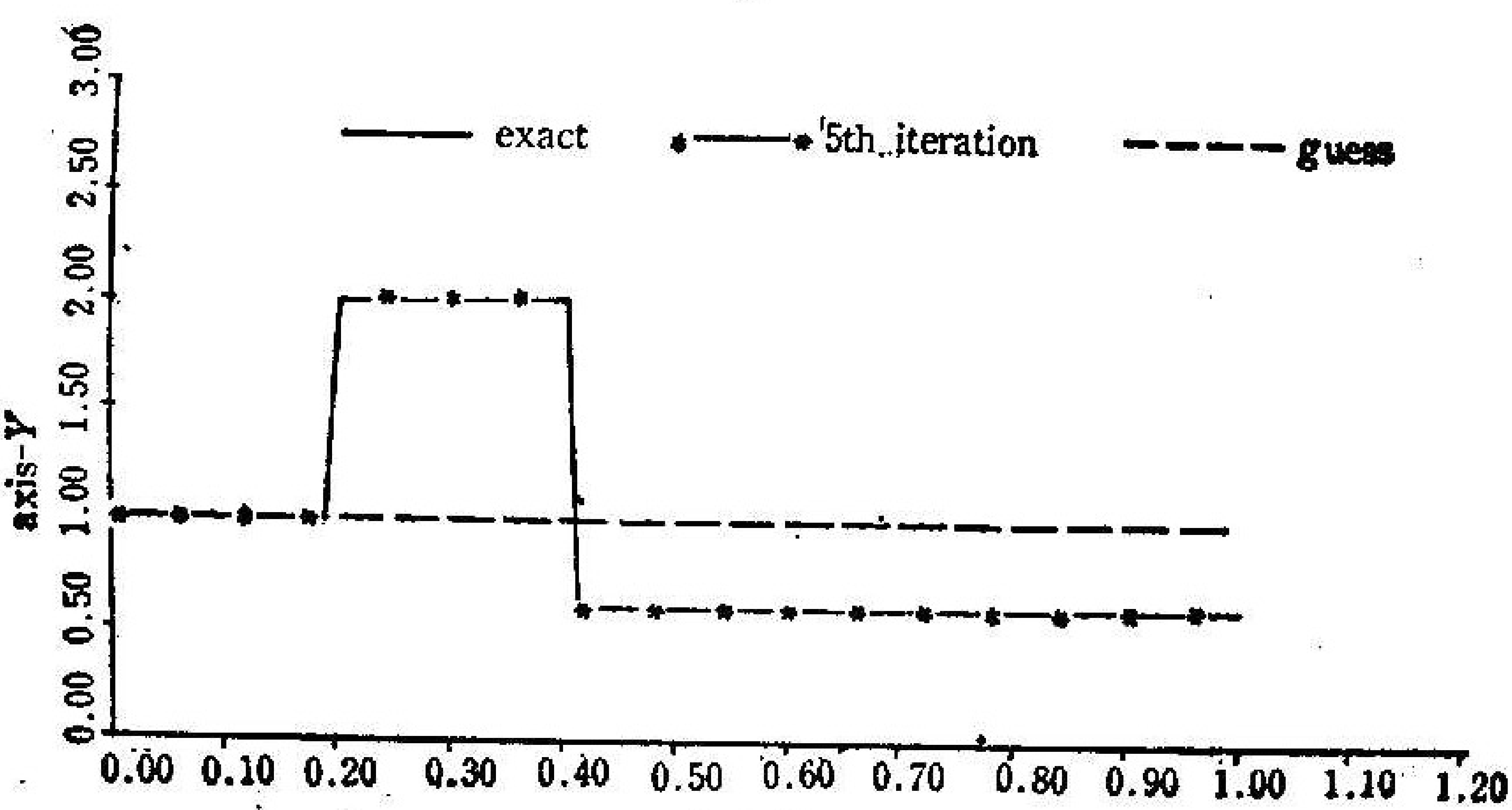

Fig. 10

Example 4. The impulse response is shown in Fig 8. The computing results for the example are plotted in Figs. 9—10.

Example 5. In this last example, the first graph (Fig. 11) shows the impulse response contaminated by some noise of 10%. The second and third graphs (Fig. 12 and Fig. 13) show the computing results of the 1st and 5th iterations respectively.

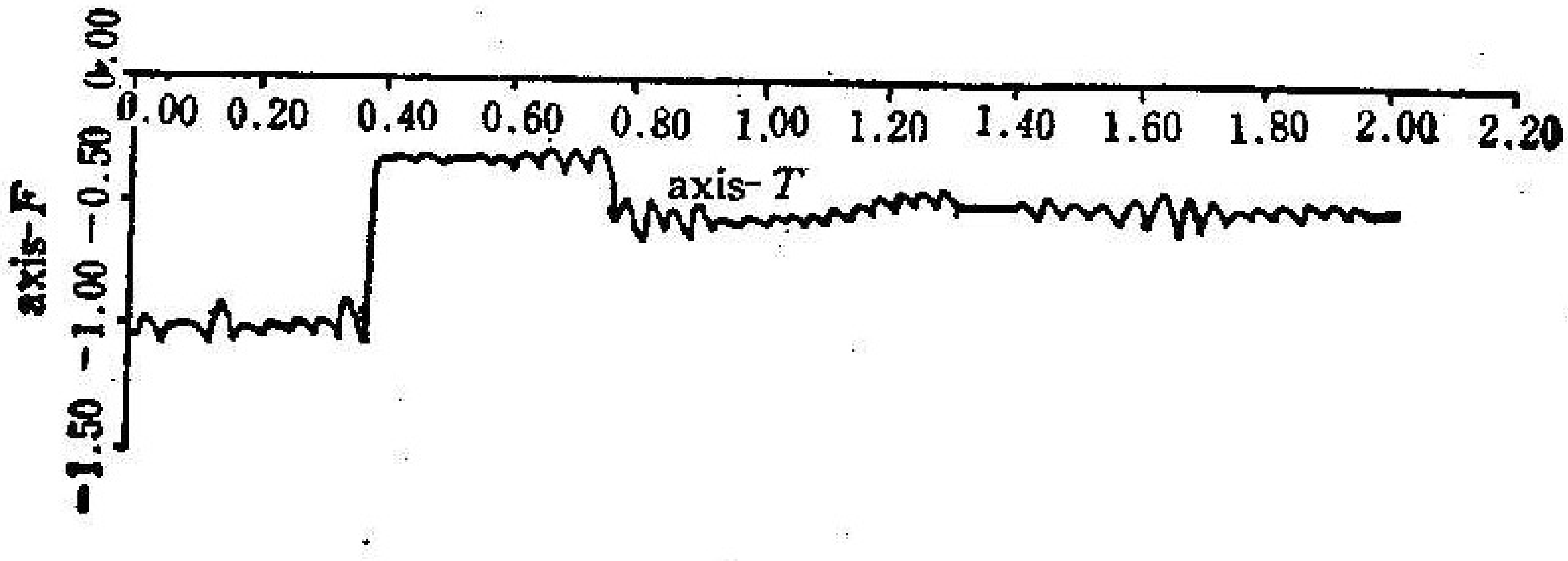

Fig. 11

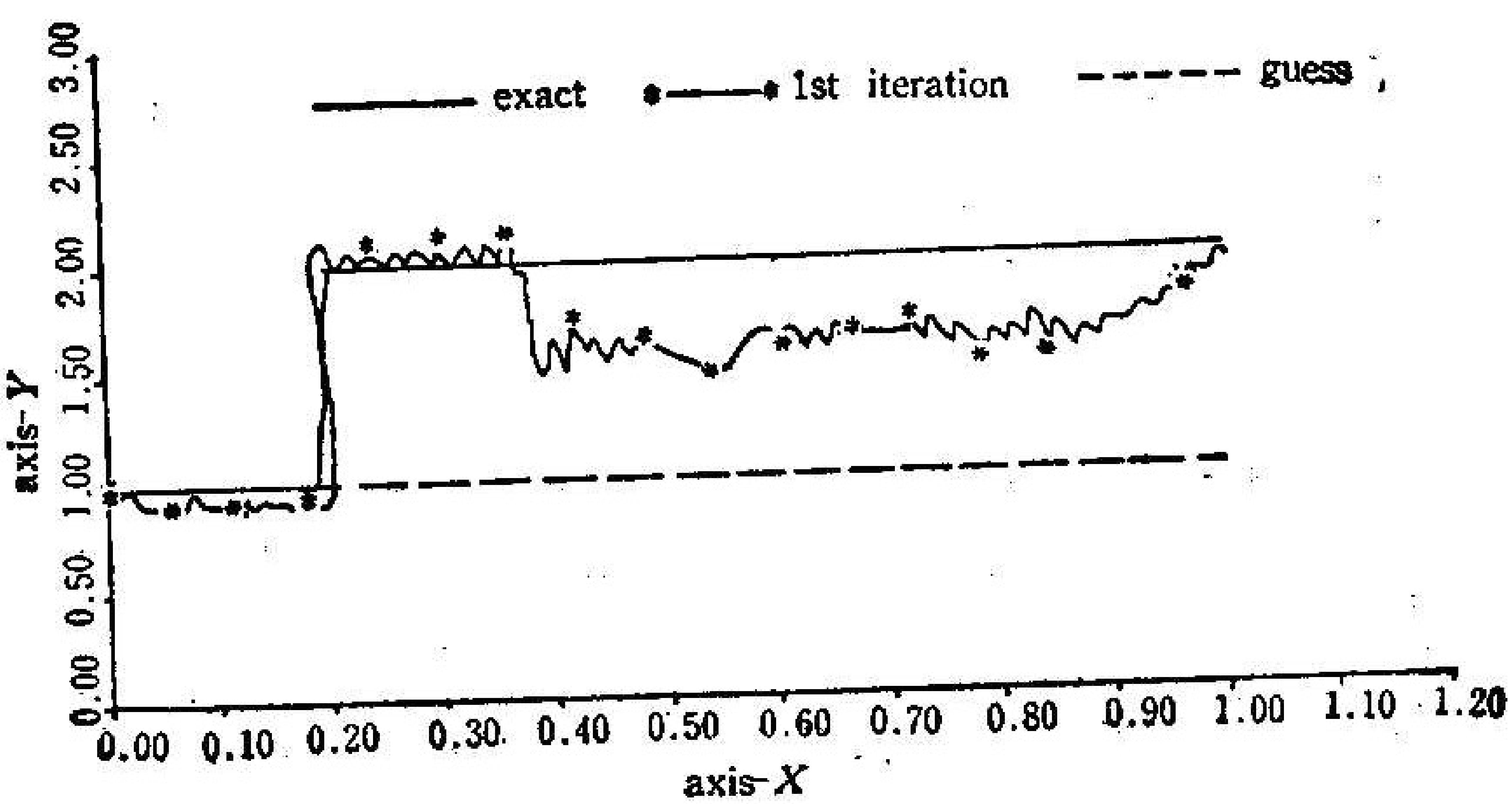

Fig. 12

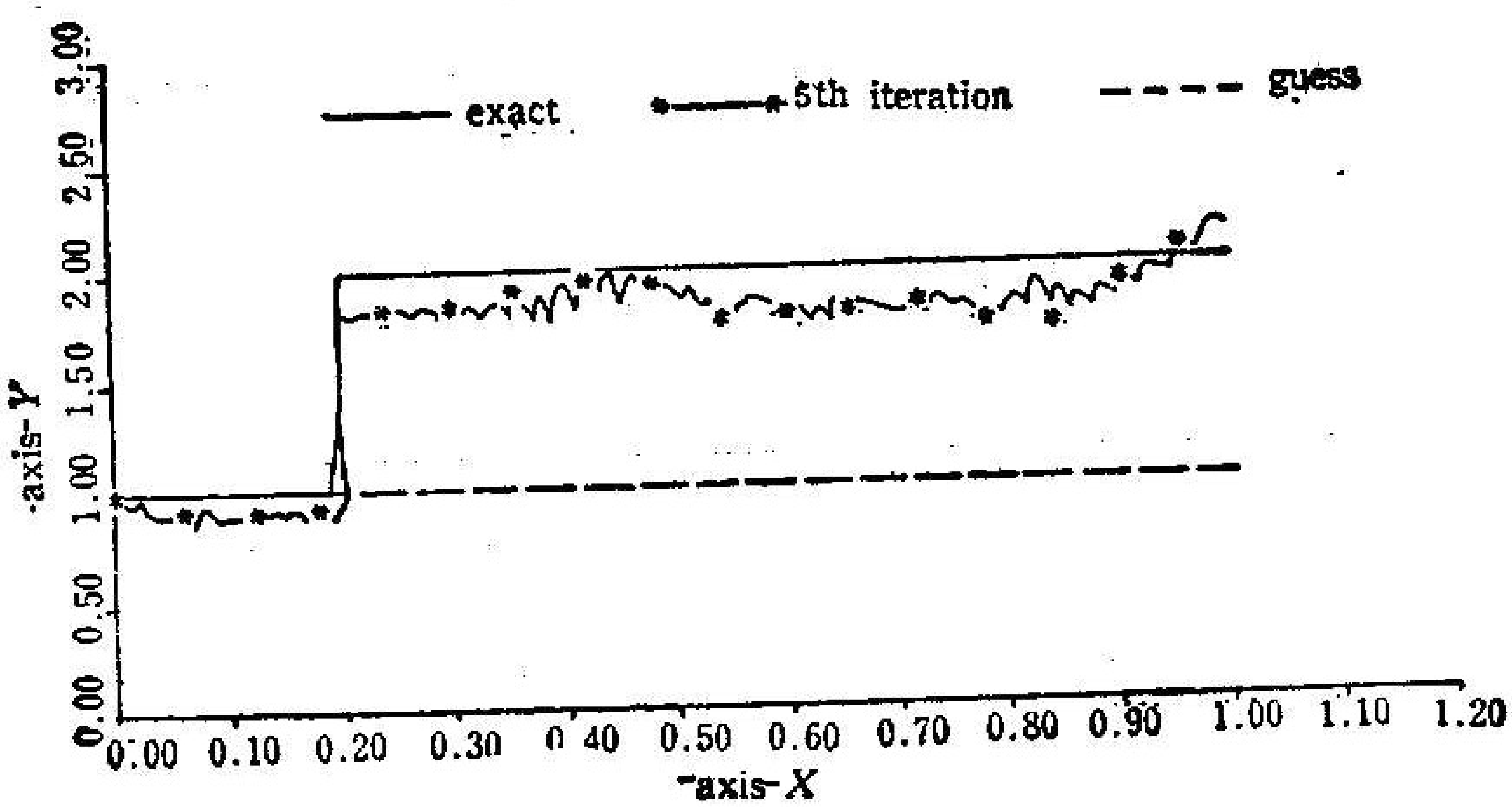

Fig. 13

5.2. Discussion

Recent contributions on the computing method of the inverse problems of a wave equation include [3]—[4] and [6]—[8]. The algorithm of [4] is very satisfactory in the one-dimensional case. However, it is difficult to extend this method to high dimensions. The method of [3] has led to excellent results for solving one-dimensional inverse problems. Especially, it is easy to be extended to the high dimensional case. For the singular integral equation, the method of [5] is efficient. This paper is a development of [3]. The method is not only stable and fast converging, but also very simple. Particularly, even if the given data are contaminated by noise, satisfactory results are also obtained by adopting the method of CI. Moreover, the method in this paper may be extended to the 2-D case, though it is necessary to use Tikhonov's regularization method[2] in the high dimensional case. When properly modified the method can be applied to solve the inverse problem of a wave equation with nonhomogeneous initial conditions.

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