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INTERVAL ITERATIVE METHODS UNDER PARTIAL ORDERING (II)*

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Abstract

Many types of nonlinear systems can be solved by using ordered iterative methods. These systems are discussed in [2] in a unified form for five different initial conditions. This paper is a continuation of [2]. Under arbitrary initial conditions, some iterative methods are given, and several theorems for the existence and uniqueness of the solution and convergence of the methods are proved.

§ 1. Introduction

In this paper we consider nonlinear systems

$$\varphi(x)=x, \quad x\in R^n. \tag{1.1}$$

Suppose there are f_i: $R^{r_i}\times R^{s_i}\to R$, such that

$$\varphi_i(x)=f_i(A_ix,\ B_ix), \quad i=1,\ 2,\ \cdots,\ n \tag{1.2}$$

where $A_i\in R^{r_i\times n}$, $B_i\in R^{s_i\times n}$, $0\leqslant r_i$, $s_i\leqslant n$, $f_i(A_ix, B_iy)$ are isotone in x and antitone in y when the latter are comparable, that is, as $x\leqslant x'$, $y\geqslant y'$, $x\leqslant y$ or $x\geqslant y$, $x'\leqslant y'$ or $x'\geqslant y'$, we have

$$f_i(A_ix,\ B_iy)\leqslant f_i(A_ix',\ B_iy'), \quad i=1,\ 2,\ \cdots,\ n.$$

Most of the functions discussed in [1] (13.2—13.5) can be written in form of (1.2). For simplicity, we suppose $A=A_i$, $B=B_i$, $i=1, 2, \cdots, n$, and consider

$$\varphi(x)=f(Ax,\ Bx)=x. \tag{1.3}$$

Clearly, (1.3) and (1.2) are equivalent.

We define some notation as follows:

$[\underline{x}, \bar{x}]=\{u\,|\,\underline{x}\leqslant u\leqslant\bar{x}\}$ is an n-dimensional interval vector, $\underline{x}$, $\bar{x}\in R^n$.

$N=\{1, 2, \cdots, n\}$.

$F[\underline{x}, \bar{x}]=[f(A\underline{x}, B\bar{x}), f(A\bar{x}, B\underline{x})]$.

$L_w[\underline{x}, \bar{x}]=[\underline{x}+w(f(A\underline{x}, B\bar{x})-\underline{x}), \bar{x}+w(f(A\bar{x}, B\underline{x})-\bar{x})]$ where $w\in R$, $w>1$.

$R[\underline{x}, \bar{x}]=[\underline{x}+Q(f(A\bar{x}, B\underline{x})-\bar{x}), \bar{x}+Q(f(A\underline{x}, B\bar{x})-\underline{x})]$ where Q is a nonnegative and nonsingular $n\times n$ matrix.

We will use the following lemmas.

Lemma 1. (1) *F is an inclusion monotonic interval extension of $\varphi(x)=f(Ax, Bx)$.*

(2) *If there exists $1>\beta>0$ such that*

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$$f(Ax, By)-f(Ax', By')\geqslant\beta(x-x'),\ y'\geqslant y,\ x\geqslant x'$$

for all comparable x, y and x', y', let $1/(1-\beta)\geqslant w>1$. Then L_w is an inclusion monotonic interval extension of $l(x)=x+w(f(Ax, Bx)-x)$.

(3) *If there exists $P\in R^{n\times n}$, such that*

$$f(Ax, By)-f(Ax', By')\leqslant P(y'-y)+(x-x'),\quad y'\geqslant y,\ x\geqslant x'$$

for all comparable x, y and x', y', let Q be a nonnegative, nonsingular, left subinverse of P. Then R is an inclusion monotonic interval extension of $r(x)=x+Q(f(Ax, Bx)-x)$.

Lemma 1 is a conclusion of several theorems in [2].

Lemma 2. *Let f: $R^n\to R^n$ be continuously differentiable on R^n. Assume that $f'(x)-I$ is nonsingular and $\|(f'(x)-I)^{-1}\|\leqslant\beta<\infty$ for all $x\in R^n$. Then for any fixed $x^0\in R^n$, there exists a unique continuously differentiable mapping x: $[0, 1]\to R^n$ such that*

$$g(x(t),t)=x(t),$$

$$x'(t)=(f'(x)-I)^{-1}(x^0-\hat{x}),\quad t\in[0, 1],\ x(0)=x^0$$

where $$g(x, t)=tf(x)+(1-t)d(x),\ d(x)=f(x)-\hat{x}+x^0,\ f(x^0)=\hat{x}.$$

§ 2. Algorithms and Convergence

Algorithm 1. Define initial interval $[\underline{x}^0, \bar{x}^0]$.

1. If $F[\underline{x}^k, \bar{x}^k]\cap R[\underline{x}^k, \bar{x}^k]\cap[\underline{x}^k, \bar{x}^k]=\varnothing$, then the algorithm is stopped.
2. $[\underline{x}^{k+1}, \bar{x}^{k+1}]=F[\underline{x}^k, \bar{x}^k]\cap R[\underline{x}^k, \bar{x}^k]\cap[\underline{x}^k, \bar{x}^k]$.

Theorem 1. *Suppose that $f(Ax, By)$ is continuous in x, $y\in[\underline{x}^0, \bar{x}^0]$ and there are $1\geqslant r>0$, $P=\mathrm{diag}(p_1, p_2, \cdots, p_n)>0$, such that*

$$f(Ax, By)-f(Ax', By')\leqslant P(y'-y)+(x-x'),\tag{2.1}$$

$$|f(Ax, Bx')-x|+|f(Ax', Bx)-x'|\geqslant r(x-x')\tag{2.2}$$

for all comparable x, y and x', y', $y\leqslant y'$, $x\geqslant x'$, x, y, x', $y'\in[\underline{x}^0, \bar{x}^0]$. Then there exists a unique solution of (1.3) in $[\underline{x}^0, \bar{x}^0]$ if and only if Algorithm 1 can be continued indefinitely. In this case it yields a sequence $\{[\underline{x}^k, \bar{x}^k]\}$ for which

(1) $$[\underline{x}^{k+1}, \bar{x}^{k+1}]\subseteq[\underline{x}^k, \bar{x}^k],\tag{2.3}$$

$$\bar{x}^{k+1}-\underline{x}^{k+1}\leqslant t(\bar{x}^k-\underline{x}^k)\tag{2.4}$$

where $0\leqslant t=\max\limits_{1\leqslant i\leqslant n}\{1-q_ir/(q_i+1)\}$, $Q=\mathrm{diag}(q_1, q_2, \cdots, q_n)>0$, $QP\leqslant I$,

$$\lim_{k\to\infty}\bar{x}^k=\lim_{k\to\infty}\underline{x}^k=x';$$

(2) *there exists a unique solution $x^*=x'$ of (1.3).*

Proof. If there exists a solution x^* of (1.3) in $[\underline{x}^0, \bar{x}^0]$, then by Lemma 1 we have

$$x^*\in F[\underline{x}^0, \bar{x}^0],\ x^*\in R[\underline{x}^0, \bar{x}^0].$$

From Algorithm 1, we have $x^*\in[\underline{x}', \bar{x}']$. We can easily show by induction that

$$x^*\in[\underline{x}^k, \bar{x}^k].$$

Hence, Algorithm 1 is continued indefinitely and not stopped.

Assume that the algorithm can be continued indefinitely.

(1) From the algorithm, (2.3) holds clearly. Now we prove that (2.4) holds. For any $i\in N$, $F_i[\underline{x}^k, \bar{x}^k]$ and $[\underline{x}^k, \bar{x}^k]$ there exist only the following four cases

$$[\underline{x}_i^k, \bar{x}_i^k]\subseteq F_i[\underline{x}^k, \bar{x}^k], \tag{2.5}$$

$$F_i[\underline{x}^k, \bar{x}^k]\subseteq[\underline{x}_i^k, \bar{x}_i^k]. \tag{2.6}$$

$$\underline{x}_i^k\le \underline{F}_i[\underline{x}^k, \bar{x}^k],\ \bar{x}_i^k\le \bar{F}_i[\underline{x}^k, \bar{x}^k], \tag{2.7}$$

$$\underline{F}_i[\underline{x}^k, \bar{x}^k]\le\underline{x}_i^k,\ \bar{F}_i[\underline{x}^k, \bar{x}^k]\le\bar{x}_i^k \tag{2.8}$$

which will be discussed respectively.

If (2.5) holds, then

$$R_i[\underline{x}^k, \bar{x}^k]\subseteq[\underline{x}_i^k, \bar{x}_i^k]\subseteq F_i[\underline{x}^k, \bar{x}^k],$$

$$\begin{aligned}\bar{x}_i^{k+1}-\underline{x}_i^{k+1}&=wR_i[\underline{x}^k, \bar{x}^k]\\&=\bar{x}_i^k-\underline{x}_i^k+q_i(f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k)-q_i(f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k)\\&=\bar{x}_i^k-\underline{x}_i^k-q_i(|f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k|+|f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k|)\\&\le(1-q_ir)(\bar{x}_i^k-\underline{x}_i^k)\le t(\bar{x}_i^k-\underline{x}_i^k).\end{aligned}$$

If (2.6) holds, then

$$F_i[\underline{x}^k, \bar{x}^k]\subseteq[\underline{x}_i^k, \bar{x}_i^k]\subseteq R_i[\underline{x}^k, \bar{x}^k],$$

$$\begin{aligned}\bar{x}_i^{k+1}-\underline{x}_i^{k+1}&=wF_i[\underline{x}^k, \bar{x}^k]=f_i(A\bar{x}^k, B\underline{x}^k)-f_i(A\underline{x}^k, B\bar{x}^k)\\&=\bar{x}_i^k-\underline{x}_i^k-(|\bar{x}_i^k-f_i(A\bar{x}^k, B\underline{x}^k)|+|f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k|)\\&\le(1-r)(\bar{x}_i^k-\underline{x}_i^k)\le t(\bar{x}_i^k-\underline{x}_i^k).\end{aligned}$$

If (2.7) holds, then

$$[\underline{x}_i^{k+1}, \bar{x}_i^{k+1}]=[f_i(A\underline{x}^k, B\bar{x}^k), \bar{x}_i^k]\cap R_i[\underline{x}^k, \bar{x}^k].$$

From $f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k\ge0$, we have

$$\underline{x}_i^{k+1}=\max\{f_i(A\underline{x}^k, B\bar{x}^k), \underline{x}_i^k+q_i(f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k)\},$$

$$\bar{x}_i^{k+1}=\bar{x}_i^k.$$

If

$$f_i(A\underline{x}^k, B\bar{x}^k)\ge\underline{x}_i^k+q_i(f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k),$$

from (2.2) and (2.7) we have

$$\begin{aligned}f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k&\ge q_i(f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k)\\&\ge q_ir(\bar{x}_i^k-\underline{x}_i^k)-q_i(f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k)\end{aligned}$$

that is

$$f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k\ge q_ir/(q_i+1)(\bar{x}_i^k-\underline{x}_i^k).$$

Hence

$$\begin{aligned}\bar{x}_i^{k+1}-\underline{x}_i^{k+1}&=\bar{x}_i^k-f_i(A\underline{x}^k, B\bar{x}^k)=\bar{x}_i^k-\underline{x}_i^k-(f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k)\\&\le(1-q_ir/(q_i+1))(\bar{x}_i^k-\underline{x}_i^k)\le t(\bar{x}_i^k-\underline{x}_i^k).\end{aligned}$$

If

$$f_i(A\underline{x}^k, B\bar{x}^k)<\underline{x}_i^k+q_i(f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k)$$

then we have

$$q_i(f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k)\geqslant f_i(A\underline{x}^k, B\bar{x}^k)-\underline{x}_i^k$$
$$\geqslant r(\bar{x}_i^k-\underline{x}_i^k)-(f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k),$$

that is

$$f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k\geqslant r/(1+q_i)(\bar{x}_i^k-\underline{x}_i^k).$$

Hence

$$\bar{x}_i^{k+1}-\underline{x}_i^{k+1}=\bar{x}_i^k-\underline{x}_i^k-q_i(f_i(A\bar{x}^k, B\underline{x}^k)-\bar{x}_i^k)$$
$$\leqslant\bar{x}_i^k-\underline{x}_i^k-q_ir/(1+q_i)(\bar{x}_i^k-\underline{x}_i^k)\leqslant t(\bar{x}_i^k-\underline{x}_i^k).$$

Following the proof of (2.7), we can prove (2.8).

Therefore, we have

$$\bar{x}^{k+1}-\underline{x}^{k+1}\leqslant t(\bar{x}^k-\underline{x}^k),\quad 1>t\geqslant 0,$$
$$\lim_{k\to\infty}\bar{x}^k=\lim_{k\to\infty}\underline{x}^k=x'.$$

(2) Because

$$F[\underline{x}^k, \bar{x}^k]\cap[\underline{x}^k, \bar{x}^k]=\varnothing,\quad k=0, 1, \cdots$$

therefore

$$\lim_{k\to\infty}|f(A\underline{x}^k, B\bar{x}^k)-\underline{x}^k|\leqslant\lim_{k\to\infty}(wF[\underline{x}^k, \bar{x}^k]+w[\underline{x}^k, \bar{x}^k])$$
$$=\lim_{k\to\infty}|f(A\bar{x}^k, B\underline{x}^k)-f(A\underline{x}^k, B\bar{x}^k)+\bar{x}^k-\underline{x}^k|=0,$$

that is

$$\lim_{k\to\infty}|f(A\underline{x}^k, B\bar{x}^k)-\underline{x}^k|=|f(Ax', Bx')-x'|=0.$$

Hence, $x'=x^*$ is the unique solution of (1.3) in $[\underline{x}^0, \bar{x}^0]$.

Algorithm 2. Define initial interval $[\underline{x}^0, \bar{x}^0]$.

1. If $L_w[\underline{x}^k, \bar{x}^k]\cap R[\underline{x}^k, \bar{x}^k]\cap[\underline{x}^k, \bar{x}^k]=\varnothing$, then the algorithm is stopped.
2. $[\underline{x}^{k+1}, \bar{x}^{k+1}]=L_w[\underline{x}^k, \bar{x}^k]\cap R[\underline{x}^k, \bar{x}^k]\cap[\underline{x}^k, \bar{x}^k]$ where $w=1/(1-\beta)$.

Theorem 2. *Suppose that the conditions of Theorem* 1 *hold and there exists* $0<\beta<1$, *such that*

$$f(Ax, By)-f(Ax', By')\geqslant\beta(x-x'),\quad y'\geqslant y,\ x\geqslant x'$$

for all comparable x, y *and* x', y'. *Then there exists a unique solution of* (1.1) *in* $[\underline{x}^0, \bar{x}^0]$ *if and only if Algorithm* 2 *can be continued indefinitely. In this case it yields a sequence* $\{[\underline{x}^k, \bar{x}^k]\}$ *for which*

(1) $$[\underline{x}^{k+1}, \bar{x}^{k+1}]\subseteq[\underline{x}^k, \bar{x}^k].$$
$$\bar{x}^{k+1}-\underline{x}^{k+1}\leqslant t(\bar{x}^k-\underline{x}^k),$$

where $0<t=\max_{1\leqslant i\leqslant n}\{1-wq_ir/(w+q_i)\}<1$,

$$\lim_{k\to\infty}\bar{x}^k=\lim_{k\to\infty}\underline{x}^k=x';$$

(2) *there exists a unique solution* $x^*=x'$ *of* (1.3) *in* $[\underline{x}^0, \bar{x}^0]$ and

$$x^*=\bigcap_{k=0}^{\infty}[\underline{x}^k, \bar{x}^k],$$

$$|x^*-\tilde{x}^k|\leqslant 1/2\cdot t(\bar{x}^k-\underline{x}^k),$$

where $\tilde{x}^k=m[\underline{x}^k, \bar{x}^k]$.

Following the proof of Theorem 1, we can prove this theorem. And from

$$\underline{L}_w[\underline{x}, \bar{x}]\geqslant\underline{F}[\underline{x}, \bar{x}], \quad \text{as } \underline{x}\leqslant\underline{F}[\underline{x}, \bar{x}], \tag{2.9}$$

$$\bar{L}_w[\underline{x}, \bar{x}]\leqslant\bar{F}[\underline{x}, \bar{x}], \quad \text{as } \bar{x}\geqslant\bar{F}[\underline{x}, \bar{x}] \tag{2.10}$$

we find, as we did for the four cases (2.5)—(2.8) in Theorem 1,

$$[\underline{x}^k, \bar{x}^k]\subseteq[\underline{y}^k, \bar{y}^k], \quad \text{as } [\underline{x}^0, \bar{x}^0]=[\underline{y}^0, \bar{y}^0]$$

where the interval sequences $[\underline{x}^k, \bar{x}^k]$ and $[\underline{y}^k, \bar{y}^k]$ are given by Algorithms 2 and 1 respectively.

From

$$1-qr/(1+q)>1-wqr/(w+q), \quad w>1$$

and (2.9), (2.10), Algorithm 2 converges more quickly than Algorithm 1.

Now we discuss the continuation-interval method.

For (1.3) let us first assume that the homotopy defining the continuation process is given in the form

$$g(x, t)=tf(Ax, Bx)+(1-t)d(x)$$

where $d(x)=f(Ax, Bx)-\hat{x}+x^0$, $f(Ax^0, Bx^0)=\hat{x}$.

Let

$$h(x, t)=x+Q(g(x, t)-x)$$

where $Q\in R^{n\times n}$ is a nonnegative, nonsingular, left subinverse of P in Lemma 1.

Clearly, the systems

$$g(x, t)=x,$$

$$h(x, t)=x$$

are equivalent. Obviously,

$$G([\underline{x}, \bar{x}], t)=F[\underline{x}, \bar{x}]-(1-t)(\hat{x}-x^0),$$

$$H([\underline{x}, \bar{x}], t)=R[\underline{x}, \bar{x}]-Q(1-t)(\hat{x}-x^0)$$

are inclusion monotonic interval extensions of $g(x, t)$ and $h(x, t)$ respectively, where

$$g(x, t)=f(Ax, Bx)-(1-t)(\hat{x}-x^0),$$

$$h(x, t)=r(x)-Q(1-t)(\hat{x}-x^0).$$

Algorithm 3. 1. Define initial point x^0, compute $f(x^0)$.

2. $[\underline{x}^{0,m_0}, \bar{x}^{0,m_0}]=x^0$,

$$[\underline{x}^{i,0}, \bar{x}^{i,0}]=[\underline{x}^{i-1,m_{i-1}}-c_i, \bar{x}^{i-1,m_{i-1}}+c_i], \quad i=1, 2, \cdots, M,$$

$$[\underline{x}^{i,k+1}, \bar{x}^{i,k+1}]=[\underline{x}^{i,k}, \bar{x}^{i,k}]\cap G([\underline{x}^{i,k}, \bar{x}^{i,k}], t_i)\cap H([\underline{x}^{i,k}, \bar{x}^{i,k}], t_i),$$

$$k=0, 1, \cdots, m_i-1, \ i=1, 2, \cdots, M-1.$$

3. $[\underline{x}^{M,k+1}, \bar{x}^{M,k+1}]=[\underline{x}^{M,k}, \bar{x}^{M,k}]\cap G([\underline{x}^{M,k}, \bar{x}^{M,k}], 1)\cap H([\underline{x}^{M,k}, \bar{x}^{M,k}], 1)$

$$=[\underline{x}^{M,k}, \bar{x}^{M,k}]\cap F[\underline{x}^{M,k}, \bar{x}^{M,k}]\cap R[\underline{x}^{M,k}, \bar{x}^{M,k}],$$

$$k=0, 1, \cdots, \text{ where } 0=t_0<t_1<\cdots<t_M=1, \ \delta_i=t_i-t_{i-1},$$

$$c_i=\beta\|x^0-\hat{x}\|\delta_i+1/2(\bar{x}^{i-1,m_{i-1}-1}-\underline{x}^{i-1,m_{i-1}-1}).$$

Theorem 3. *Suppose that the conditions of Theorem 1 hold on R^n, and $f(Ax, Bx)$ is continuously differentiable on R^n. $f'(Ax, Bx)$ is nonsingular and $\|f'(Ax, Bx)-I\| \leqslant \beta < \infty$. Then for any fixed $x^0 \in R^n$, the sequence defined by Algorithm 3 converges to the unique solution of (1.3), and*

$$x(t_i) \in [\underline{x}^{i,k}, \bar{x}^{i,k}],$$

$$[\underline{x}^{i,m_i}, \bar{x}^{i,m_i}] \subseteq [\underline{x}^{i,m_i-1}, \bar{x}^{i,m_i-1}] \subseteq \cdots \subseteq [\underline{x}^{i,0}, \bar{x}^{i,0}],$$

$$k=0, 1, \cdots, m_i,\ i=1, 2, \cdots, M-1,$$

$$x(1) \in [\underline{x}^{M,k+1}, \bar{x}^{M,k+1}] \subseteq [\underline{x}^{M,k}, \bar{x}^{M,k}], \quad k=0, 1, \cdots,$$

$$\lim_{k\to\infty} [\underline{x}^{M,k}, \bar{x}^{M,k}] = x(1),$$

$$|x(1) - \underline{x}^{M,k}| \leqslant t(\bar{x}^{M,k} - \underline{x}^{M,k}), \quad 0<t<1.$$

In order to prove Theorem 3, suppose that $Q([\underline{x}, \bar{x}], t) = G([\underline{x}, \bar{x}], t) \cap H([\underline{x}, \bar{x}], t)$ is an inclusion operator of g. Since $g(x(t), t) = x(t)$ and $h(x(t), t) = x(t)$ are equivalent, from Lemma 1, if

$$x^*(t) = g(x^*(t), t), \quad x^*(t) \in [\underline{x}, \bar{x}]$$

then

$$x^*(t) \in G([\underline{x}, \bar{x}], t) \cap H([\underline{x}, \bar{x}], t) = Q([\underline{x}, \bar{x}], t).$$

From Theorem 1, we have

$$W(Q([\underline{x}, \bar{x}], 1) \cap [\underline{x}, \bar{x}]) \leqslant tW[\underline{x}, \bar{x}], \quad 0<t<1.$$

From Theorem 3 in [3], we can prove Theorem 3.

Example 1. Let

$$f(Ax, Bx) = \begin{pmatrix} -0.001x_1^3 + x_2 + 0.008 \\ 0.1x_1 - 0.01x_2^3 + 1.88 \end{pmatrix}.$$

We compute this example using Algorithm 3.

Let $x^0 = (5, 5)^T$, $t_0 = 0$, $t_1 = 1/2$, $t_2 = 1$. We can test that the conditions of Theorem 3 hold, where $\beta = 2.03$, $f(x^0) = (4.883, 1.13)^T = \hat{x}$, $A = B = I$.

$$F[\underline{x}, \bar{x}] = \left[\begin{pmatrix} -0.001\bar{x}_1^3 + \underline{x}_2 + 0.008 \\ 0.1\underline{x}_1 - 0.01\bar{x}_2^3 + 1.88 \end{pmatrix}, \begin{pmatrix} -0.001\underline{x}_1^3 + \bar{x}_2 + 0.008 \\ 0.1\bar{x}_1 - 0.01\underline{x}_2^3 + 1.88 \end{pmatrix}\right],$$

$$R[\underline{x}, \bar{x}] = \left[\begin{pmatrix} \underline{x}_1 + 4.1753(-0.001\underline{x}_1^3 + \bar{x}_2 + 0.008 - \bar{x}_1) \\ \underline{x}_2 + 0.4175(0.1\bar{x}_1 - 0.01\underline{x}_2^3 + 1.88 - \bar{x}_2) \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \bar{x}_1 + 4.1753(-0.001\bar{x}_1^3 + \underline{x}_2 + 0.008 - \underline{x}_1) \\ \bar{x}_2 + 0.4175(0.1\underline{x}_1 - 0.01\bar{x}_2^3 + 1.88 - \underline{x}_2) \end{pmatrix}\right];$$

$$[\underline{x}^{1,0}, \bar{x}^{1,0}] = \left[\begin{pmatrix} 1.07875 \\ 1.07195 \end{pmatrix}, \begin{pmatrix} 8.2244 \\ 2.7268 \end{pmatrix}\right],$$

$$[\underline{x}^{1,1}, \bar{x}^{1,1}] = \left[\begin{pmatrix} 1.13725 \\ 1.07195 \end{pmatrix}, \begin{pmatrix} 8.2829 \\ 4.6618 \end{pmatrix}\right],$$

$$[\underline{x}^{1,2}, \bar{x}^{1,2}] = \left[\begin{pmatrix} 1.13725 \\ 2.9156 \end{pmatrix}, \begin{pmatrix} 4.7268 \\ 4.6309 \end{pmatrix}\right],$$

$$[\underline{x}^{1,3}, \bar{x}^{1,3}] = \left[\begin{pmatrix} 2.8765 \\ 2.9356 \end{pmatrix}, \begin{pmatrix} 4.6959 \\ 4.0393 \end{pmatrix}\right],$$

$$[\underline{x}^{2,0}, \bar{x}^{2,0}] = \left[\begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 9.6959 \\ 8.7495 \end{pmatrix}\right],$$

$$[\underline{x}^{2,1}, \bar{x}^{2,1}] = \left[\begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 6.0819 \\ 2.8495 \end{pmatrix}\right],$$

$$[\underline{x}^{2,2}, \bar{x}^{2,2}] = \left[\begin{pmatrix} 0.0 \\ 1.6487 \end{pmatrix}, \begin{pmatrix} 2.8578 \\ 2.4881 \end{pmatrix}\right],$$

$$[\underline{x}^{2,3}, \bar{x}^{2,3}] = \left[\begin{pmatrix} 1.6334 \\ 1.726 \end{pmatrix}, \begin{pmatrix} 2.4961 \\ 2.1209 \end{pmatrix}\right],$$

$$[\underline{x}^{2,4}, \bar{x}^{2,4}] = \left[\begin{pmatrix} 1.7185 \\ 1.9479 \end{pmatrix}, \begin{pmatrix} 2.1246 \\ 2.0782 \end{pmatrix}\right]$$

$$[\underline{x}^{2,5}, \bar{x}^{2,5}] = \left[\begin{pmatrix} 1.9463 \\ 1.9621 \end{pmatrix}, \begin{pmatrix} 2.0833 \\ 2.0185 \end{pmatrix}\right]$$

$x^* = (2, 2)^T$.

References

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