THE SPECTRAL METHOD FOR SYMMETRIC REGULARIZED WAVE EQUATIONS*

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§ 1. Introduction

A symmetric version of the regularized long wave equations (SRLWE)

$$\left(\frac{\partial^2}{\partial x^2}-1\right)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\rho+\frac{u^2}{2}\right),\qquad \frac{\partial \rho}{\partial t}+\frac{\partial u}{\partial x}=0 \tag{1.1}$$

has been investigated in [1]. The system (1.1) of equations is shown to describe weakly nonlinear ion acoustic and space-change waves. The hyperbolic secant squared solitary waves, four invariants and the numerical results have been obtained in [1]. Obviously, eliminating ρ in (1.1), we get a class of RLWE

$$u_{tt}-u_{xx}+\left(\frac{1}{2}u^2\right)_{xt}-v_{xxtt}=0. \tag{1.2}$$

Replacing the derivative for t with the derivative for x in the third and the fourth terms of (1.2), we get the Boussinesq equation. In this note we consider the periodic initial value problem for generalized nonlinear wave equations (including (1.1))

$$\begin{cases} u_t-u_{xxt}+\rho_x+f(u)_x=g(u,\ \rho,\ u_x), & (1.3)\\ \rho_t+u_x=h(\rho), & (1.4)\\ u|_{t=0}=u_0(x),\ \rho|_{t=0}=\rho_0(x), \quad -\infty<x<\infty, & (1.5)\\ u(x-\pi,\ t)=u(x+\pi,\ t),\ \rho(x-\pi,\ t)=\rho(x+\pi,\ t),\ -\infty<x<\infty,\ t\geqslant 0, & (1.6)\end{cases}$$

where $u(x, t)$, $\rho(x, t)$ are unknown real functions, and $f(u)$, $h(\rho)$ are known real functions. We propose the spectral method (continued and discrete) for the problem (1.3)—(1.6), establish the error estimates and convergence for the approximate solution, and prove the existence and uniqueness of the classical smooth solution for the system (1.3)—(1.6).

§ 2. Continued Spectral Method and Priori Estimates

First we introduce some spaces and notations. Let $C^l(\Omega)=C^l([-\pi,\ \pi])$ denote the space of functions, l times continuously differentiable over the interval $[-\pi,\ \pi]$. $L_p(\Omega)$ denotes the Lebesque space of measurable functions $u(x)$ with p-th power absolute value $|u|$ integrable over the interval $[-\pi,\ \pi]$ with the norm

$$\|u\|_{L_p}=\left(\int_{-\pi}^{\pi}|u|^p\,dx\right)^{\frac{1}{p}}.$$

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If we define the inner product

$$(u, v)=\int_{-\pi}^{\pi} u(x)v(x)dx, \quad \|u\|_{L_2}^2=(u, u),$$

then $L_2[-\pi, \pi]$ is a Hilbert space.

Let $L_\infty(\Omega)$ denote the Lebesque space of measurable functions $u(x)$ over the interval $[-\pi, \pi]$, which are essentially bounded, with the norm

$$\|u\|_{L_\infty}=\operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

Let $H^l(\Omega)$ denote the space of the functions with generalized derivatives $D^s u(|s|\leqslant l)$ with the norm $\|u\|_l^2=\sum_{|s|\leqslant l} \|D^s u\|_{L_2}^2$. $L^\infty(0, T; H^l)$ denotes the space of the functions $u(x, t)$ which belong to H^l as a function of x for every fixed $t(0\leqslant t\leqslant T)$ and $\sup_{0\leqslant t\leqslant T}\|u(\cdot, t)\|_l<\infty$. Especially,

$$\|u\|_{L^\infty(0,T;L_2)}=\sup_{0\leqslant t\leqslant T}\|u\|_{L_2} \quad \text{or} \quad \|u\|_{L_2\times L_\infty}.$$

Let $V^l=\{u\in H^l(\Omega)\,|\,u^j(x-\pi)=u^j(x+\pi),\ 0\leqslant j\leqslant l-1\}$ be a periodic functional space, where $u^j=\frac{d^j u}{dx^j}$,

$$\|u\|_V^2=\|u\|_{L_2}^2+\left\|\frac{du}{dx}\right\|_{L_2}^2, \quad V\in H^1,\ H=L_2.$$

For the backward difference quotient of $u(x, t)$ for t, we employ the following notation

$$u_{\bar t}(x, t)=\frac{1}{\Delta t}[u(x, t)-u(x, t-\Delta t)].$$

Let F_k denote the projection from H to $H_k=\mathrm{span}(v_{-k}, \cdots, v_k)$,

$$F_k g=\sum_{j=-k}^{k}(g, v_j)v_j,$$

where $v_j=\frac{1}{\sqrt{2\pi}}e^{ijx}$, $i=\sqrt{-1}$.

Set $R_k g=g-F_k g$, when $k\to\infty$, $R_k g\to 0$. From the Bessal inequality, we have

$$\|F_k g\|_{L_2}\leqslant\|g\|_{L_2}, \quad g\in H=L_2, \tag{2.1}$$

and Bernstein's estimate[9].

Suppose that the periodic function $g(x)$ is k $(k\geqslant 1)$ times differentiable and the k-th derivative is bounded, i.e.,

$$|g^{(k)}(x)|\leqslant M_k. \tag{2.2}$$

Then there exists a positive constant A, such that

$$|R_n g|\leqslant AM_k \log n/n^k, \quad n\geqslant 2. \tag{2.3}$$

In this section, we consider the continued spectral method. We construct the approximate solutions of the problem (1.3)—(1.6) as follows

$$\begin{aligned} u_k(\cdot, t)=u_k(t)&=\sum_{j=-k}^{k}\alpha_{jk}(t)v_j(x),\\ \rho_k(\cdot, t)=\rho_k(t)&=\sum_{j=-k}^{k}\beta_{jk}(t)v_j(x). \end{aligned} \tag{2.4}$$

The coefficient functions $\alpha_{jk}(t)$, $\beta_{jk}(t)$ should satisfy the equations

$$\begin{cases}(u_{kt}-u_{kxxt}+\rho_{kx}+f(u_k)_x,\ v_j(x))=(g(u_k,\ \rho_k,\ u_{kx}),\ v_j(x)) & (2.5)\\ (\rho_{kt}+u_{kx},\ v_j(x))=(h(\rho_k),\ v_j(x)), \quad j=-k,\ \cdots,\ k & (2.6)\end{cases}$$

with the initial conditions

$$\begin{cases}u_k|_{t=0}=u_{0k}(x)=F_k u_0(x),\\ \rho_k|_{t=0}=\rho_{0k}(x)=F_k\rho_0(x).\end{cases} \tag{2.7}$$

The problem (2.5)—(2.7) can be considered as an initial value problem of a system of nonlinear ordinary differential equations of first order with unknown functions $\alpha_{jk}(t)$, $\beta_{jk}(t)$. Because $\{v_j(x)\}$ is linearly independent and $v_j''(x)=-\lambda_j^2 v_j$ $\left(\lambda_j^2=\frac{j^2}{\sqrt{2\pi}}\right)$, and due to the following prior estimates, we know that there exists a global solution in the interval $[0,\ T]$ for the initial value problem (2.5)—(2.7).

Lemma 1. *Suppose that the following conditions are satisfied:*

(i) *the functions* $g(u,\ \rho,\ \eta)$, $h'(\rho)$ *are semi-bounded, i.e.,*

$$\begin{aligned}(u,\ g(u,\ \rho,\ u_x))&\leqslant C[(u,\ u)+(\rho,\ \rho)+(u_x,\ u_x)]\\ h'(\rho)&\leqslant C,\end{aligned} \tag{2.8}$$

where C is a positive constant, $h(0)=0$.

(ii) $u_0(x)\in H^1(\Omega)$, $\rho_0(x)\in L_2(\Omega)$.

Then for the solution of the problem (2.5)—(2.7) *we have the estimate*

$$\|u_k(\cdot,\ t)\|_{H^1\times L_\infty}+\|\rho_k(\cdot,\ t)\|_{L_2\times L_\infty}\leqslant E_1, \tag{2.9}$$

where the constant E_1 is independent of k.

Proof. Multiplying (2.5) by $\alpha_{jk}(t)$, and (2.6) by $\beta_{jk}(t)$, and summing for j from $-k$ to k, we have

$$(u_{kt}-u_{kxxt}+\rho_{kx}+f(u_k)_x,\ u_k)=(g(u_k,\ \rho_k,\ u_{kx}),\ u_k), \tag{2.10}$$

$$(\rho_{kt}+u_{kx},\ \rho_k)=(h(\rho_k),\ \rho_k). \tag{2.11}$$

Since

$$(\rho_{kt},\ \rho_k)=\frac{1}{2}\frac{d}{dt}\|\rho_k\|^2_{L_2}, \qquad (u_{kt},\ u_k)=\frac{1}{2}\frac{d}{dt}\|u_k\|^2_{L_2},$$

$$-(u_{kxxt},\ u_k)=\frac{1}{2}\frac{d}{dt}\|u_{kx}\|^2_{L_2},$$

$$(f(u_k)_x,\ u_k)=-(f(u_k),\ u_{kx})=(-F_x(u_k),\ 1)=0,$$

$$(\rho_{kx},\ u_k)=-(\rho_k,\ u_{kx}),$$

where $F(u)=\int_0^u f(s)ds$, adding (2.10) to (2.11), and from the hypotheses in the lemma, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\rho_k\|^2_{L_2}+\frac{1}{2}\frac{d}{dt}\|u_k\|^2_{L_2}+\frac{1}{2}\frac{d}{dt}\|u_{kx}\|^2_{L_2}\\ &=(g(u_k,\ \rho_k,\ u_{kx}),\ u_k)+(h'(\xi)\rho_k,\ \rho_k)\\ &\leqslant C[\|u_k\|^2_{L_2}+\|\rho_k\|^2_{L_2}+\|u_{kx}\|^2_{L_2}].\end{aligned} \tag{2.12}$$

Applying Gronwall's inequality to (2.12), and by inequality (2.1) of initial values, we obtain (2.9).

Lemma 2 (*Sobolev's estimates*). *Suppose that* $u\in L_q(\Omega)$, $D^m u\in L_r(\Omega)$, *where* $1\leqslant q,\ r\leqslant\infty$, $\Omega\subset R^n$. *Then there exists a contant C, such that*

$$\|D^j u\|_{L_p(\Omega)} \leq C \|D^m u\|^a_{L_r(\Omega)} \|u\|^{1-a}_{L_q(\Omega)}, \tag{2.13}$$

where $0 \leq j \leq m$, $j/m \leq a \leq 1$, $1 \leq p \leq \infty$, *and*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

Proof. See [4] or [5].

Lemma 3. *Suppose that the conditions of* Lemma 1 *are satisfied, and assume that*

(i) *for the bounded* u,

$$|g(u, \rho, u_x)| \leq C(|u_x|^3 + |\rho|^3), \tag{2.14}$$

where C *is a positive constant.*

(ii) $f(u) \in C^1$,

(iii) $u_0(x) \in H^2(\Omega)$, $\rho_0(x) \in H^1(\Omega)$.

Then for the solution of the problem (2.5)—(2.7), *we have*

$$\begin{aligned} &\|u_k(t)\|^2_{H^1 \times L_\infty} + \|\rho_k(t)\|^2_{H^1 \times L_\infty} \leq E_2, \\ &\|u_{kx}(t)\|^2_{L_\infty \times L_\infty} + \|\rho_k(t)\|^2_{L_\infty \times L_\infty} \leq E_2, \end{aligned} \tag{2.15}$$

where the constant E_2 *is independent of* k.

Proof. In view of $v_j''(x) = -\lambda_j^2 v_j(x)$, multiplying (2.5) by $\alpha_{jk}(t)$ and (2.6) by $\beta_{jk}(t)$, and summing for j from $-k$ to k, we obtain

$$(u_{kt} - u_{kxt} + \rho_{kx} + f(u_k)_x,\ -u_{kxx}) = (g(u_k, \rho_k, u_{kx}),\ -u_{kxx}), \tag{2.16}$$

$$(\rho_{kt} + u_{kx},\ -\rho_{kxx}) = (h(\rho_k),\ -\rho_{kxx}). \tag{2.17}$$

Since

$$(u_{kt},\ -u_{kxx}) = (u_{kxt},\ u_{kx}) = \frac{1}{2}\frac{d}{dt}\|u_{kx}\|^2_{L_2},$$

$$(-u_{kxxt},\ -u_{kxx}) = \frac{1}{2}\frac{d}{dt}\|u_{kxx}\|^2_{L_2},$$

$$\begin{aligned} |(f(u_k)_x,\ -u_{kxx})| &\leq \|f'(u_k)\|_{L_\infty} \cdot \frac{1}{2}(\|u_{kx}\|^2_{L_2} + \|u_{kxx}\|^2_{L_2}) \\ &\leq C(\|u_{kx}\|^2_{L_2} + \|u_{kxx}\|^2_{L_2}), \end{aligned}$$

where Sobolev's inequality has been used,

$$\|u\|_{L_\infty} \leq C(\|u_x\|_{L_2} + \|u\|_{L_2})^{\frac{1}{2}} \|u\|^{\frac{1}{2}}_{L_2}$$

$$\begin{aligned} |(g(u_k, \rho_k, u_{kx}),\ -u_{kxx})| &\leq C(|u_{kx}|^3 + |\rho_k|^3,\ |u_{kxx}|) \\ &\leq \frac{C}{2}(\|u_{kx}\|^6_{L_6} + \|\rho_k\|^6_{L_6} + 2\|u_{kxx}\|^2_{L_2}) \\ &\leq C_1[\|u_{kxx}\|^2_{L_2} + \|\rho_{kx}\|^2_{L_2} + 1]; \end{aligned}$$

here we have used Sobolev's estimate (Lemma 2)

$$\|u\|_{L_6} \leq C \|u_x\|^{\frac{1}{3}}_{L_2} \|u\|^{\frac{2}{3}}_{L_2}.$$

Since

$$(\rho_{kt},\ -\rho_{kxx}) = \frac{1}{2}\frac{d}{dt}\|\rho_{kx}\|^2_{L_2},$$

$$(u_{kx},\ -\rho_{kxx}) = (u_{kxx},\ \rho_{kx}) \leq \frac{1}{2}(\|u_{kxx}\|^2_{L_2} + \|\rho_{kx}\|^2_{L_2}),$$

$$(h(\rho_k),\ -\rho_{kx})=(h'(\xi)\rho_{kx},\ \rho_{kx})\leqslant C\|\rho_{kx}\|^2_{L_2},$$

from (2.16), (2.17) we get

$$\frac{d}{dt}[\|\rho_{kx}\|^2_{L_2}+\|u_{kx}\|^2_{L_2}+\|u_{kxx}\|^2_{L_2}]\leqslant C_2[\|\rho_{kx}\|^2_{L_2}+\|u_{kx}\|^2_{L_2}+\|u_{kxx}\|^2_{L_2}+1].$$

By Gronwall's inequality, the first inequality of (2.15) has been obtained. From Sobolev's inequality, we get the second inequality of (2.15) immediately.

Lemma 4. *If the conditions of* Lemma 3 *are satisfied, then we have*

$$\|\rho_{kt}\|_{L_2\times L_\infty}+\|u_{kt}\|_{L_2\times L_\infty}+\|u_{kxt}\|_{L_2\times L_\infty}\leqslant E_3, \tag{2.18}$$

where the constant E_3 is independent of k.

Lemma 5. *If the conditions of* Lemma 3 *are satisfied, then we have*

$$\|u_{kxxt}\|^2_{L_2\times L_\infty}\leqslant E_4, \tag{2.19}$$

where the constant E_4 is independent of k.

Lemma 6. *Suppose that the conditions of* Lemma 3 *are satisfied, and assume that $f(u)\in C^2$, $g(u,\ \rho,\ \eta)\in C^1$. Then we have*

$$\|\rho_{ktt}\|^2_{L_2\times L_\infty}+\|u_{ktt}\|^2_{L_2\times L_\infty}+\|u_{kxtt}\|^2_{L_2\times L_\infty}\leqslant E_5, \tag{2.20}$$

where the constant E_5 is independent of k.

Theorem 1. *Suppose that the following conditions are satisfied:*

(i) *functions $g(u,\ \rho,\ \eta)$, $h(\rho)$ are semi-bounded, i.e.,*

$$(u,\ g(u,\ \rho,\ u_x))\leqslant C[(u,\ u)+(\rho,\ \rho)+(u_x,\ u_x)],$$
$$h'(\rho)\leqslant C,$$

where C is a positive constant and $h(0)=0$;

(ii) $f(u)\in C^2$, $h(\rho)\in C^1$, $g(u,\ \rho,\ \eta)\in C^1$;

(iii) *for every bounded u,*

$$|g(u,\ \rho,\ u_x)|\leqslant C[|u_x|^3+|\rho|^3],\ C=\text{const.}>0;$$

(iv) $u_0(x)\in H^2(\Omega)$, $\rho_0(x)\in H^1(\Omega)$.

Then there exists the global generalized solution $u(x,\ t)$, $\rho(x,\ t)$ of the problem (1.3)—(1.6),

$$u(x,\ t)\in L^\infty(0,\ T;\ H^2),\quad u_t(x,\ t)\in L^\infty(0,\ T;\ H^2),\quad u_{tt}(x,\ t)\in L^\infty(0,\ T;\ H^1),$$
$$\rho(x,\ t)\in L^\infty(0,\ T;\ H^1),\quad \rho_t(x,\ t)\in L^\infty(0,\ T;\ H^1),\quad \rho_{tt}(x,\ t)\in L^\infty(0,\ T;\ L_2).$$

Proof. By Lemmas 1—6, for the solutions $\{u_k(x,\ t),\ \rho_k(x,\ t)\}$ of the problem (2.5)—(2.7), the estimate holds

$$\|u_k(t)\|_{H^2\times L_\infty}+\|u_{kt}\|_{H^2\times L_\infty}+\|u_{ktt}\|_{H^1\times L_\infty}+\|\rho_k(t)\|_{H^1\times L_\infty}$$
$$+\|\rho_{kt}(t)\|_{L_2\times L_\infty}+\|\rho_{ktt}\|_{L_2\times L_\infty}\leqslant E_6,$$

where the constant E_6 is independent of k. From the compactness principle, we can prove that the limiting functions $u(x,\ t)$, $\rho(x,\ t)$ of a subsequence $\{u_\nu\}$, $\{\rho_\nu\}$ of the approximate solution sequence $\{u_k\}$, $\{\rho_k\}$ are a global generalized solution of the problem (1.3)—(1.6), and $\rho(x,\ t)\in L^\infty(0,\ T;\ H^1)$. Thus the theorem has been proved.

In order to get the classical solution of problem (1.3)—(1.6), we need to estimate $\|u_{kxxxt}\|_{L_2}$, $\|u_{kxxtt}\|_{L_2}$ and $\|\rho_{kxx}\|_{L_2}$, which are uniformly bounded for k.

Lemma 7. *Suppose that the conditions of* Theorem 1 *are satisfied, and assume that* $h(\rho)\in C^2$, $u_0(x)\in H^3(\Omega)$, $\rho_0(x)\in H^2(\Omega)$. *Then for the solution of problem* (2.5)—(2.7) *we have*

$$\|u_{kxxx}\|^2_{L_2}+\|u_{kxxt}\|^2_{L_2}+\|u_{kxtt}\|^2_{L_2}+\|\rho_{kxx}\|^2_{L_2}\leqslant E_6, \tag{2.21}$$

where the constant E_6 *is independent of* k.

Theorem 2. *Suppose that the conditions of* Lemma 7 *are satisfied. Then there exists a unique global classical solution for the problem* (1.3)—(1.6).

In order to increase the smoothness of the global solution when the smoothness of the functions $f(u)$, $h(\rho)$, $g(u, \rho, \eta)$ and the initial values have been raised, we have the following more generalized estimates and theorem.

Lemma 8. *If the conditions of* Lemma 7 *are satisfied, and*

(i) $f(u)\in C^{m+1}(\Omega)$, $g(u, \rho, \eta)\in C^m(\Omega)$, $h(\rho)\in C^m(\Omega)$, $m\geqslant 2$,

(ii) $u_0(x)\in H^{m+1}(\Omega)$, $\rho_0(x)\in H^m(\Omega)$, $m\geqslant 2$.

Then for the solution of problem (2.5)—(2.7), *there are estimates*

$$\|D_x^{m+1}u_k\|^2_{L_2\times L_\infty}+\|D_x^m\rho_k\|^2_{L_2\times L_\infty}\leqslant E_7, \tag{2.22}$$

$$\|D_tD_x^{m+1}u_k\|^2_{L_2\times L_\infty}+\|D_tD_x^m\rho_k\|^2_{L_2\times L_\infty}\leqslant E_8, \tag{2.23}$$

where the constants E_7, E_8 *are independent of* k.

Theorem 3. *Suppose that the conditions of* Theorem 1 *and* Lemma 8 *are satisfied. Then there exists a unique smooth solution* $u(x, t)$, $\rho(x, t)$ *of problem* (1.3)—(1.6)

$$\begin{aligned}&u(x, t)\in L^\infty(0, T; H^{m+1}(\Omega)),\quad u_t(x, t)\in L^\infty(0, T; H^{m+1}(\Omega)),\\&u_{tt}(x, t)\in L^\infty(0, T; H^{m+1}(\Omega)),\\&\rho(x, t)\in L^\infty(0, T; H^m(\Omega)),\quad \rho_t(x, t)\in L^\infty(0, T; H^m(\Omega)).\end{aligned} \tag{2.24}$$

Furthermore, if $h(\rho)\in C^{m+1}$, *then we have*

$$\rho_{tt}(x, t)\in L^\infty(0, T; H^m(\Omega)). \tag{2.25}$$

§ 3. The Convergence of the Continued Spectral Method

In this section, we shall make the error estimation, and prove the convergence of the approximate solution $\{u_k(x, t)\}$, $\{\rho_k(x, t)\}$ of problem (1.3)—(1.6).

Suppose that the global smooth solution of problem (1.3)—(1.6) exists. Let $u-u_k=U$, $\rho-\rho_k=V$, where $u(x, t)$, $\rho(x, t)$ is a smooth solution of problem (1.3)—(1.6), and $u_k(x, t)$, $\rho_k(x, t)$ is the solution of problem (2.5)—(2.7), i.e. the approximate solution of problem (1.3)—(1.6). From (1.3)—(1.6) and (2.5)—(2.7), it follows that

$$\begin{cases}(U_t-U_{xxt}+V_x+f(u)_x-f(u_k)_x-(g(u, \rho, u_x)-g(u_k, \rho_k, u_{kx})), v_j(x))=0, & (3.1)\\(V_t+U_x-(h(\rho)-h(\rho_k)), v_j(x))=0,\quad j=-k, \cdots, k, & (3.2)\\U|_{t=0}=u_0(x)-F_ku_0(x)=R_ku_0(x), & \\V|_{t=0}=\rho_0(x)-F_k\rho_0(x)=R_k\rho_0(x). & (3.3)\end{cases}$$

Set $v=F_ku-u_k=u-R_ku-u_k=U-R_ku=\sum\limits_{j=-k}^{k}\alpha_jv_j(x)$. Then from (3.1) we have

$$(U_t-U_{xxt}+V_x+f(u)_x-f(u_h)_x-(g(u, \rho, u_x)-g(u_k, \rho_k, u_{kx})), U-R_ku)=0. \tag{3.4}$$

Since

$$(U_t, U)=\frac{1}{2}\frac{d}{dt}\|U\|^2_{L_2}, \quad -(U_{xxt}, U)=\frac{1}{2}\frac{d}{dt}\|U_x\|^2_{L_2},$$

$$|(V_x, U)|=|(V, U_x)|\leqslant\frac{1}{2}(\|V\|^2_{L_2}+\|U_x\|^2_{L_2}),$$

$$\begin{aligned}|(f(u)_x-f(u_k)_x, U)|&=|(f(u)-f(u_k), U_x)|\\&\leqslant\|f'(\xi)\|_{L_\infty}\cdot\frac{1}{2}(\|U\|^2_{L_2}+\|U_x\|^2_{L_2})\\&\leqslant C(\|U\|^2+\|U_x\|^2_{L_2}),\end{aligned}$$

$$\begin{aligned}&|g(u, \rho, u_x)-g(u_k, \rho_k, u_{kx})|\\&\quad\leqslant|g(u, \rho, u_x)-g(u_k, \rho, u_x)|+|g(u_k, \rho, u_x)-g(u_k, \rho_k, u_x)|\\&\qquad+|g(u_k, \rho_k, u_x)-g(u_k, \rho_k, u_{kx})|,\end{aligned}$$

$$\begin{aligned}&|(g(u, \rho, u_x)-g(u_k, \rho_k, u_{kx}), U)|\\&\leqslant\|g_u\|_{L_\infty}\|U\|^2_{L_2}+\|g_\rho\|_{L_\infty}\cdot\frac{1}{2}(\|U\|^2_{L_2}+\|V\|^2_{L_2})+\|g_\eta\|_{L_\infty}\frac{1}{2}(\|U\|^2_{L_2}+\|U_x\|^2_{L_2})\\&\leqslant C[\|U\|^2_{L_2}+\|V\|^2_{L_2}+\|U_x\|^2_{L_2}],\end{aligned}$$

$$(U_t, -R_ku)=\frac{d}{dt}(U, -R_ku)-(U, -R_ku_t),$$

$$(-U_{xxt}, -R_ku)=-(U_{xt}, R_ku_x)=-\frac{d}{dx}(U_x, R_ku_x)+(U_x, R_ku_{tx}),$$

$$\begin{aligned}&|(f(u)_x-f(u_k)_x, -R_ku)|=|(f(u)-f(u_k), R_ku_x)|\\&\leqslant\|f'(\xi)\|_{L_\infty}\cdot\frac{1}{2}(\|U\|^2_{L_2}+\|R_ku_x\|^2_{L_2})\leqslant C(\|U\|^2_{L_2}+\|R_ku_x\|^2_{L_2}),\end{aligned}$$

$$\begin{aligned}&|(g(u, \rho, u_x)-g(u_k, \rho_k, u_{kx})-R_ku)|\\&\leqslant C[\|U\|^2_{L_2}+\|V\|^2_{L_2}+\|U_x\|^2_{L_2}+\|R_ku\|^2_{L_2}],\end{aligned}$$

from (3.4) we get

$$\begin{aligned}\frac{d}{dt}(\|U\|^2_{L_2}+\|U_x\|^2_{L_2})\leqslant&\,2\frac{d}{dt}(U, R_ku)-2(U, R_ku_t)\\&+2\frac{d}{dt}(U_x, R_ku_x)-2(U_x, R_ku_{tx})\\&+C[\|U\|^2_{L_2}+\|V\|^2_{L_2}+\|U_x\|^2_{L_2}+\|R_ku\|^2_{L_2}+\|R_ku_x\|^2_{L_2}].\end{aligned}$$

Integrating the above inequality with respect to t, we have

$$\begin{aligned}\|U(t)\|^2_{L_2}+\|U_x(t)\|^2_{L_2}\leqslant&\,\|U(0)\|^2_{L_2}+\|U_x(0)\|^2_{L_2}+2(U(t), R_ku(t))\\&-2(U(0), R_ku(0))+2(U_x(t), R_ku_x(t))-2(U(0), R_ku_x(0))\\&-2\int_0^t[(U_x, R_ku_{tx})+(U, R_ku_{tx})]dt+2C\int_0^t[\|U(\tau)\|^2_{L_2}+\|V(\tau)\|^2_{L_2}\\&+\|U_x(\tau)\|^2_{L_2}+\|R_ku(\tau)\|^2_{L_2}+\|R_ku_x(\tau)\|^2_{L_2}]d\tau. \qquad (3.5)\end{aligned}$$

Since

$$2(U(t), R_ku(t))\leqslant\frac{1}{3}\|U(t)\|^2_{L_2}+3\|R_ku(t)\|^2_{L_2},$$

$$2(U_x(t), R_ku_x(t))\leqslant\frac{1}{3}\|U_x(t)\|^2_{L_2}+3\|R_ku_x(t)\|^2_{L_2},$$

from (3.5) it follows that

$$\|U(t)\|^2_{L_2}+\|U_x(t)\|^2_{L_2}\leqslant C_1R(t)+C_2\int_0^t[\|U(\tau)\|^2_{L_2}+\|V(\tau)\|^2_{L_2}+\|U_x(\tau)\|^2_{L_2}$$
$$+\|R_ku(\tau)\|^2_{L_2}+\|R_ku_t(\tau)\|^2_{L_2}+\|R_ku_x(\tau)\|^2_{L_2}$$
$$+\|R_ku_{tx}(\tau)\|^2_{L_2}]d\tau, \tag{3.6}$$

where $\quad R(t)=\|R_ku_0(x)\|^2_{L_2}+\|R_ku_{0x}(x)\|^2_{L_2}+\|R_ku(t)\|^2_{L_2}+\|R_ku_x(t)\|^2_{L_2}$.

Let $v=F_k\rho-\rho_k=\rho-R_k\rho-\rho_k=V-R_k\rho=\sum_j\beta_jv_j$. From (3.2) it follows that

$$(V_t+U_x-(h(\rho)-h(\rho_k)),\ V-R_k\rho)=0. \tag{3.7}$$

Since

$$(V_t,V)=\frac{1}{2}\frac{d}{dt}\|V\|^2_{L_2},$$
$$|(U_x,V)|\leqslant\frac{1}{2}(\|U_x\|^2_{L_2}+\|V\|^2_{L_2}),$$
$$|(h(\rho)-h(\rho_k),V)|\leqslant\|h'(\xi)\|_{L_\infty}\|V\|^2_{L_2},$$
$$(V_t,\ -R_k\rho)=\frac{d}{dt}(V,\ -R_k\rho)-(V,\ R_k\rho_t),$$
$$|(U_x,\ -R_k\rho)|\leqslant\frac{1}{2}(\|U_x\|^2_{L_2}+\|R_k\rho\|^2_{L_2}),$$
$$|(h(\rho)-h(\rho_k),\ -R_k\rho)|\leqslant\|h'(\xi)\|_{L_\infty}\cdot\frac{1}{2}(\|V\|^2_{L_2}+\|R_k\rho\|^2_{L_2}),$$

from (3.7) we have

$$\frac{d}{dt}\|V\|^2_{L_2}\leqslant2\frac{d}{dt}(V,\ R_k\rho)-2(V,\ R_k\rho_t)+C[\|V\|^2_{L_2}+\|U_x\|^2_{L_2}+\|R_k\rho\|^2_{L_2}]. \tag{3.8}$$

Integrating (3.8) with respect to t, it follows that

$$\|V(t)\|^2_{L_2}\leqslant\|V(0)\|^2_{L_2}+2(V(t),\ R_k\rho)-2(V(0),\ R_k\rho(0))$$
$$-2\int_0^t(V,\ R_k\rho_t)dt+C\int_0^t[\|V\|^2_{L_2}+\|U_x\|^2_{L_2}+\|R_k\rho\|^2_{L_2}]dt. \tag{3.9}$$

Noticing

$$|(2V(t),\ R_k\rho)|\leqslant\frac{1}{2}\|V(t)\|^2_{L_2}+2\|R_k\rho\|^2_{L_2},$$
$$\int_0^t(V,\ R_k\rho_t)\,dt\leqslant\frac{1}{2}\int_0^t(\|V\|^2_{L_2}+\|R_k\rho_t\|^2_{L_2})dt,$$

from (3.9) we get

$$\|V(t)\|^2_{L_2}\leqslant C_3S(t)+C_4\int_0^t[\|V\|^2_{L_2}+\|U_x\|^2_{L_2}+\|R_k\rho\|^2_{L_2}+\|R_k\rho_t\|^2_{L_2}]dt. \tag{3.10}$$

Here $\quad S(t)=2\|R_k\rho_0(x)\|^2_{L_2}+\|R_k\rho(t)\|^2_{L_2}$.

From (3.6) and (3.10) we have

$$\|U(t)\|^2_{L_2}+\|U_x(t)\|^2_{L_2}+\|V(t)\|^2_{L_2}$$
$$\leqslant C_5(R(t)+S(t))+C_6\int_0^t[\|U(\tau)\|^2_{L_2}+\|U_x(\tau)\|^2_{L_2}+V(\tau)\|^2_{L_2}$$
$$+\|R_ku(\tau)\|^2_{L_2}+\|R_ku_t(\tau)\|^2_{L_2}+\|R_ku_{xt}(\tau)\|^2_{L_2}+\|R_k\rho(\tau)\|^2_{L_2}$$
$$+\|R_k\rho_t(\tau)\|^2_{L_2}]d\tau. \tag{3.11}$$

Now suppose that the initial function $u_0(x)$ has $l+1$ order bounded derivatives, $\rho_0(x)$ has l order bounded derivatives, and the smooth solution $u(x, t)$, $\rho(x, t)$ of problem (1.3)—(1.6) and all their derivatives u_x, u_{xt}, ρ_t have l order bounded derivatives. According to Bernstein's estimate (2.3), we have

$$\sup_{0<t<T} |R(t)+S(t)| \leqslant C_l \frac{(\log k)^2}{k^{2l}}, \quad l\geqslant 2,$$

$$\sup_{0<t<T} [\|R_k u\|^2_{L_2}+\|R_k u_{xt}\|^2_{L_2}+\|R_k u_t\|^2_{L_2}+\|R_k u_x\|^2_{L_2}+\|R_k \rho\|^2_{L_2}+\|R_k \rho_t\|^2_{L_2}]$$
$$\leqslant C_l\left(\frac{\log k}{k^l}\right)^2.$$

Hence from (3.11) and Gronwall's inequality, we get

$$\|U(t)\|^2_{L_2\times L_\infty}+\|U_x(t)\|^2_{L_2\times L_\infty}+\|V(t)\|^2_{L_2\times L_\infty}=O\left(\left(\frac{\log k}{k^l}\right)^2\right).$$

Theorem 4. *Suppose that the periodic initial function $u_0(x)$ has $l+1$ order bounded derivatives ($l\geqslant 2$), and $\rho_0(x)$ has l order bounded derivatives. Assume that the smooth solution $u(x, t)$, $\rho(x, t)$ and their derivatives $u_x(x, t)$, $u_{xt}(x, t)$, $\rho_t(x, t)$ have l order bounded derivatives with respect to x. Then we have the error estimate*

$$\|U(t)\|^2_{L_2\times L_\infty}+\|U_x(t)\|^2_{L_2\times L_\infty}+\|V(t)\|^2_{L_2\times L_\infty}=O\left(\left(\frac{\log k}{k^l}\right)^2\right),$$

where $U(x, t)=u(x, t)-u_k(x, t)$, $V(x, t)=\rho(x, t)-\rho_k(x, t)$, $\{u_k(x, t), \rho_k(x, t)\}$ is the solution of problem (2.5)—(2.7).

§ 4. The Convergence of Approximate Solution of the Discrete Spectral Method

Now we consider the case where the difference quotient for t replaces the derivative for t in the above continued spectral method, and divide the domain $Q_T=\Omega\times[0, T]$ by lines $t=m\Delta t$, where $m\in\left(0, \left[\frac{T}{\Delta t}\right]\right)$.

We consider the following system of equations with periodic initial conditions

$$(u_{k\bar{t}}-u_{kxx\bar{t}}+\rho_{kx}+f(u_k)_x, v_j(x))=(g(u_k, \rho_k, u_{kx}), v_j(x)), \tag{4.1}$$

$$(\rho_{k\bar{t}}+u_{kx}, v_j(x))=(h(\rho_k), v_j(x)), \quad j=-k, \cdots, k, \tag{4.2}$$

$$u_k|_{t=0}=u_{0k}(x)=F_k u_0(x), \quad \rho_k|_{t=0}=\rho_{0k}(x)=F_k\rho_0(x), \tag{4.3}$$

$$u_k(x-\pi, t)=u_k(x+\pi, t), \quad \rho_k(x-\pi, t)=\rho_k(x+\pi, t), \tag{4.4}$$

where

$$u_{k\bar{t}}=\frac{u_k(x, t)-u_k(x, t-\Delta t)}{\Delta t}, \quad \rho_{k\bar{t}}=\frac{\rho_k(x, t)-\rho_k(x, t-\Delta t)}{\Delta t},$$

$$u_k(\cdot, t)=u_k(t)=\sum_{j=-k}^{k}\alpha_{jk}(t)v_j(x), \quad \rho_k(\cdot, t)=\rho_k(t)=\sum_{j=-k}^{k}\beta_{jk}(t)v_j(x).$$

It is easy to see that the nonlinear algebraic equations (4.1)—(4.4) are solvable by using Schauder's fixed point argument. Now we make uniform estimations for the solution of problem (4.1)—(4.4).

Lemma 9. *If the conditions of* Lemma 1 *are satisfied, then for the solution of problem* (4.1)—(4.3), *we have*

$$\|u_k(\cdot,\ t)\|_{H^1\times L_\infty}+\|\rho_k(\cdot,\ t)\|_{L_2\times L_\infty}\leqslant E_9, \tag{4.5}$$

where the constant E_9 is independent of k.

Lemma 10. *If the conditions of* Lemma 7 *are satisfied, then for the solution of problem* (4.1)—(4.3), *there is the estimate*

$$\|u_{kxxx}\|^2_{L_2\times L_\infty}+\|u_{kxxxt}\|^2_{L_2\times L_\infty}+\|u_{kxxtt}\|^2_{L_2\times L_\infty}+\|u_{kt}\|^2_{L_2\times L_\infty}+\|\rho_{kt}\|^2_{H^1\times L_\infty}$$
$$+\|\rho_{ktt}\|^2_{L_2\times L_\infty}+\|u_{ktt}\|^2_{L_2\times L_\infty}+\|\rho_k\|^2_{H^1\times L_\infty}\leqslant E_{10},$$

where the constant E_{10} is independent of k.

Theorem 5. *If the conditions of* Theorem 4 *are satisfied, then there is the error estimate*

$$\|U(t)\|^2_{L_2\times L_\infty}+\|U_x(t)\|^2_{L_2\times L_\infty}+\|V(t)\|^2_{L_2\times L_\infty}$$
$$=O\left((\Delta t)^2+\left(\frac{\log k}{k^l}\right)^2\right), \tag{4.6}$$

where $U(x,\ t)=u(x,\ t)-u_k(x,\ t)$, $V(x,\ t)=\rho(x,\ t)-\rho_k(x,\ t)$, $\{u(x,\ t),\ \rho(x,\ t)\}$ is a smooth solution of problem (1.3)—(1.6), *and $\{u_k(x,\ t),\ \rho_k(x,\ t)\}$ is the solution of problem* (4.1)—(4.3), *which is the approximate solution by using the discrete spectral method.*

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