THE ASYMPTOTIC BEHAVIOR AND THE CONVERGENCE OF THE SOLUTION OF A REACTION-DIFFUSION DIFFERENCE SCHEME IN A CIRCULAR REGION*

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§ 1. Introduction

Ludwig, Jones and Holling[1] proposed an ordinary differential equation to describe the budworm density in the forest. Ludwig, Aronson and Weinberger[2] considered the spatial effect of the budworm, by adding a diffusion term in the original model. They also studied this problem for a region of infinite strip in detail. Recently, Guo Ben-yu. Mitchell and Sleeman[3] and Guo Ben-yu, Sleeman, Mitchell[4] considered this problem for circular and rectangular regions respectively and very precise results were obtained. Guo Ben-yu and Mitchell[5] also studied the asymptotic behavior and the convergence of a reaction-diffusion difference scheme in an infinite strip.

In this paper we consider the linear and nonlinear reaction-diffusion difference equations, the existence of the positive solution of the steady problem and the asymptotic behavior of the solution of the unsteady problem. Finally, we prove the convergence of the approximate solution.

§ 2. The Difference Scheme for the Linear Problem

In this section we consider a linear model whose boundary condition means that the exterior is a lethal environment for the budworm. Assume that Ω is a bounded open domain in R^2 and $U(x, t)$ is the scaled density of the budworm population. Then $U(x, t)$ satisfies the equation

$$\begin{cases} \dfrac{\partial U}{\partial t}-\Delta U=U, & x\in \Omega,\ 0<t<\infty, \\ U(x,\ t)=0, & x\in\partial\Omega,\ 0\leqslant t<\infty, \\ U(x,\ 0)=U_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

where $U_0(x)$ is a given function and $U_0(x)=0$ on $\partial\Omega$.

If $U_0(x)=U_0(\rho)$ where $\rho=|x|$ and if Ω is a circular domain with the radius l, then

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$$\begin{cases} \dfrac{\partial U}{\partial t}+\varepsilon PU=U, & 0<\rho<1,\ 0<t<\infty \\ \dfrac{\partial U}{\partial \rho}(0, t)=0, & U(1, t)=0,\ 0\leqslant t<\infty, \\ U(\rho, 0)=U_0(\rho), & 0<\rho<1 \end{cases} \tag{2.2}$$

with $\varepsilon=\frac{1}{l^2}$ and $P=-\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}$.

Let h and τ be the mesh sizes of the space and time respectively, where $Nh=1$, N being a positive integer. We define

$$\Omega_h=\{\rho/\rho=h,\ 2h,\ \cdots,\ (N-1)h\}$$

and $\bar{\Omega}=\Omega_h+\partial\Omega_h$ where $\partial\Omega_h$ is the boundary of Ω_h.

Let $\eta^k(\rho)$ be the value of the mesh function η at the point $\rho=jh$ and $t=k\tau$, and $\eta^k_\rho(\rho)$, $\eta^k_{\bar\rho}(\rho)$ and $\eta^k_{\hat\rho}(\rho)$ denote respectively the forword, the backward and the central difference quotients of $\eta^k(\rho)$ with respect to ρ. Similarly, $\eta^k_t(\rho)$ denotes the forward difference quotient of $\eta^k(\rho)$ with respect to t. We define

$$P_h\eta^k(\rho)=-\eta^k_{\rho\bar\rho}(\rho)-\frac{1}{\rho}\eta^k_{\hat\rho}(\rho).$$

Let $u^k(\rho)$ be the approximation to $U(\rho, k\tau)$. The Crank-Nicolson scheme for solving (2.2) is

$$\begin{cases} u^k_t(\rho)+\frac{\varepsilon}{2}P_hu^k(\rho)+\frac{\varepsilon}{2}P_hu^{k+1}(\rho)=\frac{1}{2}u^k(\rho)+\frac{1}{2}u^{k+1}(\rho). & \rho\in\Omega_h,\ k\geqslant 0, \\ u^k_{\hat\rho}(0)=0,\ u^k(1)=0, & k\geqslant 0, \\ u^0(\rho)=U_0(\rho), & \rho\in\Omega_h. \end{cases} \tag{2.3}$$

The corresponding steady equation is

$$\begin{cases} \varepsilon P_hv(\rho)=v(\rho),\ \rho\in\Omega_h, \\ v_\rho(0)=0 \quad v(1)=0. \end{cases} \tag{2.4}$$

§ 3. The Discrete Green Function

To study the behavior of the solution of (2.4), we define the discrete Green function as

$$\begin{cases} P_hG_h(\rho, \rho')=\frac{1}{h^2}\delta(\rho, \rho'), & \rho\in\Omega_h, \\ G_{h,\rho}(0, \rho')=0, & G_h(1, \rho')=0, \end{cases} \tag{3.1}$$

where $\rho'\in\bar\Omega_h$ and $\delta(\rho, \rho')$ is a Kronecker function.

Let

$$G_h(\rho')=(G_h(h, \rho'),\ \cdots,\ G_h((N-1)h, \rho'))^*,$$
$$\delta(\rho')=(\underbrace{0, \cdots, 0}_{(j'-1)}, 1, \underbrace{0, \cdots, 0}_{(N-1-j')})^*,\ \rho'=j'h.$$

Then from (3.1) we obtain

$$BG_h(\rho')=\frac{1}{h^2}\delta(\rho'),$$

with

$$B=\frac{1}{h^2}\begin{bmatrix} 1+B_1 & -1-B_1 & 0 & \cdots & 0 \\ -1+B_2 & 2 & -1-B_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -1+B_{N-1} & 2 \end{bmatrix}, \quad B_j=\frac{1}{2j}. \tag{3.2}$$

Clearly, $0<B_j\leqslant\frac{1}{2}$.

If all elements of a matrix M or a vector Y are nonnegative, then we denote $M\geqslant0$ or $Y\geqslant0$. If their elements are all positive, then we say $M>0$ and $Y>0$. It can be checked easily that the matrix B is unreduced and diagonally dominant, and so it is monotone. Hence there exists $B^{-1}\geqslant0$. Therefore $G_h(\rho')$ exists. Because $\delta(\rho')\geqslant0$ and $\delta(\rho')\neq0$, it can be easily verified that

$$G_h(\rho, \rho')>0, \quad \forall\rho\in\Omega_h. \tag{3.3}$$

Let

$$\mathscr{H}=\{\eta/\eta_\rho(0)=0, \eta'1)=0\}.$$

Then

$$\eta(\rho)=h^2\sum_{\rho'\in\Omega_h}G_h(\rho, \rho')P_h\eta(\rho'). \tag{3.4}$$

In particular, we take

$$\eta(\rho)=\begin{cases}1-h^2, & \rho=0\\ 1-\rho^2, & \rho\neq0.\end{cases}$$

Then

$$P_h\eta(\rho)=\begin{cases}\frac{9}{2}, & \rho=h,\\ 4, & 2h\leqslant\rho\leqslant1-h,\end{cases} \tag{3.5}$$

and so from (3.3) and (3.4), we have

$$0<h^2\sum_{\rho'\in\Omega_h}G_h(\rho, \rho')\leqslant\frac{1-h^2}{4}. \tag{3.6}$$

§ 4. The Behavior of the Solution of the Linear Problem

We first consider the eigenvalue problem

$$\begin{cases}P_h\phi(\rho)=\lambda_h\phi(\rho), & \rho\in\Omega_h,\\ \phi_\rho(0)=0, \quad \phi(1)=0,\end{cases} \tag{4.1}$$

which is related to (2.4).

Proposition 4.1. All eigenvalues of problem (4.1) are simple and positive, arranged as

$$\frac{4}{1-h^2}\leqslant\lambda_h^{(1)}<\lambda_h^{(2)}<\cdots<\lambda_h^{(N-1)}, \quad \forall h>0.$$

The eigenvalue $\lambda_h^{(1)}$ has a corresponding eigenfunction $\phi^{(1)}(\rho)$ with $\phi^{(1)}(\rho)>0$, for all $\rho\in\Omega_h$.

Proof. Firstly, problem (4.1) is equivalent to the eigenvalue problem of B. Since B is a real Jacobi matrix, every eigenvalue of B is simple and real. It is not difficult to verify that all of them are positive.

From (3.4), we have

$$\phi(\rho)=\lambda_h h^2 \sum_{\rho'\in\Omega_h} G_h(\rho, \rho')\phi(\rho'). \tag{4.2}$$

Let

$$\begin{aligned} G_h&=h^2(G_h(\rho, \rho'))_{(N-1)\times(N-1)}, \quad \rho, \rho'\in\Omega_h, \\ \boldsymbol{\phi}&=(\phi_1, \phi_2, \cdots, \phi_{N-1})^*, \quad \phi_j=\phi(jh) \end{aligned} \tag{4.3}$$

and introduce the maximum norm, i.e. $\|\boldsymbol{\phi}\|_\infty=\max\limits_{1<j<N-1}|\phi_j|$.

Let K be the subset involving all nonnegative vectors in R^{N-1}. Then K is a normal cone. Define

$$K_1=\{\boldsymbol{\phi}\in K, \|\boldsymbol{\phi}\|_\infty\leqslant 1\},$$

$$S_1=\{\boldsymbol{\phi}\in K, \|\boldsymbol{\phi}\|_\infty=1\}.$$

Then the mapping G_h on K to K is continuous. Because $G_h(\rho, \rho')>0$, we have

$$\alpha=\inf_{\phi\in S_1}\|G_h\boldsymbol{\phi}\|_\infty>0.$$

By Theorem 17.8 of [6], G_h has a nonnegative eigenvector on S_1. Assume that $\boldsymbol{\phi}^{(s)}\geqslant 0$ with $\phi_1^{(s)}=1$ is the eigenvector associated with $[\lambda_h^{(s)}]^{-1}$. Equation (4.2) is equivalent to

$$B\boldsymbol{\phi}^{(s)}=\lambda_h^{(s)}\boldsymbol{\phi}^{(s)}. \tag{4.4}$$

Consequently it can be verified by induction that

$$\phi_1^{(s)}>\phi_2^{(s)}>\cdots>\phi_{N-1}^{(s)}>0.$$

Hence

$$\boldsymbol{\phi}^{(s)}>0.$$

If $s=1$, then $\lambda_h^{(1)}$ has a corresponding positive eigenvector; otherwise, $\lambda_h^{(1)}<\lambda_h^{(s)}$. Let $\boldsymbol{\phi}^{(1)}$ be the eigenvector associated with $\lambda_h^{(1)}$ and $\boldsymbol{z}=\boldsymbol{\phi}^{(1)}+\beta\boldsymbol{\phi}^{(s)}$. We make β sufficiently large such that $\boldsymbol{z}>0$. Then it follows that for all k,

$$G_h^k\boldsymbol{z}=(\lambda_h^{(1)})^{-k}\boldsymbol{\phi}^{(1)}+\beta(\lambda_h^{(s)})^{-k}\boldsymbol{\phi}^{(s)}>0,$$

i.e.

$$\boldsymbol{\phi}^{(1)}+\beta\left(\frac{\lambda_h^{(1)}}{\lambda_h^{(s)}}\right)^k\boldsymbol{\phi}^{(s)}>0.$$

Since k is arbitrary, we have $\boldsymbol{\phi}^{(1)}>0$. Finally, from (4.2) and (3.6) we obtain $\lambda_h^{(1)}\geqslant\dfrac{4}{1-h^2}$.

Let $l_h^*=\sqrt{\lambda_h^{(1)}}$. By Proposition 4.1 we obtain

Theorem 4.1. *If $l<l_h^*$, then the steady problem (2.4) has only the zero solution. If $l=l_h^*$, then there is a solution $v(\rho)$ with $v(0)>0$ for all $\rho\in\Omega_h$.*

We now consider the asymptotic stability of the solution of (2.3). Let

$$u^k(\rho)=\sum_{s=0}^{N-1}a_s b^k(s)\boldsymbol{\phi}^{(s)}(\rho), \quad U_0(\rho)=\sum_{s=0}^{N-1}a_s\boldsymbol{\phi}^{(s)}(\rho), \quad \rho\in\Omega_h.$$

Substituting them into (2.3), we obtain

$$b(s)=\frac{1+\dfrac{\tau}{2}-\dfrac{\varepsilon\tau}{2}\lambda_h^{(s)}}{1-\dfrac{\tau}{2}+\dfrac{\varepsilon\tau}{2}\lambda_h^{(s)}}$$

from which, we obtain the following result.

Theorem 4.2. *Let $u^k(\rho)$ be the solution of* (2.3). *Then*

(i) *if $l<l_h^*$ then for any initial value $U_0(\rho)$ and $h>0$,*

$$\lim_{k\to\infty} u^k(\rho)=0, \quad \forall\rho\in\Omega_h;$$

(ii) *if $l>l_h^*$, then for every $h>0$ there are solutions $u^k(\rho)$ with arbitrarily small initial values such that*

$$\lim_{k\to\infty} u^k(\rho)=\infty, \quad \forall\rho\in\Omega_h.$$

§ 5. The Behavior of the Solution of the Nonlinear Steady Problem

In this section we consider the following logistic model (see [2]—[4])

$$\begin{cases} \dfrac{\partial U}{\partial t}+\varepsilon PU=U-U^2, \ 0<\rho<1, \quad 0<t<\infty, \\ \dfrac{\partial U}{\partial\rho}(0, t)=0, \ U(1, t)=0, \quad 0\leqslant t<\infty, \\ U(\rho, 0)=U_0(\rho), \quad 0<\rho<1, \end{cases} \tag{5.1}$$

where $0\leqslant U_0(\rho)\leqslant M_0$. Let $M_1=\max(M_0, 1)$. Guo Ben-yu, Mitchell and Sleeman[3] proved

$$0\leqslant U(\rho, t)\leqslant M_1, \quad 0\leqslant\rho\leqslant1, \ t\geqslant0. \tag{5.2}$$

The difference scheme for solving (5.1) is

$$\begin{cases} u_t^k(\rho)+\dfrac{\varepsilon}{2}P_hu^k(\rho)+\dfrac{\varepsilon}{2}P_hu^{k+1}(\rho)=\dfrac{1}{2}u^k(\rho)+\dfrac{1}{2}u^{k+1}(\rho)-[u^k(\rho)]^2, \ \rho\in\Omega_h, \ k\geqslant0, \\ u_\rho^k(0)=0, \ u^k(1)=0, \quad k\geqslant0, \\ u^0(\rho)=U_0(\rho), \quad \rho\in\Omega_h, \end{cases} \tag{5.3}$$

with the corresponding steady equation

$$\begin{cases} \varepsilon P_hv(\rho)=v(\rho)-v^2(\rho), \quad \rho\in\Omega_h, \\ v_\rho(0)=0, \ v(1)=0. \end{cases} \tag{5.4}$$

Let

$$V=(v_1, v_2, \cdots, v_{N-1})^T, \ v_j=v(jh),$$
$$g(V)=(v_1^2, v_2^2, \cdots, v_{N-1}^2)^T.$$

Then the matrix equation of (5.4) is

$$V-\varepsilon BV-g(V)=0. \tag{5.5}$$

Proposition 5.1. The necessary condition for problem (5.5) to have a positive solution is $\varepsilon<\varepsilon_h^*$ where $\varepsilon_h^*=(\lambda_h^{(1)})^{-1}$.

Proof. Let I be the identity mapping. Since $B^{-1}\geqslant0$ and the spectral radius $r(B^{-1})=\varepsilon_h^*$, so if $\varepsilon>\varepsilon_h^*$, then

$$(I-\varepsilon B)^{-1}=-B^{-1}\sum_{\nu=0}^{\infty}\frac{(B^{-1})^\nu}{\varepsilon^{\nu+1}}\leqslant0,$$

from which and (5.5) we conclude

$$V=(I-\varepsilon B)^{-1}g(V)\leqslant0.$$

Proposition 5.2 There exists $\delta>0$, such that for all $s\in(s_h^*-\delta, s_h^*)$, problem (5.5) has a unique positive solution $V(s)$ which depends continuously on s, and $V(s)\to 0$ as $s\to s_h^*$.

Proof. (5.5) is equivalent to

$$sV-G_hV+G_hg(V)=0. \tag{5.6}$$

From (3.6), we have

$$\|G_h\|=\max_{\rho\in\Omega_h}\sum_{\rho'\in\Omega_h}h^2G_h(\rho,\rho')\leqslant\frac{1}{4},\quad \forall h>0.$$

The mapping $s_h^*-G_h$ is a linear Fredholm's operator on R^{N-1} to R^{N-1}. Let $\mathscr{N}$ and $\mathscr{Y}$ denote the null space and the range of $s_h^*-G_h$ respectively; then $R^{N-1}=\mathscr{N}\oplus\mathscr{Y}$. We take the basis of $\mathscr{N}$ to be the eigenvector $\phi^{(1)}=(\phi_1^{(1)},\cdots,\phi_{N-1}^{(1)})^*$ with $1=\phi_1^{(1)}>\phi_2^{(1)}>\cdots>\phi_{N-1}^{(1)}>0$. We take

$$\phi^{(1)},\ \phi^{(s)}=(\phi_1^{(s)},\phi_2^{(s)},\cdots,\phi_i^{(s)},\cdots,\phi_{N-1}^{(s)})^*,\ \phi_i^{(s)}=\delta_{si},\ 2\leqslant s\leqslant N-1,\ 1\leqslant i\leqslant N-1,$$

to be the basis of R^{N-1}. Take again an element $e=(1,0,\cdots,0)^*$ in the dual space $(R^{N-1})^*$ of R^{N-1}. Then $\|e\|_{(R^{N-1})^*}=1$ and $e(\phi^{(s)})=\delta_{s1}$. We define the mapping p on R^{N-1} to $\mathscr{N}$ as

$$pv=e(v)\phi^{(1)},\quad \forall v\in R^{N-1}.$$

Then p is a projection on R^{N-1} to $\mathscr{N}$. Since $s_h^*-G_h$ is a one-to-one mapping on $\mathscr{Y}$ to $\mathscr{Y}$, we can define the generalized inverse operator A of $s_h^*-G_h$ on R^{N-1} as

$$A=\frac{1}{2\pi i}\int_\Gamma\frac{1}{s_h^*-z}(z-G_h)^{-1}dz,$$

where the curve Γ is made up of a finite number of rectifiable curves. s_h^* is in the region exterior to Γ. All other eigenvalues of G_h are in its interior (see [7]). It can be proved that $\|A\|\leqslant M$ where M is independent of h and that

$$A(s_h^*-G_h)=(s_h^*-G_h)A=q, \tag{5.7}$$

where q denotes the projection of R^{N-1} onto $\mathscr{Y}$.

Let $s=s_h^*-\alpha$, $V=C(\phi^{(1)}+\psi)$ in (5.6) where $0<\alpha\leqslant\alpha_0<\frac{1}{M}$, $C=C(\alpha)$ is a real number depending on α and vector $\psi\in\mathscr{Y}$. We have

$$C(s_h^*-G_h)(\phi^{(1)}+\psi)-\alpha C(\phi^{(1)}+\psi)+G_hg(C(\phi^{(1)}+\psi))=0,$$

which is equivalent to

$$C(I-\alpha A)\psi+AG_hg(C(\phi^{(1)}+\psi))=0, \tag{5.8}$$

$$\alpha C\phi^{(1)}=pG_hg(C(\phi^{(1)}+\psi)). \tag{5.9}$$

We first obtain from (5.8)

$$\psi=-C(I-\alpha A)^{-1}AG_hg(\psi^{(1)}+\psi). \tag{5.10}$$

Let $F(\psi, C, \alpha)$ denote the right-hand term of (5.10) and $S_\sigma=\{\psi\in\mathscr{Y},\ \|\psi\|\leqslant\sigma\}$ for some $\sigma>0$. Clearly

$$\|\psi\|=\|F(\psi,C,\alpha)\|\leqslant\frac{M(1+2\sigma+\sigma^2)}{4(1-\alpha_0M)}|C|.$$

It can be proved that there is a constant $r>0$, independent of h and α, such that for $0<\alpha\leqslant\alpha_0$ and $|C|\leqslant r$, $F(\psi, C, \alpha)$ is a contractive mapping on S_σ to S_σ with respect to ψ, and Frechet's derivative operators $F'_\psi(\psi, C, \alpha)$ satisfy

$$\|F'_\psi(\psi, C, \alpha)\| \leqslant \frac{1}{2}, \quad \forall \psi \in S_\sigma \tag{5.11}$$

uniformly for C and α. Hence if $|C| \leqslant r$, then (5.8) has a unique solution in S_σ. Moreover, when $C \to 0$, $\psi(C, \alpha) \to 0$ uniformly for α.

From (5.11) and the following inequality

$$\|F(\psi^{(1)}, C, \alpha) - F(\psi^{(2)}, C, \alpha)\| \leqslant \|F'_\psi(\psi^{(2)} + \xi(\psi^{(1)} - \psi^{(2)}), C, \alpha)\| \cdot \|\psi^{(1)} - \psi^{(2)}\|,$$

where $\psi^{(1)}, \psi^{(2)} \in S_\sigma$ and $0 \leqslant \xi \leqslant 1$, it can be easily verified that $\psi(C, \alpha)$ is a uniformly continuous function for C and α. For an arbitrarily fixed α with $0 < \alpha \leqslant \alpha_0$, Let

$$f(\psi, C) = \psi + C(I - \alpha A)^{-1} A G_h g(\phi^{(1)} + \psi).$$

Clearly, it is continuously differentiable for ψ and C, and $f'_\psi(0, 0) = I$. Therefore $\psi'_C(C, \alpha)$ is existent and continuous.

We now consider (5.9). For $\alpha > 0$ we have

$$C(\alpha) = \frac{1}{\alpha} e[G_h g(C(\phi^{(1)} + \psi(C, \alpha)))] > 0,$$

from which we get

$$\alpha = Ce[G_h g(\phi^{(1)} + \psi(C, \alpha))], \quad \psi(C, \alpha) \in S_\sigma. \tag{5.12}$$

Let

$$W(\alpha, C) = \alpha - Ce[G_h g(\phi^{(1)} + \psi(C, \alpha))].$$

Clearly, it is continuously differentiable about the point (0, 0) and

$$W(0, 0) = 0,$$
$$W'_C(0, 0) = -e[G_h g(\phi^{(1)})] \neq 0.$$

By the Existence Theorem of Implicit Functions, (5.12) has a unique continuous solution $C = C(\alpha)$ for $0 < \alpha < \alpha_1 \leqslant \alpha_0$ with $C(0) = 0$ and $0 < C(\alpha) \leqslant r$ where r is independent of h provided α is sufficiently small. Therefore (5.5) has a unique continuous solution

$$V = C(\alpha)(\phi^{(1)} + \psi(C(\alpha), \alpha)).$$

Finally, $C(\alpha) \to 0$ as $\alpha \to 0$ and so $V \to 0$ as $\alpha \to 0$. If δ is sufficiently small, then $\|\psi(C(\alpha), \alpha)\| < \phi^{(1)}_{N-1}$. Hence

$$V = C(\alpha)(\phi^{(1)} + \psi(C(\alpha), \alpha)) > 0.$$

Note that $\alpha = s^*_h - \varepsilon$. The proof is completed.

From the previous propositions we obtain

Theorem 5.1. *If $l < l^*_h$, then problem (5.4) has no positive solution. If $l > l^*_h$, then it has a solution $V(\rho)$, such that for all $\rho \in \Omega_h$, $V(\rho) > 0$.*

§ 6. The Asymptotic Behavior of the Nonlinear Unsteady Problem

In this section we study the asymptotic behavior of (5.3) by using a generalized version of the technique in [5]. Let $\alpha \geqslant 0$ and

$$D_h(\varepsilon, \alpha)\eta^k(\rho) = \eta^k_t(\rho) + \frac{\varepsilon}{2} P_h \eta^k(\rho) + \frac{\varepsilon}{2} P_h \eta^{k+1}(\rho) - \frac{1}{2}\eta^k(\rho) - \frac{1}{2}\eta^{k+1}(\rho) + \alpha[\eta^k(\rho)]^2.$$

Proposition 6.1. If $0<\tau<\tau^*$, $|\xi^k(\rho)|\leqslant M_2$ and

$$\begin{cases} D_h(\varepsilon,\alpha)\eta^k(\rho)\leqslant D_h(\varepsilon,\alpha)\xi^k(\rho), & \rho\in\Omega_h,\ k\geqslant 0,\\ \eta^k_\rho(0)=\xi^k_\rho(0)=0 & k\geqslant 0,\\ \eta^k(1)\leqslant\xi^k(1), & k\geqslant 0,\\ \eta^0(\rho)\leqslant\xi^0(\rho), & \rho\in\Omega_h,\end{cases}\tag{6.1}$$

where $\tau^*=\min\left\{2,\ \dfrac{2h^2}{2\varepsilon+4\alpha M_2h^2-h^2}\right\}$, then for all $\rho\in\Omega_h$ and $k\geqslant 0$, $\eta^k(\rho)\leqslant\xi^k(\rho)$.

Proof. Let $\tilde\eta^k(\rho)=\eta^k(\rho)-\xi^k(\rho)$. Then from (6.1),

$$\left(1-\frac{\tau}{2}\right)\tilde\eta^{k+1}(\rho)+\frac{\varepsilon\tau}{2}P_h\tilde\eta^{k+1}(\rho)$$
$$\leqslant\left(1+\frac{\tau}{2}-2\alpha\tau\xi^k(\rho)\right)\tilde\eta^k(\rho)-\frac{\varepsilon\tau}{2}P_h\tilde\eta^k(\rho)-\alpha\tau[\tilde\eta^k(\rho)]^2.\tag{6.2}$$

To prove $\tilde\eta^k(\rho)\leqslant 0$ for all $\rho\in\Omega_h$ and $k\geqslant 0$, we shall apply the induction. When $k=0$, the conclusion is clear. Assume that $\tilde\eta^k(\rho)\leqslant 0$ for all $\rho\in\bar\Omega_h$ and $\tilde\eta^{k+1}(\rho_0)=\max\limits_{\rho\in\Omega_h}\tilde\eta^{k+1}(\rho)$ with $\rho_0=j_0h$. It can be verified that $P_h\tilde\eta^{k+1}(\rho_0)\geqslant 0$, $-P_h\tilde\eta^k(\rho_0)\leqslant-\dfrac{2}{h^2}\tilde\eta^k(\rho_0)$, from which and (6.2) we have

$$\left(1-\frac{\tau}{2}\right)\tilde\eta^{k+1}(\rho_0)\leqslant\left(1+\frac{\tau}{2}-2\alpha\tau\xi^k(\rho_0)-\frac{\varepsilon\tau}{h^2}\right)\tilde\eta^k(\rho_0).\tag{6.3}$$

Since $\tau<\tau^*$, hence $\tilde\eta^{k+1}(\rho_0)\leqslant 0$ and so $\tilde\eta^{k+1}(\rho)\leqslant 0$ for all $\rho\in\bar\Omega_h$. This completes the induction.

Proposition 6.2. If $u^k(\rho)$ is a solution of (5.3) and $\tau\leqslant\tau^*$, then $0\leqslant u^k(\rho)\leqslant M_1$.

Proof. Let $\eta^k(\rho)=u^k(\rho)$ and $\xi^k(\rho)=M_1$ in Proposition 6.1; then $u^k(\rho)\leqslant M_1$. To prove $u^k(\rho)\geqslant 0$, we may use the induction similar to that used in the proof of proposition 6.1, but we put $u^{k+1}(\rho_0)=\min\limits_{\rho\in\Omega_h}u^{k+1}(\rho)$.

Theorem 6.1. *Let $k>0$, $0<\tau<\tau^*$, $u^k(\rho)$ and $v(\rho)$ be the solutions of* (5.3) *and* (5.4) *respectively. Then*

(i) *if $l<l_h^*$, then for any nonne initial value $U_0(\rho)$, we have $\lim\limits_{k\to\infty}u^k(\rho)=0$;*

(ii) *if $l>l_h^*$, $U_0(\rho)\geqslant 0$ and $U_0(\rho)\not\equiv 0$, then $\lim\limits_{k\to\infty}u^k(\rho)=v(\rho)$.*

Proof. Let $w^k(\rho)$ be a solution of (2.3); then

$$\begin{cases} D_h(\varepsilon,0)u^k(\rho)=-[u^k(\rho)]^2\leqslant 0=D_h(\varepsilon,0)w^k(\rho), & \rho\in\Omega_h,\ k\geqslant 0,\\ w^k_\rho(0)=u^k_\rho(0)=0, & k\geqslant 0,\\ w^k(1)=u^k(1)=0, & k\geqslant 0,\\ w^0(\rho)=u^0(\rho)=0, & \rho\in\Omega_h.\end{cases}$$

By Propositions 6.1 and 6.2, we have

$$0\leqslant u^k(\rho)\leqslant w^k(\rho).$$

If $l<l_h^*$, then $w^k(\rho)\to 0$ as $k\to 0$. This completes the proof of (i).

Now assume $l>l_h^*$ and $0<m\leqslant U_0(\rho)\leqslant M_0$ for simplicity. Let $E^k(\rho)$ be the solution of the following problem

$$\begin{cases} D_h(\varepsilon, 1)E^k(\rho) = 0, & \rho \in \Omega_h, \quad k \geq 0, \\ E_\rho^k(0) = 0, \ E^k(1) = 0, & k \geq 0, \\ E^0(\rho) = M_1, & \rho \in \Omega_h, \end{cases} \tag{6.4}$$

where $M_1 = \max\{M_0, 1\}$. By Proposition 6.1 we have

$$u^k(\rho) \leq E^k(\rho) \leq M_1. \tag{6.5}$$

Let $l > \bar{l} > l_h^*$, which corresponds to $\varepsilon < \bar{\varepsilon} < \varepsilon_h^*$, and let $V_{\bar{\varepsilon}}(\rho)$ be a solution of the following problem

$$\begin{cases} \bar{\varepsilon} P_h V_{\bar{\varepsilon}}(\rho) + V_{\bar{\varepsilon}}(\rho)(1 - V_{\bar{\varepsilon}}(\rho)) = 0, & \rho \in \Omega_h, \\ V_{\bar{\varepsilon}, \rho}(0) = 0, \quad V_{\bar{\varepsilon}}(1) = 0. \end{cases}$$

Since $\|V_{\bar{\varepsilon}}\| \to 0$ as $\bar{\varepsilon} \to \varepsilon_h^*$, there is some $\bar{l}$ such that $\|V_{\bar{\varepsilon}}\| \leq m$.

Let $G^k(\rho)$ be the solution of the following problem

$$\begin{cases} D_h(\varepsilon, 1)G^k(\rho) = 0, & \rho \in \Omega_h, \ k \geq 0, \\ G_\rho^k(0) = 0, \ G^k(1) = 0, & k \geq 0, \\ G^0(\rho) = V_{\bar{\varepsilon}}(\rho), & \rho \in \Omega_h. \end{cases}$$

From Proposition 6.1 and (6.5), it follows that

$$G^k(\rho) \leq u^k(\rho) \leq E^k(\rho). \tag{6.6}$$

We claim that it suffices to prove that

$$\lim_{k \to \infty} E^k(\rho) = v(\rho), \tag{6.7}$$

$$\lim_{k \to \infty} G^k(\rho) = v(\rho). \tag{6.8}$$

In fact, from (6.5) we have $E^1(\rho) \leq M_1 = E^0(\rho)$. By putting $\eta^k(\rho) = E^{k+1}(\rho)$, $\xi^k(\rho) = E^k(\rho)$ and $\alpha = 1$ in Proposition 6.1, it follows that

$$0 \leq E^{k+1}(\rho) \leq E^k(\rho) \leq M_1.$$

And so there exists $\psi(\rho)$ such that

$$\psi(\rho) = \lim_{k \to \infty} E^k(\rho), \quad \forall \rho \in \bar{\Omega}_h.$$

Let $k \to \infty$ in (6.4), and so $\psi(\rho)$ is a solution of (5.4). From Proposition 5.2 we have $\psi(\rho) = v(\rho)$ which implies (6.7).

Equality (6.8) can be proved by an argument similar to that used above.

Finally, the conclusion (ii) follows from (6.6), (6.7) and (6.8).

§ 7. The Convergence of the Difference Scheme for the Nonlinear Unsteady Problem

To estimate computational error, we introduce the following notations:

$$(\eta, \xi)_{\Omega_h} = 2\pi h \sum_{\rho \in \Omega_h} \rho \eta(\rho) \xi(\rho),$$

$$\|\eta\|_{\Omega_h}^2 = (\eta, \eta)_{\Omega_h},$$

$$|\eta|_{\Omega_h, 1}^2 = \frac{1}{2}\|\eta_\rho\|_{\Omega_h}^2 + \frac{1}{2}\|\eta_{\bar{\rho}}\|_{\Omega_h}^2.$$

Proposition 7.1. For all mesh functions $\eta(\rho)$, we have

$$(\eta,\ P_h\eta)_{\Omega_h}=|\eta|^2_{\Omega_{h,1}}+\pi h^2[\eta_\rho(1-h)]^2-\pi h^2[\eta_\rho(0)]^2-\pi(1-h)\eta(1-h)\eta_\rho(1-h)$$
$$-\pi\eta(1)\eta_\rho(1-h)+\pi h\eta(h)\eta_\rho(0).$$

Proof. From Abel's formula, we obtain

$$h\sum_{\rho\in\Omega_h}Y_\rho(\rho)z(\rho)+h\sum_{\rho\in\Omega_h}Y(\rho)z_{\bar\rho}(\rho)=Y(1)z(1-h)-Y(h)z(0),$$
$$h\sum_{\rho\in\Omega_h}Y_{\bar\rho}(\rho)z(\rho)+h\sum_{\rho\in\Omega_h}Y(\rho)z_\rho(\rho)=Y(1-h)z(1)-Y(0)z(h),$$

whence

$$-(\eta,\ \eta_{\rho\bar\rho})_{\Omega_h}=-2\pi h\sum_{\rho\in\Omega_h}\rho\eta(\rho)\eta_{\rho\bar\rho}(\rho)$$
$$=2\pi h\sum_{\rho\in\Omega_h}[\rho\eta(\rho)]_\rho\eta_\rho(\rho)-2\pi\eta(1)\eta_\rho(1-h)+2\pi h\eta(h)\eta_\rho(0)$$
$$=2\pi h\sum_{\rho\in\Omega_h}\{\rho[\eta_\rho(\rho)]^2+\eta(\rho)\eta_\rho(\rho)+h[\eta_\rho(\rho)]^2\}$$
$$-2\pi\eta(1)\eta_\rho(1-h)+2\pi h\eta(h)\eta_\rho(0).$$

Similarly,

$$-(\eta,\ \eta_{\rho\bar\rho})_{\Omega_h}=2\pi h\sum_{\rho\in\Omega_h}\{\rho[\eta_{\bar\rho}(\rho)]^2+\eta(\rho)\eta_{\bar\rho}(\rho)-h[\eta_{\bar\rho}(\rho)]^2\}$$
$$-2\pi(1-h)\eta(1-h)\eta_\rho(1-h).$$

Summing them up, we complete the proof.

In particular, if $\eta_\rho(0)=g$, $\eta(1)=0$, then

$$(\eta,\ P_h\eta)_{\Omega_h}=|\eta|^2_{\Omega_{h,1}}+\frac{\pi}{h}[\eta(1-h)]^2-\pi h^2g^2+\pi hg\eta(h). \tag{7.1}$$

Proposition 7.2. If $\eta(1)=0$, then $\|\eta\|^2_{\Omega_h}\leqslant\frac{1}{2}|\eta|^2_{\Omega_{h,1}}$.

Proof. We have

$$\eta(\rho)=-h\sum_{\rho'=\rho}^{1-h}\eta_{\rho'}(\rho')$$

whence

$$[\eta(\rho)]^2\leqslant h^2\sum_{\rho'=\rho}^{1-h}\rho'[\eta_{\rho'}(\rho')]^2\cdot\sum_{\rho'=\rho}^{1-h}\frac{1}{\rho'}\leqslant-\frac{\ln\rho}{2\pi}\|\eta_\rho\|^2_{\Omega_h},$$

from which

$$\|\eta\|^2_{\Omega_h}\leqslant-\|\eta_\rho\|^2_{\Omega_h}\int_0^1\rho\ln\rho\,d\rho\leqslant\frac{1}{4}\|\eta_\rho\|^2_{\Omega_h}\leqslant\frac{1}{2}|\eta|^2_{\Omega_{h,1}}.$$

Proposition 7.3. For any mesh function $\eta^k(\rho)$, wa have

$$(\eta^k+\eta^{k+1},\ \eta^k_t)_{\Omega_h}=(\|\eta^k\|^2_{\Omega_h})_t.$$

Now consider the convergence of the difference scheme. Let $U(\rho,\ k\tau)$ and $u^k(\rho)$ be the solutions of (5.1) and (5.3) respectively, $u^k(\rho)=U(\rho,\ k\tau)+\tilde u^k(\rho)$. $R^k(\rho)$ and r^k are approximate errors of (5.3) at the interior points and the origin respectively. Then

$$\begin{cases}\tilde u^k_t(\rho)+\dfrac{s}{2}P_h\tilde u^k(\rho)+\dfrac{s}{2}P_h\tilde u^{k+1}(\rho)=\dfrac{1}{2}\tilde u^k(\rho)+\dfrac{1}{2}\tilde u^{k+1}(\rho)\\ \quad-\tilde u^k(\rho)[u^k(\rho)+U(\rho,\ k\tau)]-R^k(\rho),\quad \rho\in\Omega_h,\ k\geqslant0,\\ \tilde u^k_\rho(0)=r^k,\ \tilde u^k(1)=0,\quad k\geqslant0,\\ \tilde u^0(\rho)=0,\quad \rho\in\Omega_h.\end{cases} \tag{7.2}$$

Taking the discrete scalar product of (7.2) with $[\tilde{u}^k(\rho)+\tilde{u}^{k+1}(\rho)]$, we have from Propositions 7.1, 7.3 and (7.1).

$$\begin{aligned}
&[\|\tilde{u}^k\|^2_{\Omega_h}]_t+\frac{\varepsilon}{2}|\tilde{u}^k+\tilde{u}^{k+1}|^2_{\Omega_{h,1}}+\frac{\pi\varepsilon}{2h}[\tilde{u}^k(1-h)+\tilde{u}^{k+1}(1-h)]^2\\
&\quad+(\tilde{u}^k+\tilde{u}^{k+1},\ \tilde{u}^k(u^k+U(k\tau)))_{\Omega_h}\\
&\quad=\frac{1}{2}\|\tilde{u}^k+\tilde{u}^{k+1}\|^2_{\Omega_h}-(\tilde{u}^k+\tilde{u}^{k+1},\ R^k)_{\Omega_h}+\frac{\varepsilon\pi h^2}{2}(r^k+r^{k+1})^2\\
&\quad-\frac{\varepsilon\pi h}{2}[\tilde{u}^k(h)+\tilde{u}^{k+1}(h)][r^k+r^{k+1}].
\end{aligned}\tag{7.3}$$

From Proposition 7.2, we have

$$\begin{aligned}
\left|\frac{\varepsilon\pi h}{2}[\tilde{u}^k(h)+\tilde{u}^{k+1}(h)](r^k+r^{k+1})\right|&\leqslant 2\pi\varepsilon h^2[\tilde{u}^k(h)+\tilde{u}^{k+1}(h)]^2+\frac{\pi\varepsilon}{32}(r^k+r^{k+1})^2\\
&\leqslant\varepsilon\|\tilde{u}^k+\tilde{u}^{k+1}\|^2_{\Omega_h}+\frac{\pi\varepsilon}{32}(r^k+r^{k+1})^2\\
&\leqslant\frac{\varepsilon}{2}|\tilde{u}^k+\tilde{u}^{k+1}|^2_{\Omega_{h,1}}+\frac{\pi\varepsilon}{32}(r^k+r^{k+1})^2.
\end{aligned}$$

Since $0\leqslant u^k(\rho)\leqslant M_1$, $0\leqslant U(\rho,\ k\tau)\leqslant M_1$, we have

$$\begin{aligned}
(\tilde{u}^k+\tilde{u}^{k+1},\ \tilde{u}^k[u^k+U(k\tau)])_{\Omega_h}&=-((\tilde{u}^k)^2,\ u^k+U(k\tau))_{\Omega_h}+(\tilde{u}^{k+1},\ \tilde{u}^k[u^k+U(k\tau)])_{\Omega_h}\\
&\geqslant -M_1[\|\tilde{u}^{k+1}\|^2_{\Omega_h}+\|\tilde{u}^k\|^2_{\Omega_h}].
\end{aligned}$$

So it follows from (7.3) that

$$\begin{aligned}
\|\tilde{u}^{k+1}\|^2_{\Omega_h}\leqslant&\frac{1+2\tau+M_1\tau}{1-2\tau-M_1\tau}\|\tilde{u}^k\|^2_{\Omega_h}+\frac{\tau}{2(1-2\tau-M_1\tau)}\|R^k\|^2_{\Omega_h}\\
&+\frac{\pi\varepsilon\tau}{32(1-2\tau-M_1\tau)}(16h^2+1)(r^k+r^{k+1})^2.
\end{aligned}$$

If $\tau<\dfrac{1}{2+M_1}$, then

$$\|\tilde{u}^k\|\leqslant\rho^k e^{2(2+M_1)k\tau}$$

where

$$\rho^k=\frac{\tau}{8(1-2\tau-M_1\tau)}\sum_{s=0}^{k}(4\|R^s\|^2_{\Omega_h}+\pi\varepsilon(16h^2+1)|r^s|^2).$$

If $\rho^k\to 0$ as $h\to 0$ for all $k\tau\leqslant T$, then scheme (5.3) is convergent for $t\in(0,\ T]$.

It $\varepsilon\geqslant\dfrac{1}{2}+\delta$ where $\delta>0$, then we can obtain the following estimates

$$\frac{1}{2}\|\tilde{u}^k+\tilde{u}^{k+1}\|^2_{\Omega_h}\leqslant\frac{1}{4}|\tilde{u}^k+\tilde{u}^{k+1}|^2_{\Omega_{h,1}},$$

$$\begin{aligned}
|(\tilde{u}^k+\tilde{u}^{k+1},\ R^k)_{\Omega_h}|&\leqslant\frac{\delta}{2}\|\tilde{u}^k+\tilde{u}^{k+1}\|^2_{\Omega_h}+\frac{1}{2\delta}\|R^k\|^2_{\Omega_h}\\
&\leqslant\frac{\delta}{4}|\tilde{u}^k+\tilde{u}^{k+1}|^2_{\Omega_{h,1}}+\frac{1}{2\delta}\|R^k\|^2_{\Omega_h},
\end{aligned}$$

$$\frac{\varepsilon\pi h}{2}|(\tilde{u}^k(h)+\tilde{u}^{k+1}(h))(r^k+r^{k+1})|\leqslant\frac{\delta}{2}|\tilde{u}^k+\tilde{u}^{k+1}|^2_{\Omega_{h,1}}+\frac{\pi\varepsilon^2}{16\delta}|r^k+r^{k+1}|^2.$$

By substituting the above estimates into (7.3), we obtain

$$[\|\tilde{u}^k\|^2_{\Omega_h}]_t\leqslant M_1[\|\tilde{u}^k\|^2_{\Omega_h}+\|\tilde{u}^{k+1}\|^2_{\Omega_h}]+\frac{1}{2\delta}\|R^k\|^2_{\Omega_h}+\frac{\pi\varepsilon}{16\delta}(8\delta h^2+\varepsilon)(r^k+r^{k+1}),$$

whence

$$\|\tilde{u}^k\|^2_{\Omega_h} \leqslant \tilde{\rho}^k \rho^{2M_1 k\tau},$$

where

$$\tilde{\rho}^k = \frac{\tau}{4\delta(1-M_1\tau)} \sum_{s=0}^{k} [2\|R^s\|^2_{\Omega_h} + \pi\varepsilon(8\delta h^2 + \varepsilon)|r^s|^2].$$

If $\tilde{\rho}^k \to 0$ as $h \to 0$ uniformly for all $k>0$, then scheme (5.3) is convergent uniformly for $t>0$.

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