FINITE ELEMENT APPROXIMATION TO AXIAL SYMMETRIC STOKES FLOW*

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The finite element method for Stokes flow has been extensively and intensively studied, and the method for axial symmetric elliptic problems has also been touched, see e.g. [1]. The purpose of this paper is to discuss the finite element method for axial symmetric Stokes flow and prepare for the discussion of the infinite element approximation to axial symmetric Stokes flow, which will be published in another paper.

Let us give the classical statement of the three dimensional axial symmetric Stokes flow. Let $x=(x_1, x_2)\in\mathbb{R}^2$ and Ω be a bounded polygonal region on the half plane $x_1>0$. We consider the following problem: to find $u(x)=(u_1(x), u_2(x))$ and $p(x)$, satisfying

$$\nu(-\nabla(x_1\nabla u_1)/x_1+u_1/x_1^2)+\frac{\partial p}{\partial x_1}=f_1, \quad x\in\Omega,$$

$$-\nu\nabla(x_1\nabla u_2)/x_1+\frac{\partial p}{\partial x_2}=f_2, \quad x\in\Omega,$$

$$\frac{\partial}{\partial x_1}(x_1u_1)+\frac{\partial}{\partial x_2}(x_1u_2)=0, \quad x\in\Omega,$$

$$u=0, \quad x\in\partial\Omega\setminus\{x_1=0\},$$

$$u_1=0, \quad x\in\partial\Omega\cap\{x_1=0\}.$$

If Ω rotates around the x_2-axis, then a three-dimensional region $\tilde{\Omega}$ is formed. The above problem is a description of the incompressible viscous flow on $\tilde{\Omega}$ with low Reynold's number, where the constant $\nu>0$ is viscosity, u velocity, p pressure and $f=(f_1, f_2)$ body force.

We need some weighted Sobolev spaces for the above problem. First we define the seminorm and norm as

$$|f|_{m,\Omega}=\Big(\sum_{|\alpha|=m}\int_\Omega x_1|D^\alpha f|^2dx\Big)^{1/2},$$

$$\|f\|_{m,\Omega}=\Big(\sum_{l=0}^{m}|f|_{l,\Omega}^2\Big)^{1/2}.$$

The corresponding Hilbert spaces are donoted by $Z^m(\Omega)$, where $\alpha=(\alpha_1, \alpha_2)$, $D^\alpha=\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. Then we define the norm as

$$|f|_{1,*,\Omega}=(|f|_{1,\Omega}^2+\|f/x_1\|_{0,\Omega}^2)^{1/2},$$

$$\|f\|_{1,*,\Omega}=(|f|_{1,*,\Omega}^2+\|f\|_{0,\Omega}^2)^{1/2}.$$

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If f can be expressed as $f=f_1f_2$, where $f_1\in C^\infty(\bar{\Omega})$, $f_2\in C_0^\infty(\mathbb{R}_+^2)$, $\mathbb{R}_+^2=\{x\in\mathbb{R}^2;\ x_1>0\}$, then we denote $f\in C_*^\infty(\bar{\Omega})$. The completion of $C_*^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot\|_{1,*,\Omega}$ is denoted by $Z_*^1(\Omega)$. We define $f\in Z_+^2(\Omega)$ if and only if $f\in Z_*^1(\Omega)\cap Z^2(\Omega)$, and $\|D^{(0,2)}f/x_1\|_{0,\Omega}$ is bounded.

Let $H(\Omega)=Z_*^1(\Omega)\times Z^1(\Omega)$, $H_0(\Omega)=\{f\in H(\Omega);\quad f|_{\partial\Omega\setminus(x_1=0)}=0\}$, $M_0(\Omega)=\left\{p\in Z^0(\Omega);\ \int_\Omega x_1 p\,dx=0\right\}$. We consider the bilinear form on $H(\Omega)\times H(\Omega)$:

$$a(u,\ v)=\nu\int_\Omega x_1(\nabla u_1\cdot\nabla v_1+\nabla u_2\cdot\nabla v_2+u_1v_1/x_1^2)dx,\quad u,\ v\in H(\Omega),\tag{1}$$

and the bilinear form on $H(\Omega)\times Z^0(\Omega)$:

$$b(v,\ p)=-\int_\Omega p\left\{\frac{\partial}{\partial x_1}(x_1v_1)+\frac{\partial}{\partial x_2}(x_1v_2)\right\}dx,\quad v\in H(\Omega),\ p\in Z^0(\Omega).\tag{2}$$

Then a weak formulation of the original problem is: to find $(u,\ p)\in H_0(\Omega)\times M_0(\Omega)$, such that

$$a(u,\ v)+b(v,\ p)=F(v),\quad \forall v\in H_0(\Omega),\tag{3}$$

$$b(u,\ q)=0,\quad \forall q\in M_0(\Omega),\tag{4}$$

where

$$F(v)=\int_\Omega x_1(f_1v_1+f_2v_2)dx.$$

We see from definitions (1), (2) that a, b are bounded and

$$a(u,\ u)=\nu(|u_1|_{1,*,\Omega}^2+|u_2|_{1,\Omega}^2),$$

and we notice that the inequality of Poincaré–Friedrichs type

$$\|u_1\|_{0,\Omega}^2+\|u_2\|_{0,\Omega}^2\leqslant Ca(u,\ u)$$

holds on $H_0(\Omega)$. Throughout the paper C will always denote a positive constant. We have

$$a(u,\ u)>a_0\|u\|_{H(\Omega)}^2,\quad \forall u\in H_0(\Omega),\tag{5}$$

where $a_0>0$. Moreover, if f_1, f_2 are appropriately regular, then problem (3), (4) has a unique solution[2].

Now we consider the finite element approximation to problem (3), (4). The region Ω is divided into finite convex polygonal regions Ω_k, $k=1,\ 2,\ \cdots$, by finite broken lines. Then each subregion Ω_k is further divided into triangular elements, and it is assumed that Ω_k keep fixed in the further refinement process. It is also assumed that any two elements in Ω meet only in the entire common side, or at only a common vertex, or do not meet at all. The vertices and midpoints of the sides of all elements are taken as nodes. The element is denoted by e, and the side is denoted by s, where each e is an open set and the end points are not included in s. We make quadratic polynomial interpolation for u, and p is a constant on e. Then the subspaces $H_{0h}(\Omega)$, $M_{0h}(\Omega)$ of $H_0(\Omega)$, $M_0(\Omega)$ are obtained, and so are the subspaces $H_h(\Omega_k)$, $M_h(\Omega_k)$ of $H(\Omega_k)$, $Z^0(\Omega_k)$.

This kind of triangulation and interpolation causes loss of precision[3]. To overcome this shortcoming, there are several approaches, see e.g. [4], [5]. But for simplicity, we only consider this kind of element.

The finite element approximation to problem (3), (4) is: to find $(u_h,\ p_h)\in$

$H_{0h}(\Omega)\times M_{0h}(\Omega)$, such that

$$a(u_h, v)+b(v, p_h)=F(v), \quad \forall v\in H_{0h}(\Omega), \tag{6}$$

$$b(u_h, q)=0, \quad \forall q\in M_{0h}(\Omega). \tag{7}$$

Theorem 1. *The solution of problem* (6), (7) *exists and is unique.*

Proof. It suffices to prove that the corresponding homogeneous problem has only null solution. Let u_h, p_h be the solution of the homogeneous problem. Taking $q=p_h$ in (7) we obtain

$$b(u_h, p_h)=0,$$

and taking $v=u_h$ in (6) we obtain

$$a(u_h, u_h)=0.$$

By (5), $u_h=0$. Then, we take such a v that vanishes at all nodes except the midpoint of a side. By (2) and Green's formula we know p_h remains the same on both elements neighboring on s. But s is arbitrary; so p_h is a constant. Since $p_h\in M_0(\Omega)$, we get $p_h=0$. QED.

We now discuss the error estimation of the solutions. We will make more assumption on the geometry of the triangulation. Assume that all interior angles of all elements have a positive lower bound θ_0. Denote by h the largest length of sides, and by $h_{\min}$ the smallest length of sides. Then assume that $h/h_{\min}\leqslant C$. Let s be one side of e. Denote by e_s the isosceles triangle with s as the base and $\theta_0/2$ as the base angles. Define e_s to be the standard triangle corresponding to s. For an appropriate θ_0 and under the above assumptions, we have the following lemmas.

Lemma 1. *If* $\sup\limits_s x_1>0$, $\inf\limits_s x_1=0$, e_s *is a standard triangle corresponding to* s, *and* x_0 *is the end point of* s *at the* x_2*-axis, then for any* $x\in e_s$, *the angle between vector* $x-x_0$ *and the* x_2*-axis is greater than* $\theta_0/2$.

Lemma 2. *If* $\inf\limits_s x_1>0$, *then* $\inf\limits_s x_1\geqslant C^{-1}h$ and $\sup\limits_s x_1\leqslant C\inf\limits_s x_1$. *Similarly, if* $\inf\limits_e x_1>0$, *then* $\sup\limits_e x_1\leqslant C\inf\limits_e x_1$.

Lemma 3. *If* $\inf\limits_s x_1>0$ *and* e_s *is a standard triangle corresponding to* s, *then* $\inf\limits_s x_1\leqslant C\inf\limits_{e_s} x_1$.

Let $\lambda_i(x)$, $i=1, \cdots, 6$, be the interpolation basis functions of element e. Then we have

Lemma 4. *If the node corresponding to* λ_i *does not lie at the* x_2*-axis, then*

$$|\lambda_i(x)|\leqslant Cx_1h^{-1}.$$

The proofs of the above lemmas are easy and thus are omitted.

We consider approximation of functions in the following. Let s_i, $i=1, 2, 3$, be the three sides of element e, and $x^{(i)}$, $i=1, \cdots, 6$, be the nodes. $x^{(i)}$ are vertices if $i\leqslant 3$ or midpoints if $i\geqslant 4$.

Lemma 5. *If* $f\in Z^2(e)$ *and* f_I *is a quadratic polynomial, such that*

$$f_I(x^{(i)})=f(x^{(i)}), \quad i=1, 2, 3, \tag{8}$$

$$\int_{s_i} x_1(f-f_I)ds=0, \quad \text{as} \quad s_i\cap\{x_1=0\}=\emptyset, \tag{9}$$

$$f_I(x^{(j)})=f(x^{(j)}),\quad as\quad s_i\subset\{x_1=0\},\ x^{(j)}\in s_i, \tag{10}$$

then

$$|f-f_I|_{m,s}\leqslant Ch^{2-m}|f|_{2,s},\quad m=0,\ 1, \tag{11}$$

where ds is the differential of length.

Proof. Let $\lambda_i(x)$ be the interpolation basis functions corresponding to nodes $x^{(i)}$ and set

$$g(x)=\sum_{i=1}^{6}f(x^{(i)})\lambda_i(x). \tag{12}$$

In virtue of Taylor's formula we have

$$f(x^{(i)})=f(x)+D_if(x)+\int_0^1 tD_i^2f(\xi^{(i)}(t))\,dt,$$

where

$$D_i=(x^{(i)}-x)\cdot\nabla,\quad \xi^{(i)}(t)=tx+(1-t)x^{(i)}.$$

By substituting it into (12) we obtain

$$g(x)=\sum_{i=1}^{6}\Big(f(x)+D_if(x)+\int_0^1 tD_i^2f(\xi^{(i)}(t))\,dt\Big)\lambda_i(x).$$

We know from

$$\sum_{i=1}^{6}\lambda_i(x)=1,$$

$$\sum_{i=1}^{6}x_j^{(i)}\lambda_i(x)=x_j,\quad j=1,\ 2,$$

that

$$f(x)-g(x)=-\sum_{i=1}^{6}\int_0^1 tD_i^2f(\xi^{(i)}(t))\,dt\,\lambda_i(x). \tag{13}$$

From

$$|D^\alpha\lambda_i(x)|\leqslant Ch^{-m},\quad |\alpha|=m,\ m=0,1,$$

and after some calculation[1] we obtain the estimate

$$|f-g|_{m,s}\leqslant Ch^{2-m}|f|_{2,s},\quad m=0,\ 1. \tag{14}$$

Set

$$f_I(x)=g(x)+\sum_{i=4}^{6}c_i\lambda_i(x), \tag{15}$$

where $x^{(i)}\in s_j$ and

$$c_i=\begin{cases}\displaystyle\int_{s_j}x_1(f(x)-g(x))\,ds\Big/\int_{s_j}x_1\lambda_i(x)\,ds, & \text{as}\quad s_j\cap\{x_1=0\}=\emptyset,\\ 0, & \text{as}\quad s_j\subset\{x_1=0\}.\end{cases}$$

(15) implies

$$\int_{s_j}x_1f_I(x)\,ds=\int_{s_j}x_1f(x)\,ds.$$

Therefore (8)—(10) hold. The polynomial f_I determined by (8)—(10) is unique[3]. Hence it remains to verify (11).

Let $\hat{x}=(\hat{x}_1,\ \hat{x}_2)$ be a point on the reference plane. We construct a reference triangle $\hat{e}=\{\hat{x}\in\mathbb{R}^2;\ \hat{x}_2\ \tan^{-1}(\theta_0/2)<\hat{x}_1<1-\hat{x}_2\ \tan^{-1}(\theta_0/2),\ 0<\hat{x}_2<1/(2\tan^{-1}(\theta_0/2))\}$. As $\inf_e x_1=0$ and $s_j\subset\partial e$, if $\inf_{s_j} x_1>0$, let $e_s\subset e$ be the standard triangle corresponding to s_j. The image of $\hat{e}$ is e_s under a suitable rigid body motion and similarity transformation $x=\psi(\hat{x})$ on plane. Set $\varphi=f-g$, $\hat{\varphi}(\hat{x})=\varphi(\psi(\hat{x}))$. In virtue of the trace theorem on Sobolev space $H^1(\hat{e})$ we obtain

$$\left(\int_0^1 |\hat{\varphi}(\hat{x}_{1,0})|\,d\hat{x}_1\right)^2 \leqslant C\|\hat{\varphi}\|_{1,\hat{e}}^2.$$

Then we have

$$\left(\int_{s_j} |\varphi(x)|\,ds\right)^2 h^{-2} \leqslant C\left(\int_{e_s} |\nabla\varphi|^2\,dx + h^{-2}\int_{e_s} \varphi^2\,dx\right)$$

for the independent variable x. By Lemmas 2 and 3,

$$\left(\int_{s_j} x_1|\varphi(x)|\,ds\right)^2 \leqslant C(\inf_{s_j} x_1)^2\left(h^2\int_{e_s} |\nabla\varphi|^2\,dx + \int_{e_s} \varphi^2\,dx\right)$$
$$\leqslant C\inf_{s_j} x_1\left(h^2\int_{e_s} x_1|\nabla\varphi|^2\,dx + \int_{e_s} x_1\varphi^2\,dx\right).$$

By (14)

$$\int_{s_j} x_1|\varphi(x)|\,ds \leqslant C(\inf_{s_j} x_1)^{1/2} h^2 |f|_{2,e}.$$

Now

$$\int_{s_j} x_1\lambda_i(x)\,dx \geqslant \inf_{s_j} x_1 \cdot C^{-1}h.$$

Hence

$$|c_i| \leqslant C(\inf_{s_j} x_1)^{-\frac{1}{2}} h |f|_{2,e}. \tag{16}$$

By Lemma 2,

$$|c_i| \leqslant Ch^{1/2}|f|_{2,e}. \tag{17}$$

If $\inf_{s_j} x_1 = 0$ and $c_i \neq 0$, let $e_s \subset e$ be the standard triangle corresponding to s_j. By Lemma 1, let $x = \psi(\hat{x})$ be such a rigid body motion and similarity transformation that the image of $\hat{e}$ is e_s, and

$$C^{-1}h \leqslant \frac{x_1}{\hat{x}_1} \leqslant Ch. \tag{18}$$

Set $\hat{\varphi}(\hat{x}) = \varphi(\psi(\hat{x}))$. By rotating $\hat{e}$ around the $\hat{x}_2$-axis, we get a three-dimensional region $\tilde{e}$. In virtue of the trace theorem on Sobolev space $H^1(\tilde{e})$ we obtain

$$\left(2\pi\int_0^1 \hat{x}_1\hat{\varphi}(\hat{x})\,d\hat{x}_1\right)^2 \leqslant 2\pi C\left(\int_{\hat{e}} \hat{x}_1|\nabla\hat{\varphi}|^2\,d\hat{x} + \int_{\hat{e}} \hat{x}_1\hat{\varphi}^2\,d\hat{x}\right).$$

Then we have

$$\left(\int_{s_j} x_1\varphi(x)\,ds\right)^2 h^{-4} \leqslant C\left(h^{-1}\int_{e_s} x_1|\nabla\varphi|^2\,dx + h^{-3}\int_{e_s} x_1\varphi^2\,dx\right)$$

for the independent variable x. By (14)

$$\left|\int_{s_j} x_1\varphi(x)\,ds\right| \leqslant Ch^{5/2}|f|_{2,e}.$$

By Lemma 1

$$\int_{s_j} x_1\lambda_i(x)\,ds \geqslant C^{-1}h^2.$$

Therefore (17) also holds.

It is easy to prove

$$|\lambda_i|_{m,e} \leqslant Ch^{\frac{3}{2}-m}, \quad m=0,1. \tag{19}$$

(17), (19) imply

$$\left|\sum_{i=4}^{6} c_i\lambda_i\right|_{m,e} \leqslant Ch^{2-m}|f|_{2,e}. \tag{20}$$

Then (11) follows from (14), (15).

As $\inf_e x_1>0$, we have (16); on the other hand it is easy to prove

$$|\lambda_i|_{m,e}\leqslant C(\sup_e x_1)^{1/2}h^{1-m}.$$

By Lemma 2, (20) also holds. Then (11) follows too. QED.

Lemma 6. *If $f\in Z_+^2(e)$, then*

$$\|(f-f_I)/x_1\|_{0,e}\leqslant Ch(|f|_{2,e}+h\|D^{(0,2)}f/x_1\|_{0,e}), \tag{21}$$

where f_I is determined by Lemma 5.

Proof. There is no harm in assuming $f\in C^2(\bar{\Omega})$, and $f=0$ near the x_2-axis. Then (13) holds. Let

$$\varphi_i(x)=\int_0^1 tD_i^2f(\xi^{(i)}(t))dt\lambda_i(x). \tag{22}$$

If $x^{(i)}\overline{\in}\{x_1=0\}$, by Lemma 4

$$\int_e x_1^{-1}\varphi_i^2\,dx\leqslant Ch^2\sum_{|\alpha|=2}\int_e x_1\Big\{\int_0^1 t|D^\alpha f(\xi^{(i)}(t))|dt\Big\}^2dx$$

$$\leqslant Ch^2\sum_{|\alpha|=2}\int_e\int_0^1 t^{-1/2}\,dt\int_0^1 x_1t^{5/2}|D^\alpha f(\xi^{(i)}(t))|^2\,dt\,dx.$$

Now we change the variables. Let $\xi=(\xi_1,\xi_2)=\xi^{(i)}(t)$. Then $dx=d\xi/t^2$. By noticing $x_1t\leqslant\xi_1$ we get

$$\int_e x_1^{-1}\varphi_i^2\,dx\leqslant Ch^2\sum_{|\alpha|=2}\int_0^1 t^{-1/2}\,dt\int_e\xi_1|D^\alpha f(\xi)|^2\,d\xi,$$

that is

$$\|\varphi_i/x_1\|_{0,e}\leqslant Ch|f|_{2,e}.$$

If $x^{(i)}\in\{x_1=0\}$, we expand the differential operator in (22) and obtain

$$|\varphi_i(x)|\leqslant\int_0^1 t\{|(x_1^{(i)}-x_1)^2D^{(2,0)}f(\xi^{(i)}(t))|+2|(x_1^{(i)}-x_1)(x_2^{(i)}-x_2)D^{(1,1)}f(\xi^{(i)}(t))|$$

$$+|(x_2^{(i)}-x_2)^2D^{(0,2)}f(\xi^{(i)}(t))|\}dt|\lambda_i(x)|.$$

Since $|\lambda_i(x)|\leqslant1$, $x_1^{(i)}=0$,

$$\int_e x_1^{-1}\varphi_i^2dx\leqslant Ch^2\sum_{|\alpha|=2}\int_e x_1\Big\{\int_0^1 t|D^\alpha f(\xi^{(i)}(t))|dt\Big\}^2dx$$

$$+Ch^4\int_e x_1^{-1}\Big\{\int_0^1 t|D^{(0,2)}f(\xi^{(i)}(t))|dt\Big\}^2dx\equiv I_1+I_2.$$

The estimate of I_1 is the same as before. For I_2, $\xi_1\leqslant x_1$ yields

$$I_2\leqslant Ch^4\int_e\int_0^1 x_1^{-1}t^2|D^{(0,2)}f(\xi^{(i)}(t))|^2\,dt\,dx\leqslant Ch^4\int_0^1 dt\int_e\xi_1^{-1}|D^{(0,2)}f(\xi)|^2\,d\xi.$$

Hence

$$\|\varphi_i/x_1\|_{0,e}\leqslant Ch(|f|_{2,e}+h\|D^{(0,2)}f/x_1\|_{0,e}).$$

By (13), (22) we obtain

$$\|(f-g)/x_1\|_{0,e}\leqslant Ch(|f|_{2,e}+h\|D^{(0,2)}f/x_1\|_{0,e}).$$

To get (21), we consider (15). By Lemma 4 and $|\lambda_i|\leqslant1$, as $x^{(i)}\overline{\in}\{x_1=0\}$,

$$\|\lambda_i/x_1\|_{0,e}=\Big(\int_e x_1^{-1}\lambda_i^2\,dx\Big)^{1/2}\leqslant\Big(C\int_e h^{-1}\,dx\Big)^{1/2}\leqslant Ch^{1/2}.$$

By (17)

$$\|c_i\lambda_i/x_1\|_{0,e}\leqslant Ch|f|_{2,e}.$$

As $x^{(i)} \in \{x_1=0\}$ and $c_i=0$, the above estimate also holds. Therefore (21) holds. QED.

Lemma 7. *If $f\in Z^1(e)$, then there exists a constant f_0, such that*

$$\|f-f_0\|_{0,e}\leqslant Ch|f|_{1,e}. \tag{23}$$

Proof. If $\inf\limits_e x_1>0$, we take a constant f_0, such that

$$\int_e (f-f_0)dx=0.$$

Then[3]

$$\int_e (f-f_0)^2\,dx\leqslant Ch^2\int_e |\nabla f|^2 dx.$$

By Lemma 2

$$\int_e x_1(f-f_0)^2\,dx\leqslant C\inf_e x_1\cdot h^2\int_e |\nabla f|^2 dx\leqslant Ch^2\int_e x_1|\nabla f|^2 dx,$$

i.e. (23) holds. If $\inf\limits_e x_1=0$, we consider auxiliary triangles $\hat{e}_1=\{\hat{x}\in\mathbb{R}^2;\ 0<\hat{x}_1<\hat{x}_2,\ 0<\hat{x}_2<1\}$, $\hat{e}_2=\{\hat{x}\in\mathbb{R}^2;\ 0<\hat{x}_2<\hat{x}_1,\ 0<\hat{x}_1<1\}$. Under an appropriate affine transformation $x=\psi(\hat{x})$, the image of either $\hat{e}_1$ or $\hat{e}_2$ is e, and inequality (18) holds. Set $\hat{f}(\hat{x})=f(\psi(\hat{x}))$, and take a constant f_0, such that

$$\int_{\hat{e}_i} \hat{x}_1(\hat{f}-f_0)d\hat{x}=0,\quad i=1 \text{ or } 2.$$

Let $\tilde{e}_1$, $\tilde{e}_2$ be the locus of $\hat{e}_1$, $\hat{e}_2$ rotating around the $\hat{x}_2$-axis, respectively. In virtue of the estimate of interpolation operator on Sobolev spaces $H^1(\tilde{e}_i)$, $i=1, 2$[6],

$$\int_{\hat{e}_i} \hat{x}_1(\hat{f}-f_0)^2 d\hat{x}\leqslant C\int_{\hat{e}_i} \hat{x}_1|\Delta\hat{f}|^2 d\hat{x}.$$

By noticing (18), we obtain (23) for variable x. QED.

We now verify two Babuška-Brezzi conditions. One is related to regions Ω_k, where the subscript k will later be dropped for the sake of convenience.

Lemma 8. *Let Ω be a convex region. Then for any $p\in M_{0h}(\Omega)$, there exists a $u\in H_{0h}(\Omega)$, such that*

$$\|p\|_{0,\Omega}\leqslant C\int_\Omega p\Big(\frac{\partial}{\partial x_1}(x_1u_1)+\frac{\partial}{\partial x_2}(x_1u_2)\Big)dx/\|u\|_{H(\Omega)}. \tag{24}$$

Proof. By rotating Ω around the x_2-axis, we get a three-dimensional region $\tilde{\Omega}$. Consider p as a function defined on $\tilde{\Omega}$. Then there exists $v\in (H_0^1(\tilde{\Omega}))^3$, such that[3]

$$\operatorname{div} v=p,$$
$$\|v\|_{(H_0^1(\tilde{\Omega}))^3}\leqslant C\|p\|_{L^2(\tilde{\Omega})},$$

and v only depends on x_1, x_2.

As a function with independent variables x_1, x_2, $v\in H_0(\Omega)$ and

$$\|v\|_{H(\Omega)}\leqslant C\|p\|_{0,\Omega}.$$

Let $w\in H_{0h}(\Omega)$ be the projection of v defined by

$$a(v-w,\ z)=0,\quad \forall z\in H_{0h}(\Omega).$$

We take $u\in H_{0h}(\Omega)$, such that

$$u(x^{(i)})=w(x^{(i)}),\quad i=1, 2, 3,$$

on each element e; and

$$\int_{s_i} x_1(v-u)\,dx=0 \tag{25}$$

as $s_i\cap\{x_1=0\}=\emptyset$;

$$u|_{s_i}\equiv w|_{s_i}$$

as $s_i\subset\{x_1=0\}$, for s_i, $i=1, 2, 3$. For any $q\in M_{0h}(\Omega)$, by Green's formula,

$$\int_\Omega\Big(\frac{\partial}{\partial x_1}(x_1u_1)+\frac{\partial}{\partial x_2}(x_1u_2)-x_1p\Big)q\,dx=\sum_e\int_{\partial e}x_1qu\cdot n\,dx-\int_\Omega x_1pq\,dx,$$

where n is the unit outward normal vector of e. By (25) and Green's formula,

$$\sum_e\int_{\partial e}x_1qu\cdot n\,dx=\sum_e\int_{\partial e}x_1qv\cdot n\,dx=\int_\Omega\Big(\frac{\partial}{\partial x_1}(x_1v_1)+\frac{\partial}{\partial x_2}(x_1v_2)\Big)q\,dx=\int_\Omega x_1pq\,dx.$$

Therefore

$$\int_\Omega\Big(\frac{\partial}{\partial x_1}(x_1u_1)+\frac{\partial}{\partial x_2}(x_1u_2)-x_1p\Big)q\,dx=0. \tag{26}$$

We estimate u. Set

$$\varepsilon=v-w,\quad \varepsilon_h=u-w.$$

Then by (25)

$$\int_{s_i}x_1(\varepsilon-\varepsilon_h)\,ds=0. \tag{27}$$

As $\inf_e x_1=0$, if $s_i\cap\{x_1=0\}=\emptyset$ and $\inf_{s_i}x_1=0$, let the midpoint of s_i be $x^{(i)}$. ε_h is a quadratic polynomial on e and vanishes at the end points of s_i. By Lamma 1

$$h^2|\varepsilon_h(x^{(i)})|\leqslant C\Big|\int_{s_i}x_1\varepsilon_h\,ds\Big|. \tag{28}$$

In the same way as in proving Lemma 5, by applying the trace theorem for the three-dimensional region, we have

$$\Big(\int_{s_i}x_1|\varepsilon|\,ds\Big)^2h^{-4}\leqslant C\Big(h^{-1}\int_{e_s}x_1|\nabla\varepsilon|^2dx+h^{-3}\int_{e_s}x_1|\varepsilon|^2dx\Big). \tag{29}$$

(27)—(29) yield

$$|\varepsilon_h(x^{(i)})|^2\leqslant C\Big(h^{-1}\int_{e_s}x_1|\nabla\varepsilon|^2dx+h^{-3}\int_{e_s}x_1|\varepsilon|^2dx\Big). \tag{30}$$

If $\inf_{s_i}x_1>0$, then by [3]

$$|\varepsilon_h(x^{(i)})|^2\leqslant Ch^{-2}\Big(h^2\int_{e_s}|\nabla\varepsilon|^2dx+\int_{e_s}|\varepsilon|^2dx\Big). \tag{31}$$

By Lemma 2, (30) also holds. Set $\varepsilon_h=(\varepsilon_{h_1},\varepsilon_{h_2})$. Then

$$\|\varepsilon_{h_1}\|^2_{1,*,e}\leqslant h\sum_{i=4}^{6}|\varepsilon_{h_1}(x^{(i)})|^2,$$

$$\|\varepsilon_{h_2}\|^2_{1,e}\leqslant h\sum_{i=4}^{6}|\varepsilon_{h_2}(x^{(i)})|^2.$$

From (30) we get

$$\|\varepsilon_h\|^2_{H(e)}\leqslant C\Big(\int_e x_1|\nabla\varepsilon|^2dx+h^{-2}\int_e x_1|\varepsilon|^2dx\Big). \tag{32}$$

As $\inf_e x_1>0$, (31) holds, but

$$|e_{h_1}|^2_{1,*,e}\leqslant\sup_e x_1\sum_{i=4}^{6}|e_{h_1}(x^{(i)})|^2+\frac{h^2}{\inf_e x_1}\sum_{i=4}^{6}|e_{h_1}(x^{(i)})|^2.$$

By Lemma 2

$$\|e_{h_1}\|^2_{1,*,e}\leqslant Ch^{-2}\Big(h^2\int_e x_1|\nabla e|^2dx+\int_e x_1|e|^2dx\Big).$$

Then for the same reason

$$\|e_{h_1}\|^2_{1,e}\leqslant Ch^{-2}\Big(h^2\int_e x_1|\nabla e|^2dx+\int_e x_1|e|^2dx\Big).$$

Therefore (32) holds too. Summing inequality (32) with respect to elements, we obtain

$$\|e_h\|^2_{H(\Omega)}\leqslant C\Big(\int_\Omega x_1|\nabla e|^2dx+h^{-2}\int_\Omega x_1|e|^2dx\Big).$$

Using Aubin–Nitsche's trick, we can prove

$$\int_\Omega x_1|e|^2dx\leqslant Ch^2\int_\Omega x_1|\nabla e|^2dx.$$

Hence $$\|e_h\|_{H(\Omega)}\leqslant C\|e\|_{H(\Omega)}.$$

But w is a projection; so

$$\|e\|_{H(\Omega)}\leqslant C\|v\|_{H(\Omega)},\quad \|w\|_{H(\Omega)}\leqslant C\|v\|_{H(\Omega)}.$$

Therefore $$\|u\|_{H(\Omega)}\leqslant\|e_h\|_{H(\Omega)}+\|w\|_{H(\Omega)}\leqslant C\|v\|_{H(\Omega)}\leqslant C\|p\|_{0,\Omega}.$$

Taking $q=p$ in (26) we obtain

$$\begin{aligned}\|p\|_{0,\Omega}&\leqslant C\|p\|^2_{0,\Omega}/\|u\|_{H(\Omega)}\\&=C\int_\Omega p\Big(\frac{\partial}{\partial x_1}(x_1u_1)+\frac{\partial}{\partial x_2}(x_1u_2)\Big)dx/\|u\|_{H(\Omega)},\end{aligned}$$

which is (24). QED.

Let M_c be a finite dimensional subspace of M_0, such that if $p\in M_c$, then p is a constant on each region Ω_k, $k=1, 2, \cdots$. Set $\Gamma=\bigcup_k\partial\Omega_k$. The trace space of $H_0(\Omega)$ on Γ is denoted by $H_0(\Gamma)$. Let Y be a finite dimensional subspace of $H_0(\Gamma)$, such that u is a quadratic polynomial on each line segment of Γ as $u\in Y$.

Lemma 9. *For any $p\in M_c$, there exists a $u\in Y$, such that*

$$\|p\|_{0,\Omega}\leqslant C\sum_k\Big(p|_{\Omega k}\int_{\partial\Omega_k}x_1u\cdot n\,ds\Big)\Big/\|u\|_Y. \tag{33}$$

Proof. The proof is similar to that of Lemma 8. For p, we take a v, project it in space $H_0(\Gamma)$ on Y and get w. Then take $u\in Y$, such that $u=w$ at the end points of each line segment s on Γ, and

$$\int_s x_1(v-u)ds=0,\quad\text{as}\quad s\cap\{x_1=0\}=\emptyset,$$

$$u=w,\quad\text{as}\quad s\subset\{x_1=0\}.$$

By Green's formula

$$\int_{\partial\Omega_k}x_1u\cdot n\,ds=p|_{\Omega_k}\cdot\int_{\Omega_k}x_1\,dx.$$

Now $p \to u$ is a linear operator from M_c to Y. But the space is finite dimensional; hence
$$\|u\|_Y \leqslant C\|p\|_{M_c}.$$
Then we can get (33). QED.

Finally we obtain the error estimation for the approximate solution as follows:

Theorem 2. *Let u, p be the solution of problem (3), (4), and $u_1 \in Z^2_+(\Omega)$, $u_2 \in Z^2(\Omega)$, $p \in Z^1(\Omega)$, u_h, p_h be the solution of problem (6), (7). Then*
$$\|u-u_h\|_{H(\Omega)}+\|p-p_h\|_{0,\Omega} \leqslant Ch(|u_1|_{2,\Omega}+\|D^{(0,2)}u_1/x_1\|_{0,\Omega}+|u_2|_{2,\Omega}+|p|_{1,\Omega}).$$

Proof. Let $V_h=\{u \in H_{0h}(\Omega);\ b(u,p)=0,\ \forall p \in M_{0h}(\Omega)\}$. Then (7) implies $u_h \in V_h$, and u, u_h satisfy equations (3), (6) respectively. Hence
$$a(u, v)+b(v, p)=F(v), \quad \forall v \in V_h,$$
$$a(u_h, v)=F(v), \quad \forall v \in V_h.$$
By subtracting them we get
$$a(u-u_h, v)+b(v, p)=0, \quad \forall v \in V_h.$$
Take any $q \in M_{0h}(\Omega)$. Then $b(v, q)=0$. We have
$$a(v, v)=a(v+u-u_h, v)+b(v, p-q).$$
Inequality (5) and the boundedness of a, b leads to
$$d_0\|v\|^2_{H(\Omega)} \leqslant C(\|v+u-u_h\|_{H(\Omega)}\|v\|_{H(\Omega)}+\|p-q\|_{M_0(\Omega)}\|v\|_{H(\Omega)}),$$
that is
$$\|v\|_{H(\Omega)} \leqslant C(\|v+u-u_h\|_{H(\Omega)}+\|p-q\|_{M_0(\Omega)}).$$
Taking any $w \in V_h$ and setting $v=u_h-w$, we have
$$\|u-u_h\|_{H(\Omega)} \leqslant \|u-w\|_{H(\Omega)}+\|v\|_{H(\Omega)} \leqslant C(\|u-w\|_{H(\Omega)}+\|p-q\|_{M_0(\Omega)}). \tag{34}$$
From equation (4)
$$b(u, q)=0, \quad \forall q \in M_{0h}(\Omega),$$
According to Lemmas 5, 6, let u_{1I}, u_{2I} be the interpolation functions of u_1, u_2 respectively, and set $u_I=(u_{1I}, u_{2I})$. (8)—(10) imply
$$b(u_I, q)=b(u, q).$$
Hence $u_I \in V_h$. By Lemmas 5, 6,
$$\|u-u_I\|_{H(\Omega)} \leqslant Ch(|u_1|_{2,\Omega}+\|D^{(0,2)}u_1/x_1\|_{0,\Omega}+|u_2|_{2,\Omega}). \tag{35}$$
And by Lemma 7, there is an $r \in Z^0(\Omega)$, which is a constant on each element e, such that
$$\|p-r\|_{0,\Omega} \leqslant Ch|p|_{1,\Omega}.$$
By Schwarz's inequality,
$$\left(\int_\Omega x_1(p-r)\,dx\right)^2 \leqslant C\int_\Omega x_1(p-r)^2\,dx=C\|p-r\|^2_{0,\Omega}.$$
Set
$$\beta=\int_\Omega x_1(p-r)\,dx \Big/ \int_\Omega x_1\,dx.$$
Then $r+\beta \in M_{0h}(\Omega)$. Let $q=r+\beta$, and then we have
$$\|p-q\|_{0,\Omega} \leqslant \|p-r\|_{0,\Omega}+\|\beta\|_{0,\Omega} \leqslant C\|p-r\|_{0,\Omega} \leqslant Ch|p|_{1,\Omega}. \tag{36}$$
By substituting (35), (36) into (34), we obtain
$$\|u-u_h\|_{H(\Omega)} \leqslant Ch(|u_1|_{2,\Omega}+\|D^{(0,2)}u_1/x_1\|_{0,\Omega}+|u_2|_{2,\Omega}+|p|_{1,\Omega}). \tag{37}$$

(37) is just the desired estimation for $u-u_h$. We estimate $p-p_h$ in the following. Decompose p as $p=p_c+p_b$, where $p_c\in M_c$,

$$\int_{\Omega_k} x_1 p_b\,dx=0,\quad k=1,2,\cdots.$$

In the same way we decompose p_h as $p_h=p_{ch}+p_{bh}$. Then we take $v\in H_{0h}(\Omega)$, such that $v|_{\Omega_k}\in H_0(\Omega_k)$, $k=1,2,\cdots$. Now by Green's formula

$$b(v,p_c)=b(v,p_{ch})=0.$$

By equations (3), (6),

$$b(v,p_b-p_{bh})=a(u_h-u,v).$$

Let $M_{bh}=\left\{q\in M_{0h};\ \int_{\Omega_k}x_1 q\,dx=0,\ k=1,2,\cdots,\right\}$, then $p_{bh}\in M_{bh}$. We take a $q\in M_{bh}$. Then

$$b(v,q-p_{bh})=a(u_h-u,v)+b(v,q-p_b).$$

By Lemma 8 and taking an appropriate v, we have

$$\|q-p_{bh}\|_{0,\Omega_k}\leqslant Cb(v,q-p_{bh})/\|v\|_{H(\Omega_k)}\leqslant C(\|u-u_h\|_{H(\Omega_k)}+\|q-p_b\|_{0,\Omega_k}).$$

Hence

$$\|p_b-p_{bh}\|_{0,\Omega_k}\leqslant C(\|u-u_h\|_{H(\Omega_k)}+\|q-p_b\|_{0,\Omega_k}).$$

Taking q as in (36), we have

$$\|q-p_b\|_{0,\Omega_k}\leqslant Ch|p_b|_{1,\Omega_k}=Ch|p|_{1,\Omega_k}.$$

Summing them with respect to k and noticing (37), we get

$$\|p_b-p_{bh}\|_{0,\Omega}\leqslant Ch(|u_1|_{2,\Omega}+\|D^{(0,2)}u_1/x_1\|_{0,\Omega}+|u_2|_{2,\Omega}+|p|_{1,\Omega}).\tag{38}$$

(38) is just the desired estimation for p_b-p_{bh}. Finally, we estimate p_c-p_{ch}. Taking $v\in H_{0h}(\Omega)$, by equations (3), (6) we have

$$b(v,p_c-p_{ch})=a(u_h-u,v)+b(v,p_{bh}-p_b).$$

The trace of v on Γ is still denoted by v. Then by Green's formula we obtain

$$b(v,p_c-p_{ch})=\sum_k(p_c-p_{ch})|_{\Omega_k}\cdot\int_{\partial\Omega_k}x_1 v\cdot n\,ds.$$

By Lemma 9 and taking an appropriate v, we have

$$\|p_c-p_{ch}\|_{0,\Omega}\leqslant b(v,p_c-p_{ch})/\|v\|_Y\leqslant(a(u_h-u,v)+b(v,p_{bh}-p_b))/\|v\|_Y.$$

For $v\in Y$, we can always define its value on each region Ω_k such that $v\in H_{0h}(\Omega)$, and

$$\|v\|_{H(\Omega)}\leqslant C|v|_Y.$$

Therefore

$$\|p_c-p_{ch}\|_{0,\Omega}\leqslant C(\|u-u_h\|_{H(\Omega)}+\|p_b-p_{bh}\|_{0,\Omega}).$$

Substituting (37), (38) into it, we obtain

$$\|p_c-p_{ch}\|_{0,\Omega}\leqslant Ch(|u_1|_{2,\Omega}+\|D^{(0,2)}u_1/x_1\|_{0,\Omega}+|u_2|_{2,\Omega}+|p|_{1,\Omega}).$$

QED.

Wu Xiao-nan has considered this problem[7] and obtained the error estimation in a special case.

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