A FINITE ELEMENT METHOD OF SEMI-DISCRETIZATION WITH MOVING GRID*

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§ 1. Introduction

In solving a parabolic equation, a finite element method with variable mesh is more efficient than one with fixed mesh, if the space domain to be solved changes with time, such as the moving boundary problem, or if the peak value on the curved surface of the solution in the space domain moves with time, such as the speading of flame. In spite of the existence of this kind of methods[1-6], however, there is lack of its theoretical analysis; especially, there is hardly any proof of its optimal order accuracy.

Jamet has proved[7] that a method proposed by himself and Bonnerot, where the finite element is adopted in both space and time, has the optimal order accuracy. But his proof was made under the special condition of one dimension and uniform meshes and as a generalization of the Crank–Nicolson difference scheme, and is difficult to be extended to finite elements of more general form. Jamet also proved the convergence of their discontinuous finite element method and applied it to complex one-dimensional Stefan problem with many phases. Li[6] wrote the Stefan problem in enthalpy form so as to make his treatment of the moving boundary condition more natural when using Jamet's method. His method has strong adaptability and is fit for complex problems, but it requires many times the amount of calculation and storage than the continuous finite element method.

The purpose of this paper is to present a semi-discretization finite element method with grid moving continuously with time and to prove its optimal order accuracy. A stable difference scheme with second-order accuracy is given for the solution of an ordinary differential equation system derived from our method.

§ 2. The Semi-Discretization Finite Elements with Moving Grid

Consider solving the initial boundary value problem of second-order parabolic equation:

$$
(\mathrm{P})\qquad \begin{cases} \dfrac{\partial u}{\partial t}+Lu=f, \quad (x, t)\in \mathscr{D}, & (1)\\ u|_{t=0}=u_0(x), \quad x\in D_0, & (2)\\ u|_{\partial D_t}=0, \quad x\in \partial D_t,\ 0\leqslant t\leqslant T, & (3)\end{cases}
$$

where $\mathscr{D}=\{(x, t)\,|\,x\in D_t,\ 0\leqslant t\leqslant T\}$ is a bounded simply connected domain in $r+1$

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dimensional space, D_t is a bounded simply connected domain in r-dimensional space dependent on time, ∂D_t is the contour of D_t in r-dimensional space, L is a self-adjoint differential operator of second order.

To solve problem (P) using moving finite elements, suppose at any moment $t\in[0, T]$ there is a one to one mapping

$$x=X(y, t) \tag{4}$$

to map the simply connected bounded domain Ω in y-space into a simply connected bounded domain D_t in space x, so that the cylinder domain $\Omega\times[0, T]$ in (y, t)-space can be mapped into the bounded domain $\mathscr{D}$ in (x, t)-space.

Then subdivide Ω into finite elements, construct finite element basis functions $\psi_1(y)$, $\psi_2(y)$, $\cdots$, $\psi_m(y)$, and denote by $S_h(\Omega)$ the m-dimensional linear space expanded from these basis functions. At any moment $t\in[0, T]$, the finite element subdivision of Ω, the basis functions $\{\psi_i(y)\}_{i=1}^{i=m}$ and m-dimensional linear space $S_h(\Omega)$ are transformed into the finite element subdivision of D_t, the basis functions $\{\varphi_i(x, t)\}_{i=1}^{i=m}$ and the m-dimensional linear space $S_h(D_t)$ respectively, where the basis functions in the two spaces satisfy the following relation:

$$\varphi_i(X(y, t), t)=\psi_i(y), \quad i=1, 2, \cdots, m. \tag{5}$$

Differentiate this equation by t

$$\frac{\partial\varphi_i}{\partial x}\frac{\partial X}{\partial t}+\frac{\partial\varphi_i}{\partial t}=\frac{d\psi_i}{dt},$$

where $\frac{d}{dt}$ denotes differentiation for t under (y, t)-space coordinate and $\frac{\partial}{\partial t}$ denotes differentiation for t under the (x, t)-space coordinate,

$$\frac{\partial\varphi_i}{\partial x}=\left(\frac{\partial\varphi_i}{\partial x_1}, \frac{\partial\varphi_i}{\partial x_2}, \cdots, \frac{\partial\varphi_i}{\partial x_r}\right),$$

$$\frac{\partial X}{\partial t}=\left(\frac{\partial X_1}{\partial t}, \frac{\partial X_2}{\partial t}, \cdots, \frac{\partial X_r}{\partial t}\right)^\tau,$$

where τ denotes the transpose of a vector or a matrix. Because $\psi_i(y)$ (in (y, t)-space coordinate) is independent of t, so

$$\frac{\partial\varphi_i}{\partial t}+\frac{\partial\varphi_i}{\partial x}\frac{\partial X}{\partial t}=0, \quad i=1, 2, \cdots, m, \tag{6)'$$

or, in matrix form,

$$\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial X}\cdot\frac{\partial X}{\partial t}=0, \tag{6)''$$

where $\varphi=(\varphi_1, \varphi_2, \cdots, \varphi_m)^\tau$,

$$\frac{\partial\varphi}{\partial x}=\begin{pmatrix}\frac{\partial\varphi_1}{\partial x_1}, & \frac{\partial\varphi_1}{\partial x_2}, & \cdots, & \frac{\partial\varphi_1}{\partial x_r}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial\varphi_m}{\partial x_1}, & \frac{\partial\varphi_m}{\partial x_2}, & \cdots, & \frac{\partial\varphi_m}{\partial x_r}\end{pmatrix}. \tag{7}$$

Take the finite element approximate solution

$$u^h(x, t)=\varphi^\tau(x, t)U(t)=\sum_{i=1}^{m}U_i\varphi_i(x, t). \tag{8}$$

Apply the Galerkin method to find $u^h(x, t)\in S_h(D_t)$ that satisfies the virtual work equations

$$\left(\frac{\partial u^h(x,t)}{\partial t}, \varphi_i\right)_{D_t}+(Lu^h(x,t), \varphi_i)_{D_t}=(f, \varphi_i)_{D_t}, \quad i=1, 2, \cdots, m, \ 0\leqslant t\leqslant T. \tag{9}$$

On the other hand, from (8) and (6)″ we have

$$\frac{\partial u^h(x,t)}{\partial t}=\varphi^\tau(x,t)\frac{\partial U}{\partial t}+\frac{\partial \varphi^\tau(x,t)}{\partial t}U=\varphi^\tau(x,t)\frac{\partial U}{\partial t}-\left(\frac{\partial X}{\partial t}\right)^\tau\frac{\partial \varphi^\tau}{\partial x}U,$$

$$(Lu^h, \varphi_i)_{D_t}=(L\varphi^\tau U, \varphi_i)_{D_t}=(L\varphi^\tau, \varphi_i)_{D_t}U.$$

Put it into (9) to get

$$(\varphi^\tau, \varphi_i)_{D_t}\frac{\partial U}{\partial t}-\left(\frac{\partial X^\tau}{\partial t}\frac{\partial \varphi^\tau}{\partial x}, \varphi_i\right)_{D_t}U+(L\varphi^\tau, \varphi_i)_{D_t}U=(f, \varphi_i)_{D_t}, \quad i=1, 2, \cdots, m. \tag{10}$$

Choose

$$u^h(x, 0)=\varphi^\tau(x, 0)U(0)=R_0u_0(x), \quad x\in D_0, \tag{11}$$

where R_0 is the Riesz projection of elliptic operator L (when $t=0$) from $\mathring{H}_1(D_0)$ to $S_h(D_0)$, that is $R_0u_0(x)\in S_h(D_0)$ satisfies the equation

$$(LR_0u_0(x), \varphi_i)_{D_0}=(Lu_0(x), \varphi_i)_{D_0}, \quad i=1, 2, \cdots, m. \tag{12}$$

From this, we can get

$$(L\varphi^\tau(x, 0), \varphi_i)_{D_0}U(0)=(Lu_0(x), \varphi_i)_{D_0}, \quad i=1, 2, \cdots, m. \tag{13}$$

From (10) and (13), we get an ordinary differential equation system and its initial condition, which should be satisfied by $U(t)$, as follows:

$$M\frac{\partial U}{\partial t}+(K-Q)U=F, \tag{14}$$

$$U(0)=U_0, \tag{15}$$

where the elements of the $m\times m$ matrices M, K and Q are calculated as:

$$M_{ij}(t)=(\varphi_i, \varphi_j)_{D_t}, \tag{16}$$

$$Q_{ij}(t)=\left(\varphi_i, \frac{\partial \varphi_j}{\partial x}\cdot\frac{\partial X}{\partial t}\right)_{D_t}, \tag{17}$$

$$K_{ij}(t)=(\varphi_i, L\varphi_j)_{D_t}, \tag{18}$$

m-dimensional vectors U_0 and F are calculated as follows:

$$U_0=K^{-1}(0)(Lu_0(x), \varphi)_{D_0}, \tag{19}$$

$$F=(f, \varphi)_{D_t}. \tag{20}$$

This is a first-order ordinary differential equation system, whose coefficient, by (16)—(20), is completely determined by the basis functions $\{\psi_i(y)\}_{i=1}^{i=m}$ and mapping $X(y, t)$ (suppose L, $u_0(x)$ and $f(x, t)$ are known). $\psi_i(y)$ is determined by the finite element subdivision and the interpolating function, and there are already many results concerning its choice. $X(y, t)$ is determined by moving grid and this will be discussed in Section 5. In next section we will discuss the estimation of the error between the finite element approximate solution and the true solution and in the last section we will discuss the finite difference method for the ordinary equation system and its stability.

§ 3. Error Estimation

The purpose of this section is to prove the following result if the variable coefficient differential operator L is continuously differentiable and satisfies the uniform elliptic condition, mapping $X(y, t)$ is smooth enough and the conforming element is adopted (i.e. $S_h(\Omega)\subset \mathring{H}_1(\Omega)$), then the finite element approximate solution has an accuracy with optimal order.

Obviously, the solution of (9) and (11) is identical with the solution of the following Galerkin solution:

$$\left(\frac{\partial u^h}{\partial t}, v^h\right)_{D_t}+(Lu^h, v^h)_{D_t}=(f, v^h)_{D_t}, \quad \forall v\in S_h(D_t),\ 0\leqslant t\leqslant T, \tag{21}$$

$$u^h|_{t=0}=R_0u_0(x). \tag{22}$$

So we only have to discuss the estimate of error between the approximate solution from (21), (22) and the true solution of problem (P).

For this we suppose (H): The differential operator $L=\sum\limits_{i,j=1}^{r}\frac{\partial}{\partial x_i}a_{ij}\frac{\partial}{\partial x_j}$ satisfies the uniform elliptic condition

$$\sum_{i,j=1}^{r}a_{ij}\xi_i\xi_j\geqslant\alpha|\xi|^2, \quad \forall(x, t)\in\mathscr{D},\ \forall\xi\in R^r, \tag{23}$$

where α is a positive constant independent of (x, t) and ξ,

$$a_{ij}(x, t)\in C_2(\mathscr{D}), \quad X(y, t)\in C_2(\Omega),$$

$$\frac{\partial X}{\partial t}\in C_2(\Omega), \quad 0\leqslant t\leqslant T, \tag{24}$$

$$J^{-1} \text{ has a uniform bound on } \Omega\times[0, T], \tag{25}$$

where $J=\det\left(\frac{\partial X}{\partial y}\right)$ is the determinant of transformation $x=X(y, t)$,

$$\text{the subdivision of } \Omega \text{ is regular.} \tag{26}$$

For the sake of convenience, we denote by c the constants only dependent on Ω, α, $|a_{ij}|_{C_2(\mathscr{D})}$, $\max\limits_{0<t<T}|X|_{C_2(\Omega)}$, $\max\limits_{0<t<T}\left|\frac{\partial X}{\partial t}\right|_{C_2(\Omega)}$ and the upper bound of J^{-1} but independent of h and t. To prove the main theorem of error estimate of this section, we need the following two lemmas:

Lemma 1. *If $S_h(\Omega)\subset\mathring{H}_1(\Omega)$, mapping $X(y, t)\in C_1(\Omega\times[0, T])$, then the solution $u(x, t)$ of problem* (P) *also satisfies the virtual equation* (21).

The proof is obvious.

Lemma 2. *If $S_h(\Omega)\subset\mathring{H}_1(\Omega)$, R_t is the Riesz projective operator from $\mathring{H}_1(D_t)$ to $S_h(D_t)$ about the elliptic differential operator L (corresponding to time t), that is, for any $u(x, t)\in\mathring{H}_1(D_t)$, R_tu is the solution of the following Galerkin equation:*

$$(LR_tu, v^h)_{D_t}=(Lu, v^h)_{D_t}, \quad \forall v^h\in S^h(D_t). \tag{27}$$

If the supposition (H) *is satisfied, there is the estimate*

$$\left|\left(\frac{\partial e}{\partial t}, v^h\right)_{D_t}\right| \leqslant C\left\{\|e\|_{0,D_t} + h\|e\|_{1,D_t} + \left\|(I-R_t)\frac{du}{dt}\right\|_{0,D_t} + h\left\|(I-R_t)\frac{du}{dt}\right\|_{1,D_t}\right\}\cdot\|v^h\|_{1,D_t}, \tag{28}$$

where I is an identity operator and $e=(I-R_t)u$.

Proof. From (27)

$$(L(u-R_tu), v^h)_{D_t}=0, \quad \forall v^h\in S_h(D_t). \tag{29}$$

Under y-coordinate (29) can be written as

$$\left(\frac{\partial(u-R_tu)}{\partial y}, B\frac{\partial v^h}{\partial y}\right)_\Omega=0, \quad \forall v^h\in S_h(\Omega), \tag{30}$$

where $B=J\left(\frac{\partial y}{\partial x}\right)A\left(\frac{\partial y}{\partial x}\right)^\tau$, the component of the m-dimensional matrix A is a_{ij}, the component of $\frac{\partial y}{\partial x}$ is $\frac{\partial y_i}{\partial x_j}$, $J=\det\left(\frac{\partial x}{\partial y}\right)$.

From (23)—(25), B satisfies the uniform elliptic condition

$$\xi^\tau B\xi\geqslant\alpha'|\xi|^2, \quad \forall\xi\in R^r \text{ and } (y, t)\in\Omega\times[0, T], \tag{31}$$

where α' is a positive constant independent of y, t and ξ.

Under the coordinate in y-space, differentiate (30) on t, and we get

$$\left(\frac{\partial}{\partial y}\frac{d}{dt}(u-R_tu), B\frac{\partial v^h}{\partial y}\right)_\Omega=-\left(\frac{\partial e}{\partial y}, \frac{dB}{dt}\frac{\partial v^h}{\partial y}\right)_\Omega. \tag{32}$$

Because

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{\partial X}{\partial t},$$

$\frac{du}{dt}$ exists obviously. The existence of $\frac{d}{dt}(R_tu)$ can be got through the limit of difference quotient sequence and from this comes

$$\frac{d}{dt}(R_tu)\in S_h(\Omega). \tag{33}$$

Let $w\in\mathring{H}_1(\Omega)$ and $\varphi\in\mathring{H}_1(\Omega)$ be the solutions of the following elliptic equations respectively

$$\left(\frac{\partial w}{\partial y}, B\frac{\partial v}{\partial y}\right)_\Omega=-\left(\frac{\partial e}{\partial y}, \frac{dB}{dt}\frac{\partial v}{\partial y}\right)_\Omega, \quad \forall v\in\mathring{H}_1(\Omega) \tag{34'}$$

and

$$\left(\frac{\partial\varphi}{\partial y}, B\frac{\partial v}{\partial y}\right)_\Omega=(w, v)_\Omega, \quad \forall v\in\mathring{H}_1(\Omega). \tag{34''}$$

Then it can easily be proved from (31) and (24), (25) that

$$\|\varphi\|_{2,\Omega}\leqslant c_1\|w\|_{0,\Omega},$$

$$\|w\|^2_{0,\Omega}=\left(\frac{\partial\varphi}{\partial y}, B\frac{\partial w}{\partial y}\right)_\Omega=-\left(\frac{\partial e}{\partial y}, \frac{dB}{dt}\frac{\partial\varphi}{\partial y}\right)_\Omega\leqslant c_2\|e\|_{0,\Omega}\|\varphi\|_{2,\Omega},$$

where c_1 and c_2 are constants independent of t and h. Thus

$$\|w\|_{0,\Omega}\leqslant c\|e\|_{0,\Omega},$$
$$\|w\|_{1,\Omega}\leqslant c\|e\|_{1,\Omega}, \tag{35}$$

where c is a constant independent of t and h.

(32) minus (34) (take $v=v^h$) yields

$$\left(\frac{\partial}{\partial y}\left[\frac{du}{dt}-w-\frac{d}{dt}(R_t u)\right],\ B\frac{\partial v^h}{\partial y}\right)_\Omega=0.$$

Using the Nitsche technique, from (33) and (35) we get the L_2-estimate

$$\left\|\frac{du}{dt}-w-\frac{d(R_t u)}{dt}\right\|_{0,\Omega}\leqslant c\left\{\left\|\frac{du}{dt}-w-R_t\left(\frac{du}{dt}-w\right)\right\|_{0,\Omega}\right.$$
$$\left.+h\left\|\frac{du}{dt}-w-R_t\left(\frac{du}{dt}-w\right)\right\|_{1,\Omega}\right\}, \tag{36}$$

where c is a constant independent of t and h.

Then from (35) and (36), we get

$$\left\|\frac{d(u-R_t u)}{dt}\right\|_{0,\Omega}\leqslant\left\|\frac{du}{dt}-w-\frac{d(R_t u)}{dt}\right\|_{0,\Omega}+\|w\|_{0,\Omega}$$
$$\leqslant c\left\{\left\|(I-R_t)\frac{du}{dt}\right\|_{0,\Omega}+h\left\|(I-R_t)\frac{du}{dt}\right\|_{1,\Omega}\right.$$
$$\left.+\|e\|_{0,\Omega}+h\|e\|_{1,\Omega}\right\}.$$

From (24), (25) and the above formula,

$$\left\|\frac{d(u-R_t u)}{dt}\right\|_{0,D_t}\leqslant c\left\{\left\|(I-R_t)\frac{du}{dt}\right\|_{0,D_t}+h\left\|(I-R_t)\frac{du}{dt}\right\|_{1,D_t}\right.$$
$$\left.+\|e\|_{0,D_t}+h\|e\|_{1,D_t}\right\}. \tag{37}$$

On the other hand, as

$$\frac{\partial e}{\partial t}=\frac{de}{dt}-\frac{\partial e}{\partial x}\,\frac{\partial X}{\partial t},$$

so

$$\left(\frac{\partial e}{\partial t},\ v^h\right)_{D_t}=\left(\frac{de}{dt},\ v^h\right)_{D_t}-\left(\frac{\partial e}{\partial x}\,\frac{\partial X}{\partial t},\ v^h\right)_{D_t}. \tag{38}$$

Putting

$$\left(\frac{\partial e}{\partial x}\cdot\frac{\partial X}{\partial t},\ v^h\right)_{D_t}=-\left(e,\ \frac{\partial v^h}{\partial x}\cdot\frac{\partial X}{\partial t}\right)_{D_t}-\left(e,\ v^h\frac{\partial}{\partial x}\,\frac{\partial X}{\partial t}\right)_{D_t}$$

into (38) gives

$$\left(\frac{\partial e}{\partial t},\ v^h\right)_{D_t}=\left(\frac{de}{dt},\ v^h\right)_{D_t}+\left(e,\ \frac{\partial v^h}{\partial x}\cdot\frac{\partial X}{\partial t}\right)_{D_t}+\left(e,\ v^h\frac{\partial}{\partial x}\cdot\frac{\partial X}{\partial t}\right)_{D_t}, \tag{39}$$

where $\frac{\partial}{\partial x}\,\frac{\partial X}{\partial t}=\sum\limits_{i=1}^{n}\frac{\partial}{\partial x_i}\,\frac{\partial X_i}{\partial t}$. Because from (24) and (25) we know $\frac{\partial X}{\partial t}$ and $\frac{\partial}{\partial x}\,\frac{\partial X}{\partial t}$ are bounded uniformly, so from (37) and (39), (28) can be got.

Theorem 1. *Suppose $S_h(\Omega)\subset\overset{\circ}{H}_1(\Omega)$ is a finite element space with an accuracy of order k satisfying all the conditions of supposition* (H); *besides, if for any $t\in[0,\ T]$, $X\in C_k(\Omega)$, $\frac{\partial X}{\partial t}\in C_k(\Omega)\,(k\geqslant 2)$, the true solution $u\in H_{k+1}(D_t)$, $\frac{\partial u}{\partial t}\in H_k(D_t)$.*

Then the error between the approximate solution $u^h(x,\ t)$ and the true solution $u(x,\ t)$ of problem (P) *in L_2-norm is*

$$\max_{0\leqslant t\leqslant T}\|u-u^h\|_{0,D_t}^2\leqslant c\left\{\max_{0\leqslant t\leqslant T}\|u\|_{k,D_t}^2+\int_0^T\left(\|u\|_{k+1,D_t}^2+\left\|\frac{\partial u}{\partial t}\right\|_{k,D_t}^2\right)dt\right\}h^{2k}, \tag{40}$$

where c is a constant independent of h.

Proof. From (21) and Lamma 1 there is

$$\left(\frac{\partial(u^h-u)}{\partial t}, v^h\right)_{D_t}+(L(u^h-u), v^h)_{D_t}=0, \quad \forall v^h\in S_h(D_t).$$

Then

$$\left(\frac{\partial(u^h-R_tu)}{\partial t}, v^h\right)_{D_t}+(L(u^h-R_tu), v^h)_{D_t}=\left(\frac{\partial e}{\partial t}, v^h\right)_{D_t}, \tag{41}$$

where $e=u-R_tu$. Let $v^h=u^h-R_tu$ and put it into the above equality. We then get

$$\left(\frac{\partial v^h}{\partial t}, v^h\right)_{D_t}+(Lv^h, v^h)_{D_t}=\left(\frac{\partial e}{\partial t}, v^h\right)_{D_t}; \tag{42}$$

yet

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(v^h, v^h)_{D_t}&=\frac{1}{2}\frac{d}{dt}(v^h, v^hJ)_\Omega=\left(\frac{dv^h}{dt}, v^hJ\right)_\Omega+\frac{1}{2}\left(v^h, v^h\frac{dJ}{dt}\right)_\Omega\\&=\left(\frac{\partial v^h}{\partial t}, v^h\right)_{D_t}+\left(\frac{\partial v^h}{\partial x}\cdot\frac{\partial X}{\partial t}, v^h\right)_{D_t}+\frac{1}{2}\left(v^h, v^h\frac{dJ}{dt}J^{-1}\right)_{D_t}\\&=\left(\frac{\partial v^h}{\partial t}, v^h\right)_{D_t}-\frac{1}{2}\left(v^h, v^h\frac{\partial}{\partial x}\frac{\partial X}{\partial t}\right)+\frac{1}{2}\left(v^h, v^h\frac{dJ}{dt}\cdot J^{-1}\right)_{D_t}.\end{aligned}$$

From (24), (25) we known that $\frac{\partial}{\partial x}\frac{\partial X}{\partial t}$, $\frac{dJ}{dt}$ and J^{-1} are bounded uniformly, so substituting that into (42) we get the following inequality

$$\frac{1}{2}\frac{d}{dt}\{(v^h, v^h)_{D_t}\}+(Lv^h, v^h)_{D_t}\leqslant\left(\frac{\partial e}{\partial t}, v^h\right)_{D_t}+c(v^h, v^h)_{D_t},$$

where c is a constant independent of t and h. From Lemma 2, the elliptic condition (23) and the above inequality, the error estimate of differential inequality can be got

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{(v^h, v^h)_{D_t}\}\leqslant c\Big\{&\|e\|^2_{0,D_t}+h^2\|e\|^2_{1,D_t}+\left\|(I-R_t)\frac{du}{dt}\right\|^2_{0,D_t}\\&+h^2\left\|(I-R_t)\frac{du}{dt}\right\|^2_{1,D_t}+(v^h, v^h)_{D_t}\Big\}.\end{aligned}$$

Thus from the Gronwall inequality, for any $t\in[0, T]$ there is

$$\begin{aligned}(v^h, v^h)_{D_t}\leqslant e^{CT}(v^h, v^h)_{D_0}+Ce^{CT}\int_0^T\Big\{&\|e\|^2_{0,D_t}+h^2\|e\|^2_{1,D_t}\\&+\left\|(I-R_t)\frac{du}{dt}\right\|^2_{0,D_t}+h^2\left\|(I-R_t)\frac{du}{dt}\right\|^2_{1,D_t}\Big\}dt.\end{aligned} \tag{43}$$

As $v^h=0$ on D_0, so the first term on the right of the above inequality $(v^h, v^h)_{D_0}=0$. From (43) and the triangle inequality

$$\|u-u^h\|_{0,D_t}\leqslant\|u-R_tu\|_{0,D_t}+\|R_tu-u^h\|_{0,D_t}$$

we have

$$\begin{aligned}\|u-u^h\|^2_{0,D_t}\leqslant 2\|u-R_tu\|^2_{0,D_t}+2Ce^{CT}\int_0^T\Big\{&\|e\|^2_{0,D_t}+h^2\|e\|^2_{1,D_t}\\&+\left\|(I-R_t)\frac{du}{dt}\right\|^2_{0,D_t}+h^2\left\|(I-R_t)\frac{du}{dt}\right\|^2_{1,D_t}\Big\}dt.\end{aligned} \tag{44}$$

Thus from $\frac{du}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\cdot\frac{\partial X}{\partial t}$ and the theory of Sobolev space interpolation the estimate (40) can be got.

§ 4. Fully Discrete Scheme

This section discusses the finite difference solution of the initial value problem of the ordinary equation (14)—(15) (obtained in solving parabolic equation problems (1)—(3) by the finite element method in Section 2). By means of the Crank-Nicolson scheme we get the following system of difference equations

$$\begin{cases} M_{n+\frac{1}{2}}(U_{n+1}-U_n)+(K-Q)_{n+\frac{1}{2}}U_{n+\frac{1}{2}}\Delta t_n=F_{n+\frac{1}{2}}\Delta t_n, \quad 0\leqslant n\leqslant N-1, & (45)\\ U_0=U(0), & (46)\end{cases}$$

where $\Delta t_n=t_{n+1}-t_n$, $t_0, t_1, \cdots, t_N$ are the discrete points of the axis of time, $t_0=0$, $t_N=T$. $(\cdot)_n$ denotes the value of the function $(\cdot)$ at the moment t_n,

$$(\cdot)_{n+\frac{1}{2}}=[(\cdot)_{n+1}+(\cdot)_n]/2.$$

Lemma 3. *If the positive sequence $\{z_n\}_{n=0}^{n=N}$ satisfies the inequality*

$$z_n(1-\beta_2\Delta t)\leqslant z_{n-1}(1+\beta_1\Delta t), \quad n=1, 2, \cdots, N, \tag{47}$$

where $\Delta t=\frac{T}{N}$, β_1 and β_2 are both positive constants and satisfy the inequality

$$\beta_2\Delta t<\frac{1}{2}, \quad \beta_1\Delta t<1, \tag{48}$$

then, z_n $(n=1, 2, \cdots, N)$ satisfies the inequality

$$z_n\leqslant e^{\beta_1 T}4^{\beta_2 T}z_0. \tag{49}$$

Proof. From (47) there is

$$z_n\leqslant\frac{1+\beta_1\Delta t}{1-\beta_2\Delta t}z_{n-1}\leqslant\left(\frac{1+\beta_1\Delta t}{1-\beta_2\Delta t}\right)^n z_0,$$

$$(1+\beta_1\Delta t)^n=(1+\beta_1\Delta t)^{\frac{1}{\beta_1\Delta t}\cdot n\beta_1\Delta t}\leqslant e^{\beta_1 T},$$

$$(1-\beta_2\Delta t)^{-n}=(1-\beta_2\Delta t)^{-\frac{1}{\beta_2\Delta t}\cdot n\beta_2\Delta t}\leqslant e^{\beta_2 T}.$$

So (49) is got.

If there exists a positive constant β independent of N. such that

$$\Delta t_n\leqslant\beta\Delta t, \quad 0\leqslant n\leqslant N-1, \tag{50}$$

where $\Delta t=\frac{T}{N}$, then it can be proved that when Δt is small enough, the difference scheme (45) is unconditionally stable:

Theorem 2. *If Δt_n satisfies (50), Δt is small enough, when the right-hand term of the parabolic equation (1) $f\equiv 0$ (i.e. the right-hand term of the ordinary differential equation system (14) $F\equiv 0$), the solution U_n of the finite difference scheme (45) has the estimate*

$$U_n^T M_n U_n\leqslant cU_0^T M_0 U_0, \quad 0\leqslant n\leqslant N, \tag{51}$$

where c is a constant independent of h and N.

Proof. Multiplying both sides of the difference scheme (45) with $(U_n+U_{n+1})^\tau$ and rearranging yield (suppose $F\equiv 0$)

$$U^\tau_{n+1}M_{n+1}U_{n+1}-U^\tau_n M_n U_n-\frac{1}{2}U^\tau_{n+1}(M_{n+1}-M_n)U_{n+1}$$
$$-\frac{1}{2}U^\tau_n(M_{n+1}-M_n)U_n+\frac{1}{2}U^\tau_{n+\frac{1}{2}}K_{n+\frac{1}{2}}U_{n+\frac{1}{2}}\Delta t_n$$
$$-\frac{1}{2}U^\tau_{n+\frac{1}{2}}Q_{n+\frac{1}{2}}U_{n+\frac{1}{2}}\Delta t_n=0. \tag{52}$$

For any $v^h\in S_h(D_t)$, there is

$$\left|\left(\frac{\partial v^h}{\partial x}\cdot\frac{\partial X}{\partial t},\ v^h\right)_{D_t}\right|=\frac{1}{2}\left|\left(v^h\frac{\partial}{\partial x}\frac{\partial X}{\partial t},\ v^h\right)_{D_t}\right|\leqslant c(v^h,\ v^h)_{D_t}, \tag{53}$$

where c is a constant independent of h and t. From the above equality and the definitions of M and Q (see (16) and (17)) we have

$$-U^\tau_{n+\frac{1}{2}}Q_{n+\frac{1}{2}}U_{n+\frac{1}{2}}\leqslant\beta'(U^\tau_{n+1}M_{n+1}U_{n+1}+U^\tau_n M_n U_n), \tag{54}$$

where β' is a positive constant independent of h and N.

From definition (16) of $M(t)$, for any $v^h=\varphi^\tau V\in S_h(\Omega)$ there is

$$V^\tau(M_{n+1}-M_n)V=(v^h,\ v^h)_{D_{t_{n+1}}}-(v^h,\ v^h)_{D_{t_n}}=(v^h,\ v^h(J_{n+1}-J_n))_\Omega. \tag{55}$$

From (24) J is continuously differentiable for t and from (25) J^{-1} is uniformly bounded. So there are

$$|J_{n+1}-J_n|\leqslant\beta''\Delta t_n J_n \tag{56}$$

and

$$|J_{n+1}-J_n|\leqslant\beta''\Delta t_n J_{n+1}, \tag{57}$$

where β'' is a positive constant independent of n, N and h.

From (55), (56) and (57), we get the estimate.

$$U^\tau_{n+1}(M_{n+1}-M_n)U_{n+1}\leqslant\beta''U^\tau_{n+1}M_{n+1}U_{n+1}\Delta t_n, \tag{58}$$

$$U^\tau_n(M_{n+1}-M_n)U_n\leqslant\beta''U^\tau_n M_n U_n\Delta t_n. \tag{59}$$

And from (18) and the elliptic condition (23) of L, we get

$$U^\tau_{n+\frac{1}{2}}K_{n+\frac{1}{2}}U_{n+\frac{1}{2}}\geqslant 0. \tag{60}$$

Substitute (54), (58), (59) and (60) into (52), and we have

$$(1-\bar\beta\Delta t_n)U^\tau_{n+1}M_{n+1}U_{n+1}\leqslant(1+\bar\beta\Delta t_n)U^\tau_n M_n U_n,\quad 0\leqslant n\leqslant N-1, \tag{61}$$

where $\bar\beta=\frac{1}{2}(\beta'+\beta'')$ is independent of N and h. If Δt is small enough such that $\bar\beta\beta\Delta t\leqslant\frac{1}{2}$, then from (50) and (61), (51) can be got by Lemma 2, where c is a constant independent of N and h.

We can further prove: The finite element difference scheme (45) not only is unconditionally stable but also has an optimal order error estimate in L_2 norm.

§ 5. Determination of Mapping $X(y, t)$

Up to now, the whole calculation of the finite element is awaiting the determination of the mapping $X(y, t)$. But the choice of $X(y, t)$ is a complex problem. It is connected with the physical problem itself, and practical experience plays an important role in a good choice of mapping $X(y, t)$.

This section only discusses how to choose the mapping $X(y, t)$ after the complete determination of the finite element nodes at any time.

We first take the one-dimensional case. At this moment as the mapping $X(y, t)$ has been completely defined at nodes, only the expression of $X(y, t)$ in the domain between the two nodes has to be given. Suppose

$$x_i = X(y_i, t), \quad i=1, 2 \tag{62}$$

is the transformation expression of the nodes, where $x_1 < x_2$, $y_1 < y_2$. x and y in the domain between (x_1, x_2) and (y_1, y_2) are expressed by parameter λ as

$$\begin{cases} y = (1-\lambda)y_1 + \lambda y_2, \\ x = (1-\lambda)x_1 + \lambda x_2. \end{cases} \tag{63}$$

Then we establish from (63) one-to-one correspondence relation in the domain between $x_1 \leqslant x \leqslant x_2$ and $y_1 \leqslant y \leqslant y_2$. If (62) is known, the transformation $X(y, t)$ and its derivatives with any order in domain $[y_1, y_2]$ can be completely defined from (63); thus the calculation may proceed.

In general, the space variable is of r $(r>1)$ dimensions, and the element is simplex. If at all vertexes transformation

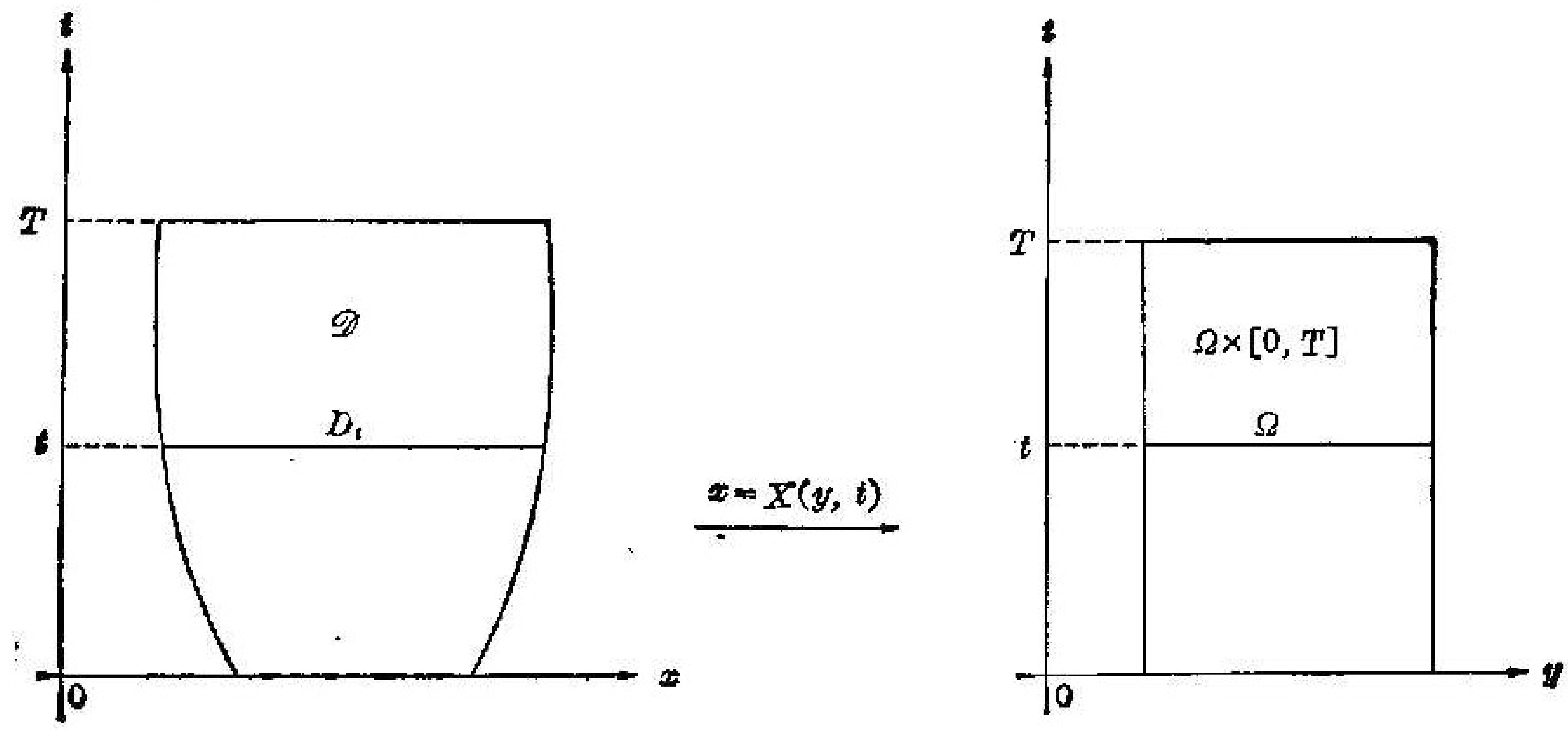
$$x_i = X(y_i, t), \quad i=1, 2, \cdots, r+1 \tag{64}$$

is given, then x and y in the simplex can be expressed by the area coordinates respectively

$$\begin{cases} y = \sum_{i=1}^{r+1} \lambda_i y_i, \\ x = \sum_{i=1}^{r+1} \lambda_i x_i, \quad \sum_{i=1}^{r+1} \lambda_i = 1, \quad \lambda_i \geqslant 0. \end{cases} \tag{65}$$

The one-to-one correspondence of the mapping $x = X(y, t)$ between simplexes with vertexes $\{x_i\}_{i=1}^{i=r+1}$ and $\{y_i\}_{i=1}^{i=r+1}$ respectively can be built up from (65); thus, the mapping $X(y, t)$ is completely defined and the calculation can go on.

Obviously for elements of other shape if the transformation $x_i = X(y_i, t)$ of all nodes is known, this finite element parametric representation can also be used to build the one-to-one correspondence relation of mapping $X(y, t)$ in elements. Thus $X(y, t)$ is completely defined.



§ 6. Concluding Remarks

The finite element method with moving grid and its error estimation presented here are only confined on the special parabolic equation and the first kind homogeneous boundary value condition merely for the convenience and the distinctness of the exposition and the whole analysis is still effective for general parabolic equations and nonhomogeneous boundary conditions of other kinds.

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