A FINITE ELEMENT APPROXIMATION OF NAVIER-STOKES EQUATIONS USING NONCONFORMING ELEMENTS*

HAN HOU-DE (韩厚德)

(Department of Mathematics, Peking University, Beijing, China)

§1. Introduction

In this paper, steady incompressible flow of viscous fluids is considered. The finite element approximation of this problem has been treated by some authores[1-3]. By means of a primitive variable formulation, the numerical treatment of Navier-Stokes equations naturally leads to the mixed finite element method, in which the Babuska-Brezzi condition is required for the conforming finite element method. It means that the finite dimensional subspace of the velocity field and the subspace of pressure must satisfy a certain matchable relationship. For example, in the two dimensional case, triangular elements are used; the subspace of the velocity field is formed by piecewise linear functions and the subspace of pressure is formed by piecewise constant functions for a conforming finite element method. It is straightforward to show that they do not satisfy Babuska-Brezzi condition. If the subspace of the velocity field is formed by piecewise quadratic functions instead of piecewise linear functions, then they satisfy the Babuska-Brezzi condition. However, a loss of precision is also incurred. The optimal error estimate cannot be obtained in this situation. M. Crouzeix and P. A. Raviart proposed to use the nonconforming triangular elements to form the approximation space of velocity field to solve stokes equations. A few years later, the nonconforming triangular elements were applied to stationary Navier-Stokes equations by R. Temam[1] and to nonstationary Navier-Stokes equations by R. Rannacher[5]. In those cases the optimal error estimate can be obtained, and therefore using nonconforming elements may be a good choice for the finite element approximation of Navier-Stokes equations. Recently a class of nonconforming rectangular elements were used for the numerical analysis of stokes equations and an optimal error estimate was given[6]. The aim of this paper is to analyse the error estimate of a finite element approximation of Navier-Stokes equations using general nonconforming elements including nonconforming rectangular elements.

Let Ω be a bounded domain of $\mathbb{R}^n (n=2, 3)$ with a Lipschitz continuous boundary $\partial\Omega$. Let $W^{m,p}(\Omega)$ denote the Sobolev space on Ω with norm $\|\cdot\|_{m,p,\Omega}$. As usual, when $p=2$, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$; when $m=0$, $W^{0,p}(\Omega)$ is denoted by $L_p(\Omega)$. Moreover, let $H_0^1(\Omega)=\{u\in H^1(\Omega),\ u=0 \text{ on } \partial\Omega\}$, $X=(H_0^1(\Omega))^n$ with norm $\|\cdot\|_X=|\cdot|_{1,2,\Omega}$, and $M=\left\{\lambda\in L_2(\Omega), \int_\Omega \lambda\, dx=0\right\}$ with norm $\|\cdot\|_M=\|\cdot\|_{0,2,\Omega}$.

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We consider the following boundary value problem of Navier-Stokes equations:

$$-\nu\Delta \boldsymbol{u}+\sum_{j=1}^{n} u_j \frac{\partial \boldsymbol{u}}{\partial x_j}+\operatorname{grad}\lambda=\boldsymbol{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u}=0, \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}=0, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\boldsymbol{u}=(u_1, \cdots, u_n)$ is the velocity vector, λ is the pressure, and ν is a positive constant, the coefficient of kinematic viscosity. Eq. (1.1) can be rewritten as

$$-\nu\Delta \boldsymbol{u}+\frac{1}{2}\sum_{j=1}^{n} u_j \frac{\partial \boldsymbol{u}}{\partial x_j}+\frac{1}{2}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(u_j\boldsymbol{u})+\operatorname{grad}\lambda=\boldsymbol{f}, \quad \text{in } \Omega. \tag{1.1$'$}$$

Then the boundary value problem (1.1)′, (1.2), (1.3) is equivalent to the following variational problem:

Find $(\boldsymbol{u}, \lambda)\in X\times M$, such that

$$a_0(\boldsymbol{u}, \boldsymbol{v})+a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, \lambda)=\langle \boldsymbol{f}, \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v}\in X, \tag{1.4}$$

$$b(\boldsymbol{u}, \mu)=0, \quad \forall \mu\in M, \tag{1.5}$$

where

$$a_0(\boldsymbol{u}, \boldsymbol{v})=\nu\sum_{i,j=1}^{n}\int_{\Omega}\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,dx, \tag{1.6}$$

$$a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})=\frac{1}{2}\sum_{i,j=1}^{n}\int_{\Omega} w_j\left(\frac{\partial u_i}{\partial x_j}v_i-\frac{\partial v_i}{\partial x_j}u_i\right)dx, \tag{1.7}$$

$$b(\boldsymbol{v}, \lambda)=-\int_{\Omega}\lambda\operatorname{div}\boldsymbol{v}\,dx, \tag{1.8}$$

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle=\sum_{i=1}^{n}\int_{\Omega} f_i v_i\,dx. \tag{1.9}$$

Obviously,

$$a_1(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v})=0, \quad \forall \boldsymbol{w}, \boldsymbol{v}\in X. \tag{1.10}$$

Let $V=\{\boldsymbol{v}\in X, \operatorname{div}\boldsymbol{v}=0\}$, and

$$N=\sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V}\frac{|a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{w}\|_X\|\boldsymbol{u}\|_X\|\boldsymbol{v}\|_X}, \tag{1.11}$$

$$\|\boldsymbol{f}\|^*=\sup_{\boldsymbol{v}\in V}\frac{|\langle \boldsymbol{f}, \boldsymbol{v}\rangle|}{\|\boldsymbol{v}\|_X}. \tag{1.12}$$

Suppose $\boldsymbol{f}\in(H^{-1}(\Omega))^n$, and $\frac{N\|\boldsymbol{f}\|^*}{\nu^2}<1$. Then problem (1.4)—(1.5) has a unique solution $(\boldsymbol{u}, \lambda)\in X\times M$ (see [3]). In this paper, we restrict ourselves within this case.

§ 2. An Abstract Error Estimate

Let $H=(L_2(\Omega))^n$. Throughout this section we suppose $\boldsymbol{f}\in H$. For each $h>0$, let M^h and X^h be two finite dimensional spaces such that $M^h\subset M$ and $X^h\subset H$, but X^h is not a subspace of X in the general case. Let $\|\cdot\|_h$ denote the norm of X^h (examples of X^h will be given in the next section). In order to discretize the variational problem (1.4)—(1.5), we first extend the definitions of $a_0(\boldsymbol{u}, \boldsymbol{v})$, $a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})$ and $b(\boldsymbol{v}, \lambda)$ to $(X\cup X^h)^2$, $(X\cup X^h)^3$ and $(X\cup X^h)\times M$ and denote them as $a_0^h(\boldsymbol{u}, \boldsymbol{v})$, $a_1^h(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})$ and $b^h(\boldsymbol{v}, \lambda)$ and

$$a_0^h(\boldsymbol{u}, \boldsymbol{v})=a_0(\boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{u}, \boldsymbol{v}\in X,$$

$$a_1^h(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})=a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}\in X,$$

$$b^h(v, \lambda) = b(v, \lambda), \quad \forall v \in X, \ \lambda \in M,$$

such that $a_0^h(u, v)$, $b^h(v, \lambda)$, $a_1^h(w; u, u)$ are bilinear forms and trilinear form respectively. Moreover, we assume

$$\|v\|_X = \|v\|_h, \quad \forall v \in X.$$

Now we consider the following approximate problem:

Find $(u_h, \lambda_h) \in (X^h \times M^h)$, such that

$$a_0^h(u_h, v) + a_1^h(u_h; u_h, v) + b^h(v, \lambda_h) = \langle f, v \rangle, \quad \forall v \in X^h, \tag{2.1}$$

$$b^h(u_h, \mu) = 0, \quad \forall \mu \in M^h. \tag{2.2}$$

Let $V^h = \{v \in X^h, \ b^h(v, \mu) = 0, \ \forall \mu \in M^h\}$, and

$$N_h = \sup_{w, u, v \in V^h} \frac{|a_1^h(w; u, v)|}{\|w\|_h \|u\|_h \|v\|_h}, \tag{2.3}$$

$$\|f\|_h^* = \sup_{v \in V^h} \frac{|\langle f, v \rangle|}{\|v\|_h}. \tag{2.4}$$

We have the following abstract error estimate for problem (2.1)—(2.2).

Theorem 2.1. *Assume that*

(i) *V^h is not empty, and*

$$a_0^h(v, v) \geq \nu \|v\|_h^2, \quad \forall v \in V^h, \tag{2.5}$$

$$a_1^h(w; v, v) = 0, \quad \forall w, \ v \in X^h; \tag{2.6}$$

(ii) *there exist two constants A_0 and B_0 independent of h, such that*

$$|a_0^h(u, v)| \leq A_0 \|u\|_h \|v\|_h, \quad \forall u, \ v \in X^h, \tag{2.7}$$

$$|b^h(v, \mu)| \leq B_0 \|v\|_h \|\mu\|_M, \quad \forall v \in X^h, \ \mu \in M^h, \tag{2.8}$$

and from (2.3), we have

$$|a_1^h(w; u, v)| \leq N_h \|w\|_h \|u\|_h \|v\|_h, \quad \forall w, \ u, \ v \in V^h; \tag{2.9}$$

(iii) *there is a constant $\beta > 0$ (independent of h) satisfying*

$$\sup_{v \in X^h} \frac{b^h(v, \mu)}{\|v\|_h} \geq \beta \|\mu\|_M, \quad \forall \mu \in M^h; \tag{2.10}$$

(iv) *there is an operator $\Pi_h: X \to X^h$ satisfying*

$$b^h(v - \Pi_h v, \mu) = 0, \quad \forall v \in X, \ \mu \in M^h; \tag{2.11}$$

(v)
$$\frac{N_h \|f\|_h^*}{\nu^2} \leq 1 - \delta_1, \tag{2.12}$$

where $\delta_1 \in (0, 1)$ is a constant independent of h.

Then the approximate problem (2.1)—(2.2) has a unique solution $(u_h, \lambda_h) \in (V^h \times M^h)$. Furthermore, we assume that

(vi)
$$\frac{N \|f\|^*}{\nu^2} \leq 1 - \delta_2, \quad \delta_2 \in (0, 1). \tag{2.13}$$

Then the following abstract error estimate holds

$$\|u - u_h\|_h \leq C \Big\{ \|u - \Pi_h u\|_h + \inf_{\mu \in M^h} \|\lambda - \mu\|_M + \sup_{w \in X^h} \frac{|G_h(u, \Pi_h u, w)|}{\|w\|_h} + \sup_{w \in X^h} \frac{|E_h(u, \lambda, w)|}{\|w\|_h} \Big\}, \tag{2.14}$$

$$\|\lambda-\lambda_h\|_M\leq C\Big\{\|u-\Pi_h u\|_h+\inf_{\mu\in M^h}\|\lambda-\mu\|_M+\sup_{w\in X^h}\frac{|G_h(u,\ \Pi_h u,\ w)|}{\|w\|_h}$$
$$+\sup_{w\in X^h}\frac{|G_h(u,\ u_h,\ w)|}{\|w\|_h}+\sup_{w\in X^h}\frac{|E_h(u,\ \lambda;\ w)|}{\|w\|_h}\Big\},\tag{2.15}$$

where $(u,\ \lambda)\in X\times M$ *is the solution of problem* (1.4)—(1.5), C *is a constant independent of* h, *and*

$$E_h(u,\ \lambda;\ w)=a_0^h(u,\ w)+a_1^h(u;\ u,\ w)+b^h(w,\ \lambda)-\langle f,\ w\rangle,\tag{2.16}$$
$$G_h(u,\ v,\ w)=a_1^h(u;\ u,\ w)-a_1^h(v;\ v,\ w).\tag{2.17}$$

Proof. From Theorem 3.1 in [3] (Chapter IV), we can obtain the existence and uniqueness of problem (2.1)—(2.2). Choosing $v=u_h$, in Eq. (2.1) we have

$$a_0^h(u_h,\ u_h)+a_1^h(u_h;\ u_h,\ u_h)+b^h(u_h,\ \lambda_h)=\langle f,\ u_h\rangle.$$

Since $a_1^h(u_h;\ u_h,\ u_h)=0,\ b^h(u_h,\ \lambda_h)=0$, the above equations are reduced to

$$a_0^h(u_h,\ u_h)=\langle f,\ u_h\rangle.$$

It yields

$$\|u_h\|_h\leq\frac{\|f\|_h^*}{\nu},\tag{2.18}$$

by (2.4) and (2.5). Let $w_h=u_h-\Pi_h u$, and

$$R=a_0^h(u_h,\ w_h)-a_0^h(\Pi_h u,\ w_h)+a_1^h(u_h;\ u_h,\ w_h)-a_1^h(\Pi_h u;\ \Pi_h u,\ w_h).$$

Then by (2.5), (2.6), (2.3), and (2.18) we have

$$R=a_0^h(w_h,\ w_h)+a_1^h(w_h;\ u_h,\ w_h)\geq\nu\|w_h\|_h^2-N_h\|u_h\|_h\|w_h\|_h^2$$
$$\geq\nu\|w_h\|_h^2\Big\{1-\frac{N_h\|f\|_h^*}{\nu^2}\Big\}\geq\delta_1\nu\|w_h\|_h^2,$$

On the other hand, from Eqs. (2.1) and (2.16), we obtain

$$R=\langle f,\ w_h\rangle-b^h(w_h,\ \lambda_h)-a_0^h(\Pi_h u,\ w_h)-a_1^h(\Pi_h u;\ \Pi_h u,\ w_h)$$
$$=a_0^h(u,\ w_h)+a_1^h(u;\ u,\ w_h)+b^h(w_h,\ \lambda)-E_h(u,\ \lambda;\ w_h)$$
$$-a_0^h(\Pi_h u,\ w_h)-a_1^h(\Pi_h u;\ \Pi_h u,\ w_h)-b^h(w_h,\ \lambda_h)$$
$$=a_0^h(u-\Pi_h u,\ w_h)+b^h(w_h,\ \lambda-\lambda_h)+a_1^h(u;\ u,\ w_h)$$
$$-a_1^h(\Pi_h u;\ \Pi_h u,\ w_h)-E_h(u,\ \lambda;\ w_h).$$

Since $w_h\in V^h$, we have $b^h(w_h,\ \mu_h)=0,\ \forall\mu_h\in M^h$. Therefore, we get

$$R=a_0^h(u-\Pi_h u,\ w_h)+b^h(w_h,\ \lambda-\mu_h)+G_h(u,\ \Pi_h u,\ w_h)$$
$$-E_h(u,\ \lambda;\ w_h),\quad\forall\mu_h\in M^h.$$

Furthermore, we obtain

$$\|w_h\|_h\leq C\Big\{\|u-\Pi_h u\|_h+\inf_{\mu\in M^h}\|\lambda-\mu\|_h$$
$$+\sup_{w\in X^h}\frac{|G_h(u,\ \Pi_h u,\ w)|}{\|w\|_h}+\sup_{w\in X^h}\frac{|E_h(u,\ \lambda;\ w)|}{\|w\|_h}\Big\}.\tag{2.19}$$

The triangle inequality yields

$$\|u-u_h\|_h\leq\|u-\Pi_h u\|_h+\|u_h-\Pi_h u\|_h.\tag{2.20}$$

Combining (2.19) and (2.20), we have (2.14). In order to estimate $\|\lambda-\lambda_h\|_M$, for any $v\in X^h$, we have

$$b^h(v, \lambda_h) = -a_0^h(u_h, v) - a_1^h(u_h; u_h, v) + \langle f, v\rangle$$
$$= a_0^h(u-u_h, v) + a_1^h(u; u, v) - a_1^h(u_h; u_h, v) + b^h(v, \lambda) - E_h(u, \lambda, v).$$

For any $v \in X^h$, $\mu_h \in M^h$, we get

$$b^h(v, \lambda_h - \mu_h) = a_0^h(u-u_h, v) + G_h(u; u_h, v) - E_h(u, \lambda; v) + b^h(v, \lambda - \mu_h).$$

Condition (2.10) yields

$$\begin{aligned}\|\lambda_h - \mu_h\|_M &\leqslant \frac{1}{\beta} \sup_{v\in X^h} \frac{b^h(v, \lambda_h-\mu_h)}{\|v\|_h} \\ &\leqslant C\left\{\|u-u_h\|_h + \|\lambda-\mu_h\|_M + \sup_{v\in X^h}\frac{|G_h(u, u_h, v)|}{\|v\|_h}\right. \\ &\quad \left. + \sup_{v\in X^h}\frac{|E_h(u, \lambda; v)|}{\|v\|_h}\right\}, \quad \forall \mu_h \in M^h. \end{aligned} \tag{2.21}$$

Then (2.15) is shown by the triangle inequality $\|\lambda-\lambda_h\|_M \leqslant \|\lambda-\mu_h\|_M + \|\lambda_h-\mu_h\|_M$, and inequalities (2.21), (2.14).

§ 3. The Nonconforming Rectangular Elements

The nonconforming rectangular element has been proposed to solve Stokes equations in two and three dimensional cases[6]. Now we recall some results of the nonconforming rectangular elements. Let $\hat{K} = ([-1, 1])^2$ denote a reference rectangular element in the two dimensional case as shown in Fig. 1. $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$, $\hat{a}_3 = (-1, 0)$, $\hat{a}_4 = (0, -1)$ are the middle points of the sides of $\hat{K}$ respectively, and $\hat{a}_5 = (0, 0)$ is the central point of $\hat{K}$. Let

$$\varphi(t) = \frac{1}{2}(5t^4 - 3t^2).$$

Then

$$\varphi(0) = 0, \ \varphi(\pm 1) = 1, \ \int_{-1}^{1} \varphi(t)dt = 0. \tag{3.1}$$

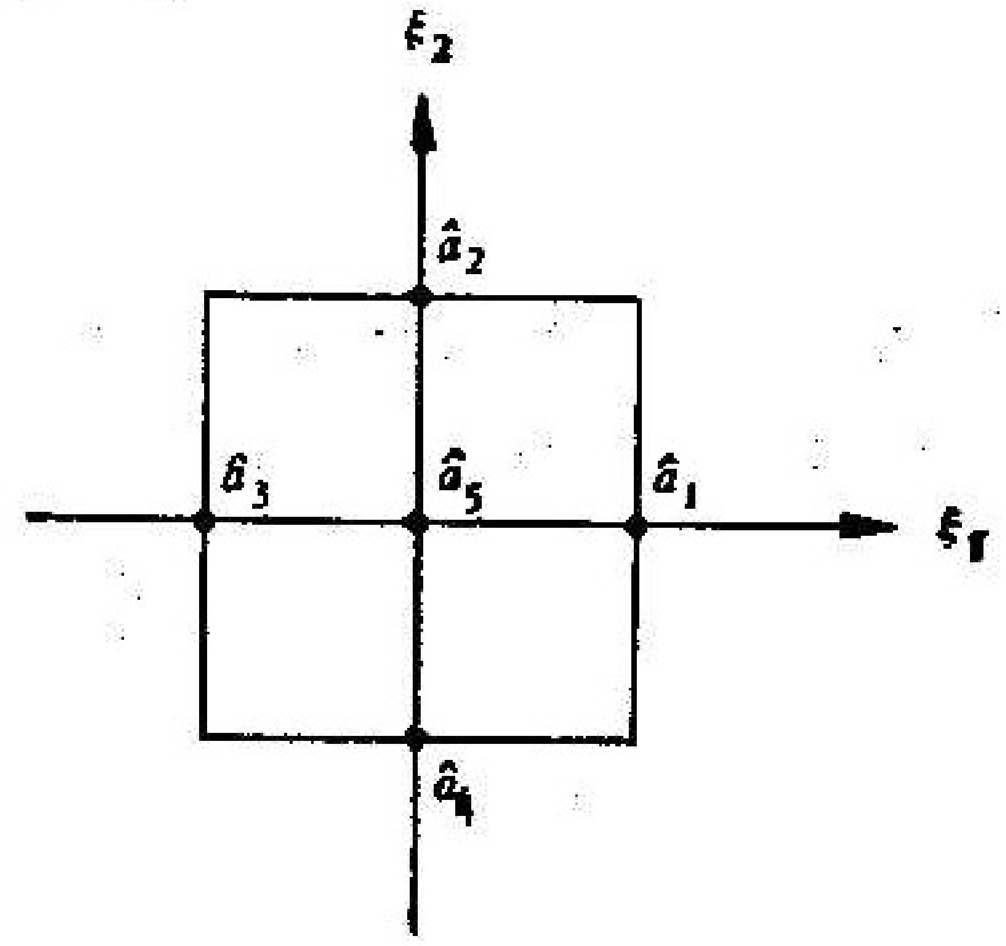

Fig. 1

Space $\hat{P}$ is defined by

$$\hat{P} = \{\hat{p} \mid \hat{p} \text{ is a linear combination of } 1, \xi_1, \xi_2, \varphi(\xi_1), \varphi(\xi_2) \text{ on } \hat{K}\}.$$

Lemma 3.1. *For any five real numbers u_1, u_2, u_3, u_4, and u_5, there exists a unique function $u(\xi) \in \hat{P}$, $\xi = (\xi_1, \xi_2)$, such that*

$$u(\hat{a}_i) = u_i, \quad i = 1, 2, \cdots, 5,$$

and

$$\frac{1}{2}\int_{-1}^{1} u(1, \xi_2)d\xi_2 = u(1, 0) = u_1, \tag{3.2$_1$}$$

$$\frac{1}{2}\int_{-1}^{1} u(-1, \xi_2)d\xi_2 = u(-1, 0) = u_3, \tag{3.2$_2$}$$

$$\frac{1}{2}\int_{-1}^{1} u(\xi_1, 1)d\xi_1 = u(0, 1) = u_2, \tag{3.2$_3$}$$

$$\frac{1}{2}\int_{-1}^{1} u(\xi_1, -1)d\xi_1 = u(0, -1) = u_4, \tag{3.2$_4$}$$

$$\frac{1}{4}\int_{\hat{K}} u(\xi)d\xi = u(0, 0) = u_5. \tag{3.2$_5$}$$

Furthermore,

$$u(\xi_1, \xi_2)=u_5+\frac{u_1-u_3}{2}\xi_1+\frac{u_2-u_4}{2}\xi_2+\frac{u_1+u_3-2u_5}{2}\varphi(\xi_1)+\frac{u_2+u_4-2u_5}{2}\varphi(\xi_2) \quad (3.3)$$

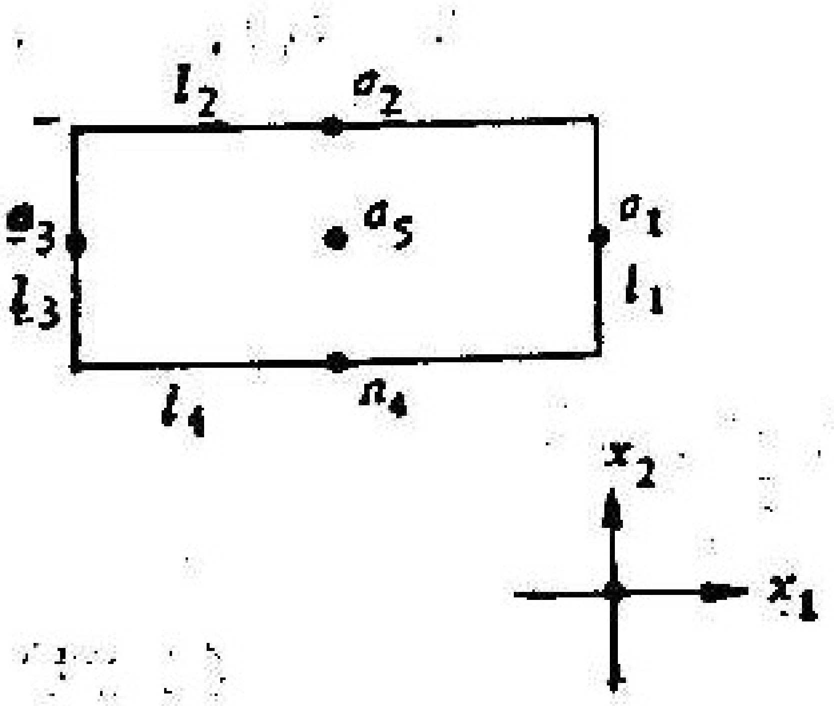

Fig. 2

Suppose K is an arbitrary rectangular element in the two dimensional case. Let l_i ($i=1, 2, 3, 4$) denote the sides of K, a_i denote the middle point of l_i, and a_5 the central point of K as shown in Fig. 2. Then there is an invariable linear mapping

$$F_K: \quad \xi\in\hat{K}\to x\in K,$$

and $\quad K=F_K(\hat{K}),\ a_i=F_K(\hat{a}_i), \quad i=1, 2, \cdots, 5.$

Taking

$$P_K=\{p=\hat{p}(F_K^{-1}(x))\,|\,\hat{p}\in\hat{P}\},\ \Sigma_K=\{p(a_i),\ 1\leqslant i\leqslant 5\},$$

we obtain the nonconforming rectangular element (K, P_K, Σ_K). Lemma 3.1 implies that for any five real numbers p_i ($i=1, \cdots, 5$) there exists a unique function $p(x)\in P_K$, such that

$$p(a_i)=p_i, \quad i=1, \cdots, 5,$$

$$\frac{1}{\text{meas}(l_i)}\int_{l_i}p\,dl=p_i, \quad i=1, \cdots, 4,$$

$$\frac{1}{\text{meas}(K)}\int_K p\,dx=p_5.$$

For any function $u(x)\in H^1(K)$, the interpolating operator Π_K is defined by $\Pi_K: H^1(K)\to P_K$, such that

$$\int_{l_i}(u-\Pi_K u)dl=0 \quad\text{and}\quad \int_K(u-\Pi_K u)dx=0. \quad (3.4)$$

Let $P_1(K)$ denote the space of all linear functions on K. Obviously, we have

$$u\equiv\Pi_K u, \quad \forall u\in P_1(K).$$

Let $\quad h_K=\max\limits_{1\leqslant i\leqslant 4}\{\text{meas}(l_i)\},\quad \rho_K=\min\limits_{1\leqslant i\leqslant 4}\{\text{meas}(l_i)\}.$

From the interpolating approximation theorem[7], we obtain

Lemma 3.2. *For function $u\in H^2(K)$, and*

$$\frac{h_K}{\rho_K}\leqslant\sigma, \quad \sigma>0, \quad (3.5)$$

then there is a constant C independent of h_K, and such that

$$\|u-\Pi_K u\|_{l,2,K}\leqslant Ch_K^{2-l}|u|_{2,2,K}, \quad l=0, 1, \quad (3.6)_1$$

$$\|u-\Pi_K u\|_{0,4,K}\leqslant Ch_K^{\frac{5}{4}}|u|_{2,2,K}. \quad (3.6)_2$$

For our case, inequality

$$\|u-\Pi_K u\|_{1,2,K}\leqslant C\|u\|_{1,2,K} \quad (3.6)_3$$

can be established in the same manner, where C is a constant independent of h_K.

From now on, suppose Ω is a rectangular domain. Then we describe the finite dimensional space associated with the nonconforming element (K, P_K, Σ_K) for the approximating space $H_0^1(\Omega)$. The rectangular domain Ω is subdivided into a finite number of subrectangles K. Let $\mathcal{T}_h$ denote this triangulation. The following properties are satisfied:

(i) $\bar{\Omega}=\bigcup_{K\in\mathcal{T}_h} K$,

(ii) for each distinct $K_1,\ K_2\in\mathcal{T}_h$, $K_1\cap K_2$ is empty, or a common vertix of K_1 and K_2, or a common side of K_1 and K_2.

(iii) let $h=\max_{K\in\mathcal{T}_h}\{h_K\}$, $\rho=\min_{K\in\mathcal{T}_h}\{\rho_K\}$. Assume

$$h/\rho\leqslant\sigma, \tag{3.5$'$}$$

where $\sigma>0$ is a constant.

Let N_a denote the set of all middle points of the sides of $K\in\mathcal{T}_h$, namely, $N_a=\bigcup_{K\in\mathcal{T}_h}\{a_1,\ a_2,\ a_3,\ a_4\}$, and $N_0=N_a\cap\partial\Omega$, $N_1=N_a\backslash N_0$. The finite dimensional space associated with the nonconforming element $(K,\ P_K,\ \Sigma_K)$ is defined by

$$S^h(\Omega)=\{v_h\,|\,u_h|_K\in P_K,\quad \forall K\in\mathcal{T}_h;\ v_h \text{ is continuous at } N_1;\text{ and } v_h(a)=0,\quad \forall a\in N_0\}.$$

Obviously, $S^h(\Omega)$ is not a subspace of $H_0^1(\Omega)$. Therefore we need to equip the space $S^h(\Omega)$ with a norm $\|\cdot\|_h$. Let

$$\|v_n\|_h=\Big(\sum_{K\in\mathcal{T}_h}|v_h|_{1,2,K}^2\Big)^{\frac{1}{2}}, \tag{3.7}$$

For any $v\in H^1(\Omega)$, we have

$$\|v\|_h=|v|_{1,2,\Omega}. \tag{3.8}$$

Moreover, the interpolating operator Π_h is defined by

$$\Pi_h: H_0^1(\Omega)\rightarrow S^h(\Omega),$$

such that $\qquad (\Pi_h u)|_K=\Pi_K(u|_K),\quad \forall u\in H_0^1(\Omega),$

where $u|_K$ denotes the restriction of u on K.

Lemma 3.2 yields the following result[7].

Lemma 3.3. *For any $u\in H^2(\Omega)$, there exists a constant C independent of h and u, such that*

$$\|u-\Pi_h u\|_{0,2,\Omega}\leqslant Ch^2|u|_{2,2,\Omega}, \tag{3.9$_1$}$$

$$\|u-\Pi_h u\|_{0,4,\Omega}\leqslant Ch^{\frac{5}{4}}|u|_{2,2,\Omega}, \tag{3.9$_2$}$$

$$\|u-\Pi_h u|_h\leqslant Ch|u|_{2,2,\Omega}, \tag{3.10$_1$}$$

$$\|u-\Pi_h u\|_h\leqslant C\|u\|_{1,2,\Omega}. \tag{3.10$_2$}$$

In the three dimensional case, let $\hat{K}=([-1,\ 1])^3$, $\hat{P}=\{\hat{p}\,|\,\hat{p}$ is a linear combination of functions $1,\ \xi_1,\ \xi_2,\ \xi_3,\ \varphi(\xi_1);\ \varphi(\xi_2),\ \varphi(\xi_3)$ on $\hat{K}\}$, and $\hat{\Sigma}=\{\hat{p}(\hat{a}_i),\ 1\leqslant i\leqslant 7\}$, where $\hat{a}_i\,(i=1,\ 2,\ \cdots,\ 6)$ is the central point of two-face of $\hat{K}$ respectively, and a_7 is the central point of $\hat{K}$. Similarly, we obtain the finite dimensional space $S^h(\Omega)$, the interpolating operator Π_h associated with nonconforming 3-rectangular element $(\hat{K},\ \hat{P},\ \hat{\Sigma})$, and the results in Lemma 3.2, 3.3 for the three dimensional case.

For the finite dimensional space $S^h(\Omega)$, the inverse inequalities hold. We have

Lemma 3.4. *For any $v_h\in S^h(\Omega)$, and positive real numbers $q\geqslant r$, $n=2$ (or $n=3$), there exists a constant C independent of h, such that*

$$\|v_h\|_{0,q,\Omega}\leqslant Ch^{-n(\frac{1}{r}-\frac{1}{q})}\|v_h\|_{0,r,\Omega},\quad \forall v_h\in S^h(\Omega).$$

The proof can be found in [7].

The remainder of this section will be devoted to properties of $S^h(\Omega)$, which are crucially important for the finite element approximation of N–S equations by nonconforming elements. For any $\vartheta \in W^{0,2}(\Omega)$, we consider the following auxiliary problem

$$-\Delta\chi=\vartheta, \quad \text{in } \Omega, \tag{3.12}$$

$$\chi=0, \quad \text{in } \partial\Omega. \tag{3.13}$$

Problem (3.12), (3.13) has a unique solution $\chi \in H_0^1(\Omega)\cap H^2(\Omega)$, and we have

$$\|\chi\|_{1,2,\Omega} \leqslant C\|\vartheta\|_{-1,2,\Omega}, \tag{3.14}$$

$$\|\chi\|_{2,2,\Omega} \leqslant C\|\vartheta\|_{0,2,\Omega}, \tag{3.15}$$

where C is a constant independent of ϑ.

For any $v_h \in S^h(\Omega)$, let

$$D_h(\chi, v_h) \equiv \sum_{K\in\mathcal{T}_h}\int_K \sum_{i=1}^{n} \frac{\partial\chi}{\partial x_i}\frac{\partial v_h}{\partial x_i}\,dx-(\vartheta, v_h). \tag{3.16}$$

Then we have

Lemma 3.5. *There exists a constant C independent of h, such that*

$$|D_h(\chi, v_h)| \leqslant Ch\|v_h\|_h|\chi|_{2,2,\Omega}, \quad \forall v_h\in S^h(\Omega). \tag{3.17}$$

This lemma is a corollary of Lemma 4.6 of the next section.

Lemma 3.6. *For any $\vartheta\in W^{0,2}(\Omega)$ and $v_h\in S^h(\Omega)$, then there is a constant C independent of h and v_h, such that*

$$\left|\int_\Omega v_h\vartheta\,dx\right| \leqslant C\{\|\vartheta\|_{-1,2,\Omega}+h\|\vartheta\|_{0,2,\Omega}\}\|v_h\|_h. \tag{3.18}$$

Proof. By (3.16), we have

$$\int_\Omega v_h\vartheta\,dx=\sum_{K\in\mathcal{T}_h}\int_K \sum_{i=1}^{n}\frac{\partial\chi}{\partial x_i}\frac{\partial v_h}{\partial x_i}\,dx-D_h(\chi, v_h).$$

Furthermore, we obtain

$$\left|\int_\Omega v_h\vartheta\,dx\right| \leqslant \|\chi\|_{1,2,\Omega}\|v_h\|_h+|D_h(\chi, v_h)| \leqslant C\{\|\vartheta\|_{-1,2,\Omega}+h\|\vartheta\|_{0,2,\Omega}\}\|v_h\|_h$$

by (3.14), (3.15), and (3.17).

Lemma 3.7 (discrete imbedding theorem). *For a positive integer k ($k=1, 2, \cdots$ if $n=2$; $k=1, 2, 3$ if $n=3$); there exists a constant $C(k)$ independent of h, such that*

$$\|v_h\|_{0,2k,\Omega} \leqslant C(k)\|v_h\|_h, \quad \forall v_h\in S^h(\Omega). \tag{3.19}$$

Proof. Taking $\vartheta=v_h^{2k-1}$, we obtain

$$\|v_h\|_{0,2k,\Omega}^{2k} \leqslant C\{\|v_h^{2k-1}\|_{-1,2,\Omega}+h\|v_h^{2k-1}\|_{0,2,\Omega}\}\|v_h\|_h$$

by inequality (3.18). Now we estimate $\|v_h^{2k-1}\|_{-1,2,\Omega}$ and $h\|v_h^{2k-1}\|_{0,2,\Omega}$. First, we have

$$h\|v_h\|_{0,4k-2,\Omega}^{2k-1}=h\left\{\int_\Omega v_h^{4k-2}\,dx\right\}^{\frac{1}{2}}=h\|v_h\|_{0,4k-2,\Omega}^{2k-1}.$$

Using the inverse inequality (3.11), we get

$$\|v_h\|_{0,4k-2,\Omega}^{2k-1} \leqslant Ch^{-\frac{n(k-1)}{2k}}\|v_h\|_{0,2k,\Omega}^{2k-1}.$$

Therefore, for $0<h\leqslant 1$, and $k=1, 2, \cdots$ (if $n=2$) or $k=1, 2, 3$ (if $n=3$), we have

$$h\|v_h^{2k-1}\|_{0,2,\Omega} \leqslant Ch^{1-\frac{n(k-1)}{2k}}\|v_h\|_{0,2k,\Omega}^{2k-1} \leqslant C\|v_h\|_{0,2k,\Omega}^{2k-1}. \tag{3.20}$$

Secondly, by application of the Hölder inequality with

$$q=\frac{2k}{2k-1},\quad p=2k,$$

we obtain

$$|(v_h^{2k-1},\ \psi)|=\left|\int_\Omega v_h^{2k-1}\psi\,dx\right|\leqslant\left\{\int_\Omega |v_h|^{2k}\,dx\right\}^{\frac{2k-1}{2k}}\left\{\int_\Omega |\psi|^{2k}\,dx\right\}^{\frac{1}{2k}}$$
$$=\|v_h\|_{0,2k,\Omega}^{2k-1}\|\psi\|_{0,2k,\Omega}.$$

By the imbedding theorem from $W^{1,2}(\Omega)\to W^{0,2k}(\Omega)$, there exists a constant $C(k)$, such that

$$\|\psi\|_{0,2k,\Omega}\leqslant C(k)\|\psi\|_{1,2,\Omega}$$

(for $k=1,\ 2,\ \cdots$ if $n=2$; or for $k=1,\ 2,\ 3$ if $n=3$). Combining the previous two inequalities, we find that

$$\|v_h^{2k-1}\|_{-1,2,\Omega}=\sup_{\substack{\psi\in W^{1,2}(\Omega)\\ \psi\neq 0}}\frac{|(v_h^{2k-1},\ \psi)|}{\|\psi\|_{1,2,\Omega}}\leqslant C(k)\|v_h\|_{0,2k,\Omega}^{2k-1}. \tag{3.21}$$

By (3.20), (3.21), we eventually get (3.19).

Remark 1. When $k=1$, inequality (3.19) is reduced to the discrete Poincáre inequality

$$\|v_h\|_{0,2,\Omega}\leqslant C(1)\|v_h\|_h,\quad \forall v_h\in S^h(\Omega).$$

Remark 2. In fact, (3.19) holds for general nonconforming elements if (3.17) holds. A similar result for nonconforming triangular elements has been given by Temam[1].

§ 4. The Error Estimates of Finite Element Approximate Solutions of N–S Equations

We now estimate the error of finite element approximate solutions by the nonconforming rectangular elements discussed in section 3. We take $X^h=(S^h(\Omega))^n$ with norm $\|\cdot\|_h$,

$$\|\boldsymbol{v}\|_h=\left\{\sum_{i=1}^n\|v_i\|_h^2\right\}^{\frac{1}{2}},\quad \forall \boldsymbol{v}\in X_h, \tag{4.1}$$

and

$$M^h=\left\{\mu\,\big|\,\mu|_K \text{ is a constant},\ \int_\Omega \mu\,dx=0\right\}. \tag{4.2}$$

Let

$$a_0^h(\boldsymbol{u},\ \boldsymbol{v})=\nu\sum_{i,j=1}^n\sum_{K\in\mathcal{T}_h}\int_K\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,dx, \tag{4.3}$$

$$a_1^h(\boldsymbol{w};\boldsymbol{u},\ \boldsymbol{v})=\frac{1}{2}\sum_{i,j=1}^n\sum_{K\in\mathcal{T}_h}\int_K\left[w_j\frac{\partial u_i}{\partial x_j}v_i-w_j\frac{\partial v_i}{\partial x_j}u_i\right]dx, \tag{4.4}$$

$$b^h(\boldsymbol{v},\ \lambda)=-\sum_{K\in\mathcal{T}_h}\int_K\lambda\,\mathrm{div}\,\boldsymbol{v}\,dx, \tag{4.5}$$

and

$$V^h=\{\boldsymbol{v}\in X^h\,|\,b^h(\boldsymbol{v},\ \mu)=0,\ \ \forall\mu\in M^h\}. \tag{4.6}$$

For any $\boldsymbol{v}\in(H_0^1(\Omega))^n$, the operator Π_h is defined by $\Pi_h:(H_0^1(\Omega))^n\to X^h$, such that

$$\Pi_h\boldsymbol{v}=(\Pi_h v_1,\ \cdots,\ \Pi_h v_n).$$

Therefore, from (3.4), we obtain

$$b^h(\boldsymbol{v}-\Pi_h\boldsymbol{v},\ \mu)=0,\quad \forall\boldsymbol{v}\in V,\ \mu\in M^h. \tag{4.7}$$

By (3.9) and (3.10), we get

$$\|v-\Pi_h v\|_{0,2,\Omega}\leqslant Ch^2|v|_{2,2,\Omega},\quad \forall v\in(H^2(\Omega))^n, \tag{4.8$_1$}$$

$$\|v-\Pi_h v\|_{0,4,\Omega}\leqslant Ch^{\frac{5}{4}}|v|_{2,2,\Omega},\quad \forall v\in(H^2(\Omega))^n, \tag{4.8$_2$}$$

$$\|v-\Pi_h v\|_h\leqslant Ch|v|_{2,2,\Omega},\quad \forall v\in(H^2(\Omega))^n, \tag{4.9$_1$}$$

$$\|v-\Pi_h v\|_h\leqslant C\|v\|_{1,2,\Omega},\quad \forall v\in X, \tag{4.9$_2$}$$

where C is a constant independent of h. For subspace M^h we have

$$\inf_{\mu\in M^h}\|\lambda-\mu\|_M\leqslant Ch|\lambda|_{1,2,\Omega},\quad \forall\lambda\in H^1(\Omega), \tag{4.10}$$

by the interpolating approximate theorem[7]. From Lemma 3.7, we obtain the following

Lemma 4.1. *For a positive integer k ($k=1, 2, \cdots$ if $n=2$; $k=1, 2, 3$ if $n=3$), there exists a constant $C(k)$ independent of h, such that*

$$\|v\|_{0,2k,\Omega}\leqslant C(k)\|v\|_h,\quad \forall v\in X^h. \tag{4.11}$$

Application of Lemma 4.1, gives

Lemma 4.2. *There exist two constants N_0 and F_0 independent of h, such that*

$$N_h\leqslant N_0,\quad \forall 1\geqslant h>0, \tag{4.12}$$

$$\|f\|_h^*\leqslant F_0,\quad \forall 1\geqslant h>0. \tag{4.13}$$

Lemma 4.3.

$$N\leqslant \varliminf_{h\to 0} N_h, \tag{4.14}$$

$$\|f\|^*\leqslant \varliminf_{h\to 0}\|f\|_h^*. \tag{4.15}$$

Proof. For arbitrary $\varepsilon>0$ there are $w_\varepsilon,\ u_\varepsilon,\ v_\varepsilon\in V\cap(H^2(\Omega))^n$, and $\|w_\varepsilon\|_h=\|u_\varepsilon\|_h=\|v_\varepsilon\|_h=1$, such that

$$N-\varepsilon\leqslant|a_1^h(w_\varepsilon;u_\varepsilon,v_\varepsilon)|.$$

On the other hand, we know that $\Pi_h w_\varepsilon,\ \Pi_h u_\varepsilon,\ \Pi_h v_\varepsilon\in V^h$ by equality (4.7), and $\|w_\varepsilon-\Pi_h w_\varepsilon\|_h\to 0,\ \|u_\varepsilon-\Pi_h u_\varepsilon\|_h\to 0,\ \|v_\varepsilon-\Pi_h v_\varepsilon\|_h\to 0$, when $h\to 0$. Hence we obtain that $\|\Pi_h w_\varepsilon\|_h\to 1,\ \|\Pi_h u_\varepsilon\|_h\to 1,\ \|\Pi_h v_\varepsilon\|\to 1$, when $h\to 0$, and

$$N-\varepsilon\leqslant N_h+N_0|1-\|\Pi_h w_\varepsilon\|_h\|\Pi_h u_\varepsilon\|_h\|\Pi_h v_\varepsilon\|_h|$$
$$+|a_1^h(w_\varepsilon,u_\varepsilon,v_\varepsilon)-a_1^h(\Pi_h w_\varepsilon,\Pi_h u_\varepsilon,\Pi_h v_\varepsilon)|.$$

For the fixed $\varepsilon>0$, let $h\to 0$ we obtain $N-\varepsilon\leqslant\varliminf_{h\to 0}N_h$. Then let $\varepsilon\to 0$, and (4.14) is shown. (4.15) can be established in the same way.

Corollary.

$$N\leqslant N_0, \tag{4.12$'$}$$

$$\|f\|^*\leqslant F_0. \tag{4.13$'$}$$

Lemma 4.4. *There exists a constant β independent of h, such that*

$$\sup_{v_h\in X^h}\frac{b^h(v_h,\mu_h)}{\|v_h\|_h}\geqslant\beta\|\mu_h\|_M,\quad \forall\mu_h\in M^h. \tag{4.16}$$

Proof. For any $\mu_h\in M^h$, there is a vector function $v\in X$ satisfying[8]

$$\operatorname{div} v=\mu_h,\quad\text{and}\quad \|v\|_X=\|v\|_h\leqslant C\|\mu_h\|_M.$$

For $v\in X$, we have $\Pi_h v\in X^h$ and

$$b^h(v-\Pi_h v,\mu)=0,\quad \forall\mu\in M^h,$$

$$\|\Pi_h v\|_h \leqslant C\|v\|_X \leqslant C\|\mu_h\|_M \tag{4.17}$$

by (4.7) and $(4.9)_2$. Therefore, we have, by (4.17),

$$\sup_{v_h\in X^h} \frac{b^h(v_h,\ \mu_h)}{\|v_h\|_h} \geqslant \frac{b^h(\Pi_h v,\ \mu_h)}{\|\Pi_h v\|_h} = \frac{b(v,\ \mu_h)}{\|\Pi_h v\|_h} = \frac{\|\mu_h\|_M^2}{\|\Pi_h v\|_h} \geqslant \beta\|\mu_h\|_M,$$

where β is a constant independent of h, and the conclusion follows.

Now we consider the approximate problem (2.1), (2.2), where X^h, M^h, $a_0^h(\cdot,\cdot)$, $a_1^h(\cdot,\ \cdot,\ \cdot)$, and $b^h(\cdot,\ \cdot)$ are given by (4.1)—(4.5). In order to estimate the error of finite element solutions of N–S equations from Theorem 2.1, we need to estimate $G_h(u,\ \Pi_h u,\ w_h)$, $G_h(u,\ u_h,\ w_h)$ and $E_h(u,\ \lambda;\ w_h)$ we have

Lemma 4.5. *For any $u\in X\cap(H^2(\Omega))^n$ there is a constant C independent of h and u, such that*

$$\sup_{w_h\in X^h} \frac{|G_h(u,\ \Pi_h u,\ w_h)|}{\|w_h\|_h} \leqslant Ch|u|_{2,2,\Omega}|u|_{1,2,\Omega}. \tag{4.18}$$

And for any $u_h\in X^h$, there exists a constant C independent of h, u, u_h, such that

$$\sup_{w_h\in X^h} \frac{|G_h(u,\ u_h,\ w_h)|}{\|w_h\|_h} \leqslant C[h+\|\Pi_h u-u_h\|_h]\{\|u\|_h+\|u_h\|_h\}. \tag{4.19}$$

Proof. For any $u\in X\cap(H^2(\Omega))^n$ and $w_h\in X^h$, we have

$$\begin{aligned}|G_h(u,\ \Pi_h u,\ w_h)| &= |a_1^h(u;\ u,\ w_h)-a_1^h(\Pi_h u;\ \Pi_h u,\ w_h)|\\ &\leqslant |a_1^h(u;\ u-\Pi_h u,\ w_h)|+|a_1^h(u-\Pi_h u,\ \Pi_h u,\ w_h)|\\ &\leqslant C[\|u-\Pi_h u\|_{0,4,\Omega}+\|u-\Pi_h u\|_h][|u|_{1,2,\Omega}+\|\Pi_h u\|_h]\|w_h\|_h\\ &\leqslant Ch|u|_{2,2,\Omega}|u|_{1,2,\Omega}\|w_h\|_h\end{aligned}$$

by (4.8)—(4.11), and inequality (4.18) is proved. (4.19) can be established in the same manner.

Lemma 4.6. *Suppose $(u,\ \lambda)$ is the solution of problem (1.4)—(1.5), and $u\in X\cap(H^2(\Omega))^n$, $\lambda\in M\cap H^1(\Omega)$; then there exists a constant C independent of h, such that*

$$\sup_{w_h\in X^h} \frac{|E_h(u,\ \lambda;\ w_h)|}{\|w_h\|_h} \leqslant Ch\{\|u\|_{2,2,\Omega}(|u|_{1,2,\Omega}+1)+|\lambda|_{1,2,\Omega}\}. \tag{4.20}$$

Proof. By the definition of $E_h(u,\ \lambda;\ w_h)$ and equation (1.1)′, we have

$$\begin{aligned}E_h(u,\ \lambda;\ w_h) &= \nu\sum_{K\in\mathcal{T}_h}\sum_{i=1}^n\int_{\partial K}\frac{\partial u_i}{\partial n}w_i^h\,ds-\sum_{K\in\mathcal{T}_h}\sum_{i,j=1}^n\int_{\partial K}\frac{1}{2}u_j u_i\cos(n,\ x_j)w_i^h\,ds\\ &\quad-\sum_{K\in\mathcal{T}_h}\sum_{i=1}^n\int_{\partial K}\lambda\cos(n,\ x_i)w_i^h\,ds\equiv I_1+I_2+I_3,\end{aligned}$$

where $w_h=(w_1^h,\ \cdots,\ w_n^h)\in X^h$, and $\dfrac{\partial u_i}{\partial n}$ denotes the outer normal derivative of u_i on ∂K. From the proof of Lemma 4.2 in [6], we know that

$$|I_1|\leqslant Ch|u|_{2,2,\Omega}\|w_h\|_h, \tag{4.21}$$

$$|I_3|\leqslant Ch|\lambda|_{1,2,\Omega}\|w_h\|_h. \tag{4.22}$$

Taking $u_i u_j$ instead of λ in (4.22), we obtain

$$\begin{aligned}|I_2| &\leqslant Ch\Big(\sum_{i,j=1}^n|u_i u_j|_{1,2,\Omega}\Big)\|w_h\|_h\leqslant Ch\|u\|_{0,\infty,\Omega}|u|_{1,2,\Omega}\|w_h\|_h\\ &\leqslant Ch\|u\|_{2,2,\Omega}|u|_{1,2,\Omega}\|w_h\|_h.\end{aligned} \tag{4.23}$$

Combining (4.21), (4.22), and (4.23), we come to the conclusion.

Corollary. Inequality (3.17) holds.

Finally, we obtain

Theorem 4.1. *Suppose*

(i)
$$N_0F_0/\nu^2\leqslant 1-\delta, \tag{4.24}$$
where $\delta\in(0,1)$ *is a constant.*

(ii) $(\boldsymbol{u},\lambda)$ *is the solution of problem* (1.4)—(1.5), *and*
$$\boldsymbol{u}\in X\cap(H^2(\Omega))^n,\quad \lambda\in M\cap H^1(\Omega).$$
Then approximate problem (2.1)—(2.2) *has a unique solution* $(\boldsymbol{u}_h,\lambda_h)\in X^h\times M^h$, *and*
$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_h\leqslant Ch\{\|\boldsymbol{u}\|_{2,2,\Omega}(|\boldsymbol{u}|_{1,2,\Omega}+1)+|\lambda|_{1,2,\Omega}\}, \tag{4.25}$$
$$\|\lambda-\lambda_h\|_M\leqslant Ch\{\|\boldsymbol{u}\|_{2,2,\Omega}(|\boldsymbol{u}|_{1,2,\Omega}+1)^2+|\lambda|_{1,2,\Omega}(|\boldsymbol{u}|_{1,2,\Omega}+1)+(1+|\boldsymbol{u}|_{1,2,\Omega})\}. \tag{4.26}$$

Proof. First we point out that condition $N_0F_0/\nu^2\leqslant 1-\delta$ implies conditions (v) and (vi) in Theorem 2.1. Then it is straightforward to check that conditions (i)—(iv) in Theorem 2.1 hold. By Theorem 2.1, we obtain error estimates (2.14) and (2.15). Then inequality (4.25) follows immediately from (2.14), (4.10), (4.18), (4.20), (4.9). From (2.18) and (4.13), we have $\|\boldsymbol{u}_h\|_h\leqslant C$, and
$$\sup_{\boldsymbol{w}_h\in X^h}\frac{|G_h(\boldsymbol{u},\boldsymbol{u}_h,\boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|}\leqslant C[h+\|\boldsymbol{u}-\boldsymbol{u}_h\|_h][\|\boldsymbol{u}\|_h+1], \tag{4.27}$$
by (4.19). Combining (2.15), (4.9), (4.10), (4.18), (4.20), (4.25), (4.27), and (3.8), we can derive (4.26) and this completes the proof.

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