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THE ESTIMATES OF $\|M^{-1}N\|_\infty$ AND THE OPTIMALLY SCALED MATRIX*

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§ 1. Introduction

It is well-known that, if the elements m_{ij} of an $n\times n$ matrix M satisfy the inequality

$$|m_{ii}|-\sum_{j\neq i}|m_{ij}|\geqslant\delta>0, \qquad i=1, 2, \cdots, n, \tag{1}$$

where δ is a constant, then the inequality

$$\|M^{-1}\|_\infty\leqslant 1/\delta \tag{2}$$

holds[1,2]. But sometimes it is necessary to estimate the norm $\|M^{-1}N\|_\infty$, where N is an $n\times n$ or $n\times m$ matrix, and the use of the estimate (2), i.e. the estimate $\|M^{-1}N\|_\infty\leqslant\|M^{-1}\|_\infty\|N\|_\infty\leqslant\|N\|_\infty/\delta$, does not result satisfactorily. In this paper, we give an upper and a lower bound of $\|M^{-1}N\|_\infty$ for some matrices M and N. In [3], James and Riha applied the scaling transformation to prove the convergence of some iterative schemes for solving systems of linear algebraic equations. In this paper, we define an "optimally scaled matrix" by means of the scaling transformation. Our estimates of $\|M^{-1}N\|_\infty$ and the optimally scaled matrix are very useful in the discussion of the convergence of some iterative matrices. For, in the literature up to now, in order to prove the convergence of an iterative matrix $G(A)$ of a matrix A, such as Jacobi iterative matrix, SOR iterative matrix, etc., it is a common procedure to construct a dominant matrix $H(A)$, such that $|G(A)|\leqslant H(A)$ and, consequently,

$$\rho(G(A))\leqslant\rho(H(A)), \tag{3}$$

where $\rho(\cdot)$ is the spectral radius of the matrix enclosed in the brackets; thus, we need only to prove the convergence of the iterative matrix $H(A)$. Now for the optimally scaled matrix $\widetilde{A}$ of the matrix A we have

$$\rho(G(A))=\rho(G(\widetilde{A})). \tag{4}$$

Evidently, (4) is better than (3), since from (4) $G(A)$ is convergent, if and only if $G(\widetilde{A})$ is so, and this may be obtained easily by our estimates of $\|M^{-1}N\|_\infty$. We will discuss in this way the convergence of some splittings of a matrix. Besides, we will give some other applications of the estimates of $\|M^{-1}N\|_\infty$ and the optimally scaled matrix.

§ 2. The Estimates of $\|M^{-1}N\|_\infty$ and the Optimally Scaled Matrix

Theorem 1. *If $M=(m_{ij})$ is an $n\times n$ matrix, $N=(n_{ij})$ is an $n\times m$ matrix and*

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$$|m_{ii}|>\sum_{j\neq i}|m_{ij}|, \qquad i=1, 2, \cdots, n, \tag{5}$$

then we have

$$\|M^{-1}N\|_\infty\leqslant\max_i(\sum_j|n_{ij}|/(|m_{ii}|-\sum_{j\neq i}|m_{ij}|)), \tag{6}$$

where
$$\|M^{-1}N\|_\infty=\max_i\sum_j|(M^{-1}N)_{ij}|.$$

Proof. First let $N=(n_i)$ be an $n\times 1$ matrix, that is, an n-dimensional vector, and $M^{-1}N=X$. Thus, $MX=N$. If $X=(x_1, x_2, \cdots, x_n)^T$ and

$$\|M^{-1}N\|_\infty=\|X\|_\infty=\max_i|x_i|=|x_{i_0}|,$$

then
$$m_{i_0i_0}x_{i_0}=n_{i_0}-\sum_{j\neq i_0}m_{i_0j}x_j.$$

Hence,
$$|m_{i_0i_0}|\,|x_{i_0}|\leqslant|n_{i_0}|+|x_{i_0}|\sum_{j\neq i_0}|m_{i_0j}|$$

and
$$|x_{i_0}|\leqslant|n_{i_0}|/(|m_{i_0i_0}|-\sum_{j\neq i_0}|m_{i_0j}|)\leqslant\max_i(|n_i|/(|m_{ii}|-\sum_{j\neq i}|m_{ij}|)).$$

Thus we have proved (6) when N is an n-dimensional vector. Now, let $N=(n_{ij})$ be an $n\times m$ matrix, $|N|=(|n_{ij}|)$, $D=\operatorname{diag}M$, $B=D-M$, $\widetilde{M}=|D|-|B|$ and $\widetilde{N}=|N|$. It is easily seen that

$$\rho(D^{-1}B)\leqslant\rho(|D|^{-1}|B|)<1.$$

Therefore
$$M^{-1}=(D-B)^{-1}=(I+D^{-1}B+(D^{-1}B)^2+\cdots)D^{-1},$$
$$\widetilde{M}^{-1}=(|D|-|B|)^{-1}=(I+|D|^{-1}|B|+(|D|^{-1}|B|)^2+\cdots)|D|^{-1}.$$

Hence
$$|M^{-1}|\leqslant\widetilde{M}^{-1}, \qquad |M^{-1}N|\leqslant\widetilde{M}^{-1}\widetilde{N}$$

and
$$\|M^{-1}N\|_\infty\leqslant\|\widetilde{M}^{-1}\widetilde{N}\|_\infty$$

(we also have $\rho(M^{-1}N)\leqslant\rho(\widetilde{M}^{-1}\widetilde{N})$, provided that N is a square matrix). Now, if $\widetilde{N}_j$ is the vector composed of the elements of the jth column of $\widetilde{N}$, from $\widetilde{M}^{-1}\geqslant 0$ we have

$$\|\widetilde{M}^{-1}\widetilde{N}\|_\infty=\max_i\sum_j(\widetilde{M}^{-1}\widetilde{N})_{ij}=\max_i\sum_j(\widetilde{M}^{-1}\widetilde{N}_j)=\max_i(\widetilde{M}^{-1}\sum_j\widetilde{N}_j).$$

Taking $\sum_j\widetilde{N}_j$ as the n-dimensional vector N and $\widetilde{M}$ as M mentioned above, we have proved our theorem.

In Theorem 1 taking $N=I$ (the unit matrix), we get the estimate (2). Thus (2) is a special case of (6).

Theorem 2. *Under the conditions of Theorem 1, if, furthermore, M is an L-matrix and $N\geqslant 0$ then*

$$\min_i\sum_j(M^{-1}N)_{ij}\geqslant\min_i(\sum_j|n_{ij}|/(|m_{ii}|-\sum_{j\neq i}|m_{ij}|))=\min_i(\sum_j n_{ij}/\sum_j m_{ij}). \tag{7}$$

The proof of this theorem is similar to that of Theorem 1 and is therefore omitted, but it may be noticed that, under the conditions of Theorem 2, $M=\widetilde{M}$, $N=\widetilde{N}$, $M^{-1}\geqslant 0$ and $M^{-1}N\geqslant 0$.

It is well-known that[4], if $A=(a_{ij})\geqslant 0$, then

$$\min_i\sum_j a_{ij}\leqslant\rho(A)\leqslant\max_i\sum_j a_{ij}. \tag{8}$$

Therefore, we have

Corollary 1. If $M=(m_{ij})$ is an $n\times n$ L-matrix, $N=(n_{ij})$ is an $n\times n$ nonnegative matrix and $\sum_j m_{ij}>0$ $(i=1, 2, \cdots, n)$, then

$$\min_i \left(\sum_j n_{ij}/\sum_j m_{ij}\right) \leqslant \rho(M^{-1}N) \leqslant \max_i \left(\sum_j n_{ij}/\sum_j m_{ij}\right). \tag{9}$$

It is interesting that, under the conditions of the corollary, if $\sum_j n_{ij}/\sum_j m_{ij}$ is a constant independent of i, then this constant is just $\rho(M^{-1}N)$. Hence, however we change the elements of M and N, $\rho(M^{-1}N)$ will not change its value, only if $\sum_j n_{ij}/\sum_j m_{ij}$ remains unchanged. Thus, for example,

$$\begin{bmatrix} 5 & -1 & -1 \\ -2 & 10 & -2 \\ -1 & -1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 10 & -3 & -1 \\ -2 & 5 & 0 \\ 0 & -4 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix},$$

and

$$\begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -2 \\ 0 & -2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

have the same spectral radius 2/3. Applying this to the special case $N=I$, we have

$$\rho(M^{-1}) = \|M^{-1}\|_\infty = 1/\delta,$$

provided that M is an L-matrix and $\sum_j m_{ij} = \delta\ (i=1, 2, \cdots, n)$. This, in some way, supplements (2).

In [5] we obtained some other estimates of $\|M^{-1}N\|_\infty$. Now, we explain the concept of the optimally scaled matrix. We have

Theorem 3. *Let $A=(a_{ij})$ be an irreducible $n\times n$ matrix, $D=\operatorname{diag} A$, $\det D\neq 0$ and $B=D-A$. Then there exists a diagonal matrix $Q=\operatorname{diag}(q_1, q_2, \cdots, q_n)$ with positive diagonal elements, such that the matrix $\tilde{A}=(\tilde{a}_{ij})=AQ$ satisfies the equality*

$$\sum_{j\neq i} |\tilde{a}_{ij}|/|\tilde{a}_{ii}| = \rho(|D|^{-1}|B|), \qquad i=1, 2, \cdots, n. \tag{10}$$

Furthermore, for $\bar{A}=(\bar{a}_{ij})=A\bar{Q}$, where $\bar{Q}=\operatorname{diag}(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_n)$, $\bar{q}_i>0$ $(i=1,2,\cdots,n)$, if $\bar{Q}\neq \mathrm{const}\times Q$, then

$$\min_i \left(\sum_{j\neq i} |\bar{a}_{ij}|/|\bar{a}_{ii}|\right) < \rho(|D|^{-1}|B|) < \max_i \left(\sum_{j\neq i} |\bar{a}_{ij}|/|\bar{a}_{ii}|\right). \tag{11}$$

Proof. This is a direct consequence of Lemma 2.5 in Chapter 2 of [4] and the Perron–Frobenius theorem about the nonnegative matrix. Here we give a simple proof.

Our problem is to find a positive vector $q=(q_1, q_2, \cdots, q_n)^T$, such that

$$\sum_{j\neq i} |a_{ij}| q_j / |a_{ii}| q_i = k \text{ (const.)} \tag{12}$$

and to prove that $k=\rho(|D|^{-1}|B|)$. But from (12), we have

$$|D|^{-1}|B| q = kq.$$

Since A is irreducible, so is $|D|^{-1}|B|$. From the Perron–Frobenius theorem, $\rho(|D|^{-1}|B|)$ and the corresponding positive eigenvector may be taken as the k and q mentioned above respectively. On the other hand, applying Lemma 2.5 in [4] to the matrix $(|D|\bar{Q})^{-1}(|B|\bar{Q})$, we can easily prove the rest of the theorem.

According to [3], $\tilde{A}=AQ$ is a scaled transformation of A. We call $\tilde{A}$, which satisfies (10), the optimally scaled matrix of A. $\tilde{A}$ has many excellent properties, for example:

Corollary 2. One and only one of the following three propositions holds:

1° $|\tilde{a}_{ii}|>\sum_{j\neq i}|\tilde{a}_{ij}|$, $i=1, 2, \cdots, n$, i.e. $\tilde{A}$ is strictly diagonally dominant.

2° $|\tilde{a}_{ii}|<\sum_{j\neq i}|\tilde{a}_{ij}|$, $i=1, 2, \cdots, n$. We call such $\tilde{A}$ a strictly diagonally inferior matrix.

3° $|\tilde{a}_{ii}|=\sum_{j\neq i}|\tilde{a}_{ij}|$, $i=1, 2, \cdots, n$. We call such $\tilde{A}$ a diagonally equilibrous matrix.

This corollary follows immediately from Theorem 5. In fact, its three propositions correspond to $\rho(|D|^{-1}|B|)<1$, $\rho(|D|^{-1}|B|)>1$ and $\rho(|D|^{-1}|B|)=1$ respectively. Using the notation of Theorem 3, we have

Corollary 3. The following propositions are equivalent:

1° $\rho(|D|^{-1}|B|)<1$.

2° $\rho(|\tilde{D}|^{-1}|\tilde{B}|)<1$, where $\tilde{D}=\operatorname{diag}\tilde{A}$ and $\tilde{B}=\tilde{D}-\tilde{A}$.

3° A is a strictly diagonally dominant matrix.

4° A is an H-matrix.

5° $\tilde{A}$ is an H-matrix.

This corollary is a direct consequence of Theorem 7.2 in Chapter 2 of [6] and the above theorem. Here and in § 3 and § 4, we call A an H-matrix, if $|D|-|B|$ is an M-matrix defined in [6].

§ 3. The Convergence of Some Iterative Matrices

From the theorems and corollaries in § 2, we can easily obtain the theorems for the convergence of a number of iterative matrices, such as Jacobi, JOR, SOR[7], GSOR (Generalized SOR)[8], AOR (Accelerated Overrelaxation)[9], SSOR (Symmetric SOR)[10], SAOR (Symmetric AOR) and other iterative matrices.

Lemma 1. *If $A=M-N$ is a splitting of A and $\tilde{A}=\tilde{M}-\tilde{N}$, a splitting of $\tilde{A}$, $\tilde{M}=PMQ$ and $\tilde{N}=PNQ$, where P and Q are nonsingular matrices, then, $\rho(\tilde{M}^{-1}\tilde{N})=\rho(M^{-1}N)$, provided that M is invertible.*

This lemma is similar to Theorem 3 in [3]. Its proof is very simple and is thus omitted.

Definition. *The set of equimodular matrices of a matrix A is defined as*

$$\Omega(A)=\{G=(g_{ij}):\ |g_{ij}|=|a_{ij}|,\qquad i, j=1, 2, \cdots, n\}.$$

As a rule, given G, let $D_G=\operatorname{diag} G$ and $B_G=D_G-G$. Now, we assume that E_G is a matrix composed of some elements of B_G and zeros, and that $F_G=B_G-E_G$. We define the generalized iterative matrix as

$$T_1(G)=(D_G-RE_G)^{-1}((I-\Omega)D_G+(\Omega-R)E_G+\Omega F_G), \tag{13}$$

where

$$R=\operatorname{diag}(r_1, r_2, \cdots, r_n), \qquad \Omega=\operatorname{diag}(\omega_1, \omega_2, \cdots, \omega_n) \tag{14}$$
$$0\leqslant r_i\leqslant\omega_i, \quad \omega_i\neq 0, \quad i=1, 2, \cdots, n,$$

and D_G-RE_G is nonsingular. Let $T_2(G)$ be the matrix obtained from $T_1(G)$ by interchanging the matrices E_G and F_G and replacing R and Ω with

$$\tilde{R}=\operatorname{diag}(\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_n) \quad\text{and}\quad \tilde{\Omega}=\operatorname{diag}(\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)$$

respectively, where

$$0\leqslant\tilde{r}_i\leqslant\tilde{\omega}_i, \quad \tilde{\omega}_i\neq 0, \quad i=1, 2, \cdots, n.$$

We call the product $T_1(G)T_2(G)$ the symmetric generalized iterative matrix. The iterative matrices mentioned in the beginning of this paragraph are all special cases of $T_1(G)$ or $T_1(G)T_2(G)$. Now, we can state our theorem.

Theorem 4. *The following three propositions are equivalent:*

1. *A is an H-matrix.*

2. *For arbitrary $G\in\Omega(A)$ and $\omega_i(i=1, 2, \cdots, n)$ in $(0, 2/[1+(|D|^{-1}|B|)])$, the generalized iterative matrix $T_1(G)$ of G converges.*

3. *For arbitrary $G\in\Omega(A)$ and ω_i and $\tilde{\omega}_i$ in $(0, 2/[1+\rho(|D|^{-1}|B|)])$, the symmetric generalized iterative matrix $T_1(G)T_2(G)$ converges.*

Proof. First, let A be an H-matrix. From Theorem 3, the optimally scaled matrix $\tilde{G}=(\tilde{g}_{ij})=GQ$ of $G\in\Omega(A)$ is strictly diagonally dominant, i.e.

$$|\tilde{g}_{ii}|>\sum_{j\neq i}|\tilde{g}_{ij}|, \quad i=1, 2, \cdots, n. \tag{15}$$

By Lemma 1, in order to prove the convergence of $T_1(G)$, we need only to prove that of $T_1(\tilde{G})$; $T_1(\tilde{G})$ is obtained from $T_1(G)$ by replacing D_G, E_G and F_G with $D_{\tilde{G}}=D_GQ$, $E_{\tilde{G}}=E_GQ$ and $F_{\tilde{G}}=F_GQ$ respectively. Now, according to Theorem 1, we have

$$T_1(\tilde{G})\leqslant\max_i\frac{|1-\omega_i||\tilde{g}_{ii}|+(\omega_i-r_i)\sum_{j\in E}|\tilde{g}_{ij}|+\omega_i\sum_{j\in F}|\tilde{g}_{ij}|}{|\tilde{g}_{ii}|-r_i\sum_{j\in E}|\tilde{g}_{ij}|}, \tag{16}$$

where $\sum_{j\in E}|\tilde{g}_{ij}|$ and $\sum_{j\in F}|\tilde{g}_{ij}|$ denote the sum of all the elements lying on the ith row of $E_{\tilde{G}}$ and $F_{\tilde{G}}$ respectively. Under the conditions of Theorem 4, we obtain at once $\|T_1(\tilde{G})\|_\infty<1$, so that $T_1(G)$ converges. 3° follows immediately from

$$\rho(T_1(G)T_2(G))=\rho(T_1(\tilde{G})T_2(\tilde{G}))\leqslant\|T_1(\tilde{G})T_2\tilde{G})\|_\infty \leqslant\|T_1(\tilde{G})\|_\infty\|T_2(\tilde{G})\|_\infty<1.$$

Thus, from 1° we have proved 2° and 3°.

Conversely, suppose A is not an H-matrix; so neither is $G\in\Omega(A)$, and the optimally scaled matrix $\tilde{G}=(\tilde{g}_{ij})=GQ$ of G satisfies the inequalities

$$|\tilde{g}_{ii}|\leqslant\sum_{j\neq i}|\tilde{g}_{ij}|, \qquad i=1, 2, \cdots, n. \tag{17}$$

Let G be an L-matrix; then so is $\tilde{G}$. From Lemma 1 and Theorem 2, we have

$$\begin{aligned}\rho(T_1(G))&=\rho(T_1(\tilde{G}))\geqslant\min_i\sum_j(T_1(\tilde{G}))_{ij}\\&\geqslant\min_i\{[(1-\omega_i)|\tilde{g}_{ii}|+(\omega_i-r_i)\sum_{j\in E}|\tilde{g}_{ij}|\\&\quad+\omega_i\sum_{j\in F}|\tilde{g}_{ij}|]/[|\tilde{g}_{ii}|-r_i\sum_{j\in E}|\tilde{g}_{ij}|]\},\end{aligned} \tag{18}$$

where we assume r_i is so small that $|\tilde{g}_{ii}|-r_i\sum_{j\in E}|\tilde{g}_{ij}|>0$ and $0<\omega_i\leqslant1$. From (17), the right-hand side of (18) is not less than 1. Thus $T_1(G)$ is not convergent. On the other hand,

$$\begin{aligned}\rho(T_1(G)T_2(G))&=\rho(T_1(\tilde{G})T_2(\tilde{G}))\geqslant\min_i\sum_j(T_1(\tilde{G})T_2(\tilde{G}))_{ij}\\&\geqslant\min_i\sum_j(T_1(\tilde{G}))_{ij}\cdot\min_i\sum_j(T_2(\tilde{G}))_{ij}\geqslant1.\end{aligned}$$

Therefore $T_1(G)T_2(G)$ is not convergent. Our theorem is thus established.

If G is a five-diagonal matrix:

$$G=\begin{bmatrix} c_1 & -d_1 & & -e_1 & & \mathbf{0} \\ -b_2 & c_2 & -d_2 & & \ddots & \\ \ddots & \ddots & \ddots & \mathbf{0} & & -e_{n-m} \\ -a_{m+1} & & \ddots & \ddots & \ddots & \\ & \ddots\ \mathbf{0} & & \ddots & \ddots & -d_{n-1} \\ \mathbf{0} & & -a_n & & -b_n & c_n \end{bmatrix},$$

and
$$E_G=\begin{bmatrix} 0 & d_1 & & \mathbf{0} \\ b_2 & 0 & d_2 & \\ & \ddots & \ddots & d_{n-1} \\ \mathbf{0} & & b_n & 0 \end{bmatrix}, \quad F_G=\begin{bmatrix} & e_1 & & \mathbf{0} \\ & & \ddots & e_{n-m} \\ a_{m+1} & \mathbf{0} & & \\ \mathbf{0} & \ddots & a_n & \end{bmatrix},$$

then, $T_1(G)T_2(G)$ is an alternating direction iterative scheme, which is much similar to the alternating direction difference equation for solving an elliptic partial differential equation of second order. In [11], another alternating direction scheme for solving a system of linear algebraic equations is introduced. For this scheme we can also apply Theorems 1, 2 and 3 to prove its convergence.

Let A be an irreducible $n\times n$ matrix, $D=\operatorname{diag}A$, $\det D\neq 0$ and $B=D-A$. Similarly, for the matrix H and V, we define $D_H=\operatorname{diag}H$, $D_V=\operatorname{diag}V$, $B_H=D_H-H$, $B_V=D_V-V$ and

$$\Omega(A)=\{G:G=H+V,\ D_H+D_V=|D|,\ |B_H|+|B_V|=|B|\}.$$

Moreover, we denote

$$T_3(G)=(R+H)^{-1}(R-V), \qquad T_4(G)=(\tilde{R}+V)^{-1}(\tilde{R}-H),$$

where $R=\operatorname{diag}(r_1, r_2, \cdots, r_n)$ and $\tilde{R}=\operatorname{diag}(\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_n)$. Thus, the product $T_3(G)T_4(G)$ is a generalization of the alternating direction scheme in [11]. We have

Theorem 5. *The following three propositions are equivalent:*

1° *A is an H-matrix.*

2° *For arbitrary $G\in(A)$ and r_i in $[\tau_i, \infty)$, where $\tau_i=\max((D_V)_{ii}, (D_H)_{ii})$, $T_3(G)$ converges.*

3° *For arbitrary $G\in(A)$ and r_i and $\tilde{r}_i$ in $[\tau_i, \infty)$, where $\tau_i=\max((D_V)_{ii}, (D_H)_{ii})$, the generalized alternating direction iterative matrix $T_3(G)T_4(G)$ converges.*

Proof. The proof is similar to that of Theorem 4; so we explain it briefly.

If 1° holds, we have $Q=\operatorname{diag}(q_1, q_2, \cdots, q_n)$, mentioned in Theorem 3, such that

$$|D_{ii}|q_i>\sum_{j\neq i}|B_{ij}|q_j, \quad i=1, 2, \cdots, n.$$

Hence
$$(H_{ii}+V_{ii})q_i>\sum_{j\neq i}(|(B_H)_{ij}|+|(B_V)_{ij}|)q_j, \quad i=1, 2, \cdots, n.$$

But $T_3(G)$ is similar to

$$Q^{-1}T_3(G)Q=(RQ+D_HQ-B_HQ)^{-1}(RQ-D_VQ+B_VQ).$$

From

$$\rho(T_3(G))=\rho(Q^{-1}T_3(G)Q)\leqslant\|Q^{-1}T_3(G)Q\|_\infty$$
$$\leqslant\max_i\{[|r_i-V_{ii}|q_i+\sum_j|(B_V)_{ij}|q_j]/[|r_i+H_{ii}|q_i-\sum_j|(B_H)_{ij}|q_j]\},$$
$$(r_i+H_{ii})Q_i\geqslant(V_{ii}+H_{ii})q_i>\sum_j|(B_H)_{ij}|q_j\geqslant0$$

and

$$(r_i-V_{ii})\geqslant\tau-V_{ii}\geqslant0,$$

we can easily prove $T_3(G)$ and $T_3(G)T_4(G)$ to be convergent.

Conversely, if A is not an H-matrix, there exists a $Q=\mathrm{diag}(q_1, q_2, \cdots, q_n)$ with positive diagonal elements such that

$$|D_{ii}|q_i\leqslant\sum_j|B_{ij}|\cdot q_j,\quad i=1, 2, \cdots, n.$$

We choose such H and V that

$$D_H+D_V=|D|,\ B_H\geqslant0,\ B_V\geqslant0\quad\text{and}\quad B_H+B_V=|B|.$$

Hence

$$(H_{ii}+V_{ii})q_i\leqslant\sum_j((B_H)_{ij}+(B_V)_{ij})q_j,\quad i=1, 2, \cdots, n$$

and

$$\sum_j(QT_3(G)Q)_{ij}\geqslant\min_i\frac{(r_i-V_{ii})q_i+\sum_j(B_V)_{ij}q_j}{(r_i+H_{ii})q_i-\sum_j(B_H)_{ij}q_j}\geqslant1.$$

Here, we take r_i so large that $(r_i+H_{ii})q_i>\sum_j(B_H)_{ij}q_j$ and $r_i>V_{ii}$ $(i=1, 2, \cdots, n)$. From this the result follows.

§ 4. Other Applications of the Estimates of $\|M^{-1}N\|_\infty$ and the Optimally Scaled Matrix

The theorems in § 2 have many applications. At least, they may simplify the proofs of a lot of theorems. For example,

Theorem 6. *If $A\geqslant0$, $Q=\mathrm{diag}(q_1, q_2, \cdots, q_n)$, $q_i>0(i=1, 2, \cdots, n)$, then*

$$\min_i(\sum_j a_{ij}q_j/q_i)\leqslant\rho(A)\leqslant\max_i(\sum_j a_{ij}q_j/q_i).\tag{19}$$

Proof. Since A is similar to

$$Q^{-1}AQ=Q^{-1}(a_{ij}q_j),$$

(19) follows directly from Corollary 1.

It may be seen that, if A is not nonnegative, then we may still obtain (from Theorem 1)

$$\rho(A)\leqslant\max_i(\sum_j|a_{ij}|q_j/q_i).$$

Theorem 7. *If G is diagonally dominant and*

$$0<\omega_i,\quad \tilde{\omega}_i<2|g_{ii}|/\sum_j|g_{ij}|,\qquad i=1, 2, \cdots, n,\tag{20}$$

then $T_1(G)$ and $T_1(G)T_2(G)$ defined in § 3 are convergent.

Proof. From Theorem 1, it follows that

$$\|T_1(G)\|_\infty\leqslant\max_i\frac{|1-\omega_i||g_{ii}|+(\omega_i-r_i)\sum_{j\in E}|g_{ij}|+\omega_i\sum_{j\in F}|g_{ij}|}{|g_{ii}|-r_i\sum_{j\in E}|g_{ij}|}.\tag{21}$$

Under the conditions of our theorem, we can easily prove that the right-hand side of (21) is less than 1. Similarly, we have $\|T_2(G)\|_\infty<1$. Hence

$$\|T_1(G)T_2(G)\|_\infty<1.$$

This theorem is a generalization of Theorem 2 in [8]. It should be noticed that $2|g_{ii}|/\sum_j|g_{ij}|$ in (20) may be greater than or less than $2/(1+\rho(|D_G|^{-1}|B_G|))$ in Theorem 6. But if G is an optimally scaled matrix,

$$2|g_{ii}|/\sum_j|g_{ij}|=2/(1+\rho(|D_G|^{-1}|B_G|)),\quad i=1,\ 2,\ \cdots,\ n.$$

This is why we can deduce Theorem 6 by means of the optimally scaled matrix.

Obviously, not only the convergence but also the rate of convergence can be discussed from Theorems 1 and 2 of § 2. Perhaps, the results obtained in this manner are better than those obtained from (2). If we first use Theorem 3 and then use Theorem 1 or Theorem 2, we can obtain even better results. For example,

Theorem 8. *If G is an H-matrix, and $0<\omega_i<2/(1+\rho(|D_G|^{-1}|B_G|))$, then*

$$\rho(T_1(G))\leqslant\max_i(|1-\omega_i|+\omega_i\rho(|D_G|^{-1}|B_G|))<1.$$

Proof. Under the conditions of our theorem, we have at once

$$\begin{aligned}\rho(T_1(G))&=\rho(T_1(\tilde G))\leqslant\|T_1(\tilde G)\|_\infty\\&\leqslant\max_i\{[|1-\omega_i|\,|\tilde g_{ii}|+(\omega_i-r_i)\sum_{j\in E}|\tilde g_{ij}|\\&\quad+\omega_i\sum_{j\in F}|\tilde g_{ij}|]/[|\tilde g_{ii}|-r_i\sum_{j\in E}|\tilde g_{ij}|]\}\\&\leqslant\max_i\{[|1-\omega_i|+\omega_i\rho(|D_G|^{-1}|B_G|)\\&\quad-r_i\sum_{j\in E}|\tilde g_{ij}|/|\tilde g_{ii}|]/[1-r_i\sum_{j\in E}|\tilde g_{ij}|/|\tilde g_{ii}|]\}\\&\leqslant\max_i(|1-\omega_i|+\omega_i\rho(|D_G|^{-1}|B_G|))<1.\end{aligned}$$

Besides the results in § 3 and § 4, we have also proved theorems about M-matrix and other problems by the theorems and corollaries in § 2. This indicates that the estimates of $\|M^{-1}N\|_\infty$ and the optimally scaled matrix are indeed good tools for discussing some problems.

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